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ON THE INJECTIVE DOMINATION POLYNOMIAL OF GRAPHS

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Abstract. A subset S of vertices in a graph G is called injective dominating set if for every vertex v not in S there exists a vertex $u \in S$ such that $|\Gamma(u,v)| \geq 1$, where $|\Gamma(u,v)|$ is the number of common neighbors between the vertices u and v. The injective domination number $\gamma_{in}(G)$ of G is the minimum cardinality of such dominating sets. In this article, we introduce the injective domination polynomial of a graph G of order p as $D_{in}(G,x) = \sum_{j=\gamma_{in}(G)}^{p} d_{in}(G,j)x^j$, where $d_{in}(G,j)$ is the number of the injective dominating sets of G of size j. We obtain some properties of $D_{in}(G,x)$ and compute this polynomial for some specific graphs.

1 Introduction

All graphs considered here are finite, undirected without loops and multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of all vertices and edges of G, respectively. The open neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are defined by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The cardinality of $N(v)$ is called the degree of the vertex v and denoted by $deg(v)$ in G. The maximum and the minimum degrees in G are denoted respectively by $\Delta(G)$ and $\delta(G)$. That is $\Delta(G) = \max_{v\in V} |N(u)|$, $\delta(G) = \min_{v\in V} |N(u)|$. The distance between two vertices u and v in G is the number of edges in a shortest path connecting them, this is also known as the geodesic distance. The eccentricity of a vertex v is the greatest geodesic distance between v and any other vertex and denoted by $e(v)$. For more terminology and notations about graph, we refer the reader to [11, 12].
A subset D of $V(G)$ is called dominating set if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that v is adjacent to u. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. For more details about domination of graphs, we refer to [13].
The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with the vertex set $V(G)$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G [6].

Proposition 1.1 ([6]).

(i) $con(K_p) = K_p$.

(ii) $con(\overline{K_p}) = \overline{K_p}$.

(iii) $con(K_{r,m}) = K_r \cup K_m$.

(iv) $con(W_p) = K_p$.

(v) $con(P_p) \cong P_{\lceil \frac{p}{2} \rceil} \cup P_{\lfloor \frac{p}{2} \rfloor}$.

(vi) $con(C_p) \cong \begin{cases} C_p, & \textit{if } p \textit{ is odd and } p \geq 3; \\ P_2 \cup P_2, & \textit{if } p = 4; \\ C_{\frac{p}{2}} \cup C_{\frac{p}{2}}, & \textit{if } p \textit{ is even.} \end{cases}$

The common neighborhood (CN-neighborhood) of a vertex $v \in V(G)$ denoted by $N_{cn}(v)$ is defined by $N_{cn}(v) = \{u \in V(G) : uv \in E(G) \text{ and } |\Gamma(u,v)| \geq 1\}$, where $|\Gamma(u,v)|$ is the number of common neighborhood between the vertices u and v, [10]. The concept of injective domination in graph has introduced in [8]. For a graph G, a subset S of $V(G)$ is called injective dominating set if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $|\Gamma(u,v)| \geq 1$. The minimum cardinality of such dominating set denoted by $\gamma_{in}(G)$ and is called the injective domination number of G. The injective neighborhood $N_{in}(v)$ of a vertex $v \in V(G)$ is defined by $N_{in}(v) = \{u \in V(G) : |\Gamma(u,v)| \geq 1\}$. The cardinality of $N_{in}(v)$ is called the injective degree of the vertex v and denoted by $deg_{in}(v)$ in G, and $N_{in}[v] = N_{in}(v) \cup \{v\}$. A vertex v in G is called injective isolated (Inj-isolated) vertex if and only if v is isolated or a center vertex of a star component of G. For more details about the injective domination of graphs, we refer to [1, 7].

Proposition 1.2 ([8]). *For any graph G, $\gamma_{in}(G) = \gamma(con(G))$.*

Proposition 1.3 ([8]). *Let G be a graph with p vertices. Then $\gamma_{in}(G) = p$ if and only if G is a forest with $\Delta(G) \leq 1$.*

Proposition 1.4 ([8]). *Let G be a nontrivial connected graph. Then $\gamma_{in}(G) = 1$ if and only if there exists a vertex $v \in V(G)$ such that $N(v) = N_{cn}(v)$ and $e(v) \leq 2$.*

Let $\mathcal{D}(G,j)$ be the family of dominating sets of a graph G of size j and let $d(G,j) = |\mathcal{D}(G,j)|$. The domination polynomial $D(G,x)$ of G is defined by $D(G,x) = \sum_{j=\gamma(G)}^{p} d(G,j)x^j$, [5]. The dominating sets and the domination polynomial of graphs have been studied extensively, for example in [5, 3, 4, 2]. Recently, the neighborhood polynomial of graphs has studied in [9].

There are many graph polynomial have introduced and studied extensively like Characteristic polynomial, Chromatic polynomial, Matching polynomial, Tutte polynomial...etc. The graph polynomial is one of the ways for algebraic graph representation. By the analysis of graph polynomial and studied its properties we can get some information about the graph, that motivated us to introduce a new type of graph polynomial is called injective domination polynomial of graphs. In this paper, we introduce the injective domination polynomial of graphs. Some properties of $D_{in}(G,x)$ are obtained and exact formulas for some specific graphs are computed.

2 Injective domination polynomial of graphs

In this section, we define the injective domination polynomial of a graph G and study some of its properties.

Definition 2.1. Let G be a graph on p vertices. The injective domination polynomial of G is denoted by $D_{in}(G,x)$ and defined as

$$D_{in}(G,x) = \sum_{j=\gamma_{in}(G)}^{p} d_{in}(G,j)x^j,$$

where $\gamma_{in}(G)$ is the injective domination number of G, and $d_{in}(G,j)$ is the number of injective dominating sets of G of size j.

For instance, the cycle C_4 has one Inj-dominating set of size four and four Inj-dominating sets of size three and two, then the injective domination polynomial of C_4 is $D_{in}(C_4,x) = x^4 + 4x^3 + 4x^2$. From Proposition 1.2 and Definition 2.1, it is easy to check the following proposition.

Proposition 2.2. *For any graph G, $D_{in}(G,x) = D(con(G),x)$.*

Theorem 2.3. *Let G be a graph on $p \geq 2$ vertices. Then $D_{in}(G,x) = \big(1+x\big)^p - 1$ if and only if for every vertex $v \in V(G)$, $N_{cn}(v) = N(v)$ and $e(v) \leq 2$.*

Proof. Suppose $D_{in}(G,x) = \big(1+x\big)^p - 1$. Then $D_{in}(G,x) = \sum_{j=1}^{p} \binom{n}{j} x^j$, which means that any vertex v in G has a full Inj-degree. Hence by Proposition 1.4, $N_{cn}(v) = N(v)$ and $e(v) \leq 2$, $\forall v \in V(G)$. The converse is clear. □

Corollary 2.4. *For any complete graph K_p with $p \geq 3$, $D_{in}(K_p, x) = \big(1+x\big)^p - 1$.*

Lemma 2.5 ([5]). *If a graph G consists of m components $G_1, \dots, G_m$, then*

$$D(G,x) = D(G_1,x)\dots D(G_m,x)$$

.

By the same argument, Lemma 2.5 is also true for the injective domination polynomial of any graph G with m components.

Lemma 2.6. *If a graph G consists of m components $G_1, \dots, G_m$, then*

$$D_{in}(G,x) = D_{in}(G_1,x)\dots D_{in}(G_m,x)$$

.

Proof. The proof is straightforward from Proposition 2.2 and Lemma 2.5. □

Lemma 2.7 ([5]). *Let $\overline{K_p}$ be the empty graph with p vertices. Then $D(\overline{K_p}, x) = x^p$.*

Theorem 2.8. *Let G be a graph on p vertices. Then $D_{in}(G,x) = x^p$ if and only if G is a forest with $\Delta(G) \leq 1$.*

Proof. Suppose $D_{in}(G,x) = x^p$. Then by Proposition 2.2 and Lemma 2.7, $con(G) \cong \overline{K_p}$. But $\overline{K_p} \cong con(nK_1 \cup mK_2)$ for some $n, m \in \mathbb{Z}^+ \cup \{0\}$, where $p = n + 2m$. Hence, G is a forest with $\Delta(G) \leq 1$. The converse is clear. □

In the following theorem we obtain the injective domination polynomial of the join graph $G_1 + G_2$ of two graphs G_1 and G_2.

Theorem 2.9. *Let G_1 and G_2 be any two graphs of orders p_1 and p_2, respectively.*

(i) If G_1 or G_2 is an isolated-free graph, then

$$D_{in}(G_1 + G_2, x) = \big(1+x\big)^{p_1+p_2} - 1.$$

(ii) If G_1 and G_2 have isolated vertices, then

$$D_{in}(G_1+G_2,x) = \begin{cases} \prod\limits_{k=1}^{2}\Big((1+x)^{p_k}-1\Big) + \sum\limits_{k=1}^{2}\Big((1+x)^{p_k}-(1+x)^{r_k}\Big), & \text{if } G_1 \neq \overline{K_{p_1}} \text{ and } G_2 \neq \overline{K_{p_2}}\,; \\ \prod\limits_{k=1}^{2}\Big((1+x)^{p_k}-1\Big) + (1+x)^{p_1}-(1+x)^{r_1}, & \text{if } G_1 \neq \overline{K_{p_1}} \text{ and } G_2 = \overline{K_{p_2}}; \\ \prod\limits_{k=1}^{2}\Big((1+x)^{p_k}-1\Big) + (1+x)^{p_2}-(1+x)^{r_2}, & \text{if } G_1 = \overline{K_{p_1}} \text{ and } G_2 \neq \overline{K_{p_2}}; \\ \prod\limits_{k=1}^{2}\Big((1+x)^{p_k}-1\Big), & \text{if } G_1 = \overline{K_{p_1}} \text{ and } G_2 = \overline{K_{p_2}}, \end{cases}$$

where r_1 and r_2 are the number of isolated vertices of G_1 and G_2, respectively.

Proof.

(i) Suppose G_1 or G_2 is an isolated-free graph. Then for any vertex $v \in V(G_1+G_2)$, $N_{cn}(v) = N(v)$ and $e(v) \leq 2$. Hence by Theorem 2.3, $D_{in}(G_1 + G_2, x) = \big(1+x\big)^{p_1+p_2} - 1$.

(ii) Suppose $G_1 \neq \overline{K_{p_1}}$. Then any non-isolated vertex of G_1 forms an injective dominating set of $G_1 + G_2$. Thus any subset of vertices of G_1 contains at least one non-isolated vertex forms an injective dominating set of $G_1 + G_2$. Hence, we have $\binom{p_1}{j} - \binom{r_1}{j}$ injective dominating sets of $G_1 + G_2$ of size $1 \leq j \leq p_1$ (we can do the same for G_2). On the other hand, suppose $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ be any subsets of vertices of G_1 and G_2, respectively, such that $|D_1| + |D_2| = j$. Clearly that, $D_1 \cup D_2$ is an Inj-dominating set of $G_1 + G_2$ of size j. Hence the result.

□

As a corollary of Theorem 2.9, we have the following formula for the injective domination polynomial of the complete bipartite graph $K_{r,m}$, the wheel graph W_p and the star S_p.

Corollary 2.10.

(i) $D_{in}(K_{r,m},x) = \Big((1+x)^r - 1\Big)\Big((1+x)^m - 1\Big)$.

(ii) *If* $p \geq 4$, $D_{in}(W_p,x) = (1+x)^p - 1$.

(iii) $D_{in}(S_p,x) = x(1+x)^{p-1} - x$.

Corollary 2.11. *Let* $G = G_1 + G_2 + \cdots + G_n$ *for any graphs* G_k, $k = 1,2,\ldots,n$, *where* $n \geq 3$. *Then*

$$D_{in}(G,x) = (1+x)^{\sum_{k=1}^{n} p_k} - 1.$$

Proof. Since $G = G_1 + G_2 + \cdots + G_n$, where $n \geq 3$, then for any vertex $v \in V(G)$, $N_{cn}(v) = N(v)$ and $e(v) \leq 2$. Hence by Theorem 2.3, $D_{in}(G,x) = (1+x)^{\sum_{k=1}^{n} p_k} - 1$. □

The following proposition is an easy consequence from the definition of the injective domination polynomial of graphs.

Proposition 2.12. *Let* G *be a graph on* p *vertices. Then*

(i) If G *is a connected graph and* $G \neq S_p$, *then* $d_{in}(G,p) = 1$ *and* $d_{in}(G,p-1) = p$.

(ii) $d_{in}(G,j) = 0$ *if and only if* $j < \gamma_{in}(G)$ *or* $j > p$.

(iii) $D_{in}(G,x)$ *has no constant term.*

(iv) $D_{in}(G,x)$ *is a strictly increasing function in* $[0,\infty)$.

(v) The only polynomial of degree two can $D_{in}(G,x)$ *be equal is* x^2 *if and only if* $G \cong K_2$ *or* $G \cong \overline{K_2}$.

(vi) Let H *be any induced subgraph of* G. *Then*

$$deg\big(D_{in}(G,x)\big) \geq deg\big(D_{in}(H,x)\big).$$

(vii) Zero is a root of $D_{in}(G,x)$, *with multiplicity* $\gamma_{in}(G)$.

Theorem 2.13. *Let* G *be a graph of order* p *with* t *vertices of Inj-degree one and* r *Inj-isolated vertices. If* $D_{in}(G,x) = \sum_{j=\gamma_{in}(G)}^{p} d_{in}(G,j)x^j$ *is its domination polynomial, then the following hold:*

(i) $r = p - d_{in}(G,p-1)$.

(ii) If G *has* s *path* P_3*-components, then* $d_{in}(G,p-2) = \binom{p}{2} - t + s - r(p-1) + \binom{r}{2}$.

(iii) If G *has no Inj-isolated vertices and* $D_{in}(G,-2) \neq 0$, *then* $t = \binom{p}{2} - d_{in}(G,p-2)$.

(iv) $d_{in}(G,1) = |\{v \in V(G) : |deg_{in}(v) = p-1\}|$.

Proof.

(i) Suppose $B \subseteq V(G)$ be the set of all Inj-isolated vertices of G. Then by assumption, $|B| = r$. It is clear that, for any vertex $v \in V(G) - B$, the set $V(G) - \{v\}$ is an Inj-dominating set of G of size $p-1$. Hence, $d_{in}(G,p-1) = |V(G) - B| = p - r$.

(ii) Suppose $D \subseteq V(G)$ be a set of G of size $p-2$ which is not Inj-dominating set. To compute how many D sets in G, we have two cases:
Case 1. $D = V(G) - \{u,v\}$, where u or v is an Inj-isolated vertex in G. So, for every Inj-isolated vertex u in G, there are $p-1$ vertices such that $V(G) - \{u,v\}$ is not an Inj-dominating set of G. Therefore, the total number of $(p-2)$-subsets of vertices of G of the form $V(G) - \{u,v\}$ which is not Inj-dominating set, where u or v is an Inj-isolated vertex is $r(p-1) - \binom{r}{2}$, since if u and v are Inj-isolated vertices, then we count $V(G) - \{u,v\}$ for both u and v.
Case 2. $D = V(G) - \{u,v\}$, where u and v are Inj-adjacent (u and v have at least a common neighbor) and $deg_{in}(u) = 1$. Since we have s P_3-components, then the number of such Inj-edges $\{u,v\}$ is $t-s$. Hence the result.

(iii) Since $D_{in}(G,-2) \neq 0$, then by Lemma 2.6, G has no P_3-components. Hence by Part (ii), $t = \binom{p}{2} - d_{in}(G,p-2)$.

(iv) For any vertex $v \in V(G)$, the set $\{v\}$ is an Inj-dominating set of G if and only if $N(v) = N_{cn}(v)$ and $e(v) \leq 2$ (Proposition 1.4), which means that $deg_{in}(v) = p-1$.

□

Lemma 2.14 ([5]). *Let G be a graph of order p. Then for every $0 \leq j < \frac{p}{2}$, we have $d(G,j) \leq d(G,j+1)$.*

Proposition 2.15. *Let G be a graph of order p. Then for every $0 \leq j < \frac{p}{2}$, we have $d_{in}(G,j) \leq d_{in}(G,j+1)$.*

Proof. The proof follows from Proposition 2.2 and Lemma 2.14. □

3 Injective domination polynomial of some specific graphs

In this section, we compute the injective domination polynomial of the path P_p, cycle C_p, firefly graph and the corona product $K_n \circ \overline{K_m}$.

3.1 Injective domination polynomial of paths and cycles

Lemma 3.1 ([4]). *For any $p \geq 4$,*

$$D(P_p,x) = x\big[D(P_{p-1},x) + D(P_{p-2},x) + D(P_{p-3},x)\big],$$

with the initial values $D(P_1,x) = x$, $D(P_2,x) = x^2 + 2x$ and $D(P_3,x) = x^3 + 3x^2 + x$.

Theorem 3.2. *For any $p \geq 2$, the injective domination polynomial of the path P_p is given by*

$$D_{in}(P_p,x) = \begin{cases} \big[D(P_{\frac{p}{2}},x)\big]^2, & \text{if } p \text{ is even;} \\ D(P_{\frac{p+1}{2}},x)D(P_{\frac{p-1}{2}},x), & \text{if } p \text{ is odd,} \end{cases}$$

with the initial values $D(P_1,x) = x$, $D(P_2,x) = x^2 + 2x$ and $D(P_3,x) = x^3 + 3x^2 + x$.

Proof. The proof is straightforward from Proposition 1.1 and Lemmas 2.5, 3.1. □

Lemma 3.3 ([3]). *For any $p \geq 4$,*

$$D(C_p,x) = x\big[D(C_{p-1},x) + D(C_{p-2},x) + D(C_{p-3},x)\big],$$

with the initial values $D(C_1,x) = x$, $D(C_2,x) = x^2 + 2x$ and $D(C_3,x) = x^3 + 3x^2 + 3x$.

Theorem 3.4. *For any $p \geq 3$, the injective domination polynomial of the cycle C_p is given by*

$$D_{in}(C_p,x) = \begin{cases} D(C_p,x), & \text{if } p \text{ is odd;} \\ \big[D(P_2,x)\big]^2, & \text{if } p = 4; \\ \big[D(C_{\frac{p}{2}},x)\big]^2, & \text{if } p \text{ is even and } p \geq 6, \end{cases}$$

with the initial values $D(C_1,x) = x$, $D(C_2,x) = x^2 + 2x$ and $D(C_3,x) = x^3 + 3x^2 + 3x$.

Proof. The proof is straightforward from Proposition 1.1 and Lemmas 2.5, 3.3. □

3.2 Injective domination polynomial of the firefly graph

A firefly graph $F_{s,t,p-2s-2t-1}$ ($s \geq 0$, $t \geq 0$ and $p-2s-2t-1 \geq 0$) is a graph of order p that consists of s triangles, t pendent paths of length 2 and $p-2s-2t-1$ pendent edges, sharing a common vertex.

Let $\mathfrak{F}_p$ be the set of all firefly graphs $F_{s,t,p-2s-2t-1}$. Note that $\mathfrak{F}_p$ contains the stars S_p ($\cong F_{0,0,p-1}$), stretched stars ($\cong F_{0,t,p-2t-1}$), friendship graphs ($\cong F_{\frac{p-1}{2},0,0}$) and butterfly graphs ($\cong F_{s,0,p-2s-1}$), [14]. In the following, we will discuss the injective domination polynomial of the firefly graph in cases $t = 0$, $t \neq 0$.

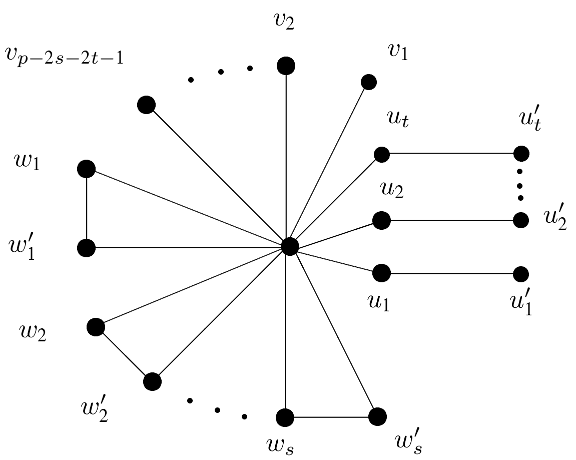

Figure 1. Firefly graph $F_{s,t,p-2s-2t-1}$

Lemma 3.5. *For the firefly graph $F_{s,t,p-2s-2t-1}$,*

$$\gamma_{in}(F_{s,t,p-2s-2t-1}) = \begin{cases} 1, & \text{if } t = 0 \text{ and } s > 0; \\ 2, & \text{otherwise.} \end{cases}$$

Proof. In general, any set contains the center and another adjacent vertex will Inj-dominate all the other vertices in $F_{s,t,p-2s-2t-1}$, then $\gamma_{in}(F_{s,t,p-2s-2t-1}) \leq 2$. Now, if $t = 0$ and $s > 0$, then $F_{s,0,p-2s-2t-1}$ contains a vertex v satisfies $N(v) = N_{cn}(v)$ and $e(v) \leq 2$, so by Proposition 1.4, $\gamma_{in}(F_{s,0,p-2s-2t-1}) = 1$. Otherwise, $F_{s,t,p-2s-2t-1}$ does not contain a vertex satisfies Proposition 1.4. Hence, $\gamma_{in}(F_{s,t,p-2s-2t-1}) = 2$. □

Theorem 3.6. *For the firefly graph $F_{s,0,p-2s-1}$, where $t = 0$, $s > 0$ and $p > 2s+1$,*

$$d_{in}(F_{s,0,p-2s-1}, j) = \begin{cases} 2s, & \text{if } j = 1; \\ \sum_{i=1}^{2s+1} \binom{p-i}{j-1}, & \text{if } j \geq 2. \end{cases}$$

Hence, $D_{in}(F_{s,0,p-2s-1}, x) = 2sx + \sum_{j=2}^{p} \left[\sum_{i=1}^{2s+1} \binom{p-i}{j-1} \right] x^j.$

Proof. In this case by Lemma 3.5, $\gamma_{in}(F_{s,0,p-2s-1}) = 1$ (since $F_{s,0,p-2s-1}$ contains a vertex satisfies Proposition 1.4). Actually, $F_{s,0,p-2s-1}$ contains $2s$ vertices satisfy Proposition 1.4 which they are all the vertices of the triangles except the center vertex. On the other hand, any Inj-dominating set of size $j \geq 2$ must contain at least one vertex from the triangles, thus we have $\sum_{i=1}^{2s+1} \binom{p-i}{j-1}$ possibilities. □

Proposition 3.7. *For the friendship graph $F_{\frac{p-1}{2},0,0}$,*

$$D_{in}(F_{\frac{p-1}{2},0,0}, x) = (1+x)^{\frac{p-1}{2}} - 1.$$

Proof. It is easy to see that $N_{cn}(v) = N(v)$ and $e(v) \leq 2$, $\forall v \in V(F_{\frac{p-1}{2},0,0})$. Hence by Theorem 2.3, the result is obtained. □

Now, in case $t \neq 0$, we divide the Inj-dominating sets of $F_{s,t,p-2s-2t-1}$ with respect to the center vertex to Inj-dominating sets contain and do not contain the center vertex.

Lemma 3.8. *The number of Inj-dominating sets which contain the center vertex and of size j, where $2 \leq j \leq p$ in a firefly graph $F_{s,t,p-2s-2t-1}$ is*

$$d_{in}(F_{s,t,p-2s-2t-1}, j) = \begin{cases} p-t-1, & \text{if } j = 2; \\ \displaystyle\sum_{i=1}^{p-t-1} \binom{p-1-i}{j-2}, & \text{if } j \geq 3. \end{cases}$$

Proof. From the proof of Lemma 3.5, any set contains the center and another adjacent vertex in $F_{s,t,p-2s-2t-1}$ is an Inj-dominating set of size two, thus we have $p-t-1$ Inj-dominating set of size two. Otherwise, any Inj-dominating set of size $j \geq 3$ containing the center vertex must contain at least on edge joining the center vertex with any other vertex, so we have $\sum_{i=1}^{p-t-1} \binom{p-1-i}{j-2}$ possibilities. □

Lemma 3.9. *The number of Inj-dominating sets which do not contain the center vertex and of size j, where $t+1 \leq j \leq p-1$ in a firefly graph $F_{s,t,p-2s-2t-1}$ is*

$$d_{in}(F_{s,t,p-2s-2t-1}, j) = \begin{cases} p-t-1, & \text{if } j = t+1; \\ \displaystyle\sum_{i=1}^{p-t-2} \binom{p-i-(t+1)}{j-(t+1)}, & \text{if } t+2 \leq j \leq p-1. \end{cases}$$

Proof. It is clear that, any Inj-dominating set of $F_{s,t,p-2s-2t-1}$ does not contain the center vertex must contain all the end vertices of the t pendant P_3 paths in $F_{s,t,p-2s-2t-1}$ and at least one vertex from the other vertices. Therefore, there are $p-t-1$ Inj-dominating sets of size $t+1$ and $\sum_{i=1}^{p-t-2} \binom{p-i-(t+1)}{j-(t+1)}$ possibilities of Inj-dominating sets of size $t+2 \leq j \leq p-1$. □

Theorem 3.10. *For the firefly graph $F_{s,t,p-2s-2t-1}$, where $t \neq 0$,*

$$D_{in}(F_{s,t,p-2s-2t-1}, x) = (p-t-1)(x^2 + x^{t+1}) + \sum_{j=3}^{p} \left[\sum_{i=1}^{p-t-1} \binom{p-1-i}{j-2} \right] x^j + \sum_{j=t+2}^{p-1} \left[\sum_{i=1}^{p-t-2} \binom{p-i-(t+1)}{j-(t+1)} \right] x^j.$$

Proof. The proof is straightforward by the definition of the injective domination polynomial of graphs and Lemmas 3.8, 3.9. □

3.3 Injective domination polynomial of $K_n \circ \overline{K_m}$

We start by the following proposition:

Proposition 3.11. *Let $G \cong K_n \circ \overline{K_m}$. Then $\gamma_{in}(G) = 2$.*

Proof. It is easy to see that any two adjacent vertices in G Inj-dominate all the other vertices, so $\gamma_{in}(G) \leq 2$. But G has no a vertex of full Inj-degree (a vertex v satisfies $N_{cn}(v) = N(v)$ and $e(v) \leq 2$, Proposition 1.4). Hence, $\gamma_{in}(G) = 2$. □

According to Proposition 3.11, $d_{in}(K_n \circ \overline{K_m}, j) = 0$ for $j < 2$ or $j > n(m+1)$. Thus, we will compute $d_{in}(K_n \circ \overline{K_m}, j)$ for $2 \leq j \leq n(m+1)$. To make more simplicity, in the following two lemmas we compute $d_{in}(K_n \circ \overline{K_m}, j)$ for the independent Inj-dominating sets and the non independent Inj-dominating sets, respectively.

Lemma 3.12. *For any independent Inj-dominating set of size j ($n \leq j \leq mn$),*

$$d_{in}(K_n \circ \overline{K_m}, j) = \sum_{b_1+b_2+\cdots+b_m=n} \binom{n}{b_1, b_2, \ldots, b_m} \prod_{i=1}^{m} \binom{m}{i}^{b_i},$$

where $j = b_1 + 2b_2 + \cdots + mb_m$ and $0 \leq b_1, b_2, \ldots, b_m \leq n$.

Proof. It is not difficult to see that any independent Inj-dominating set of $K_n \circ \overline{K_m}$ must contain at least one vertex from each copy of $\overline{K_m}$. Thus the number of all independent Inj-dominating sets of $K_n \circ \overline{K_m}$ is given by the multinomial

$$\left[\sum_{i=1}^{m} \binom{m}{i}\right]^n = \sum_{b_1+b_2+\cdots+b_m=n} \binom{n}{b_1, b_2, \ldots, b_m} \prod_{i=1}^{m} \binom{m}{i}^{b_i},$$

where $0 \leq b_1, b_2, \ldots, b_m \leq n$. So, to determine $d_{in}(K_n \circ \overline{K_m}, j)$ for each $n \leq j \leq mn$, we need to know how many vertex should be chosen from each copy of $\overline{K_m}$ by determine the numbers $b_1, b_2, \ldots, b_m$ such that $\sum_{i=1}^{m} b_i = n$ and $j = \sum_{i=1}^{m} ib_i$. Hence,

$$d_{in}(K_n \circ \overline{K_m}, j) = \sum_{b_1+b_2+\cdots+b_m=n} \binom{n}{b_1, b_2, \ldots, b_m} \prod_{i=1}^{m} \binom{m}{i}^{b_i},$$

where $j = b_1 + 2b_2 + \cdots + mb_m$. □

Lemma 3.13. *For any non independent Inj-dominating set of size j,*

$$d_{in}(K_n \circ \overline{K_m}, j) = n\sum_{i=1}^{m} \binom{mn-i}{j-2} + \sum_{i=2}^{n} \binom{n}{i}\binom{mn}{j-i},$$

where $2 \leq j \leq n(m+1)$.

Proof. Suppose S is a non independent Inj-dominating set of $K_n \circ \overline{K_m}$ of size j ($2 \leq j \leq n(m+1)$). We divide the Inj-dominating sets of $K_n \circ \overline{K_m}$ here into two parts, Inj-dominating sets contain exactly one vertex of K_n which they have $n\sum_{i=1}^{m}\binom{mn-i}{j-2}$ possibilities and Inj-dominating sets contain more than one vertex of K_n which they have $\sum_{i=2}^{n}\binom{n}{i}\binom{mn}{j-i}$ possibilities. Hence,

$$d_{in}(K_n \circ \overline{K_m}, j) = n\sum_{i=1}^{m} \binom{mn-i}{j-2} + \sum_{i=2}^{n} \binom{n}{i}\binom{mn}{j-i}.$$

□

Theorem 3.14. *For any $n, m \geq 1$,*

$$\begin{aligned} D_{in}(K_n \circ \overline{K_m}, x) = &\sum_{j=2}^{n(m+1)} \left[n\sum_{i=1}^{m} \binom{mn-i}{j-2} + \sum_{i=2}^{n} \binom{n}{i}\binom{mn}{j-i}\right] x^j \\ &+ \sum_{\sum_{i=1}^{m} ib_i = n}^{nm} \left[\sum_{\sum_{i=1}^{m} b_i = n} \binom{n}{b_1, b_2, \ldots, b_m} \prod_{i=1}^{m} \binom{m}{i}^{b_i}\right] x^{\sum_{i=1}^{m} ib_i}, \end{aligned}$$

where $0 \leq b_1, b_2, \ldots, b_m \leq n$.

Proof. The proof is straightforward by the definition of the injective domination polynomial of graphs and Lemmas 3.12, 3.13. □

Example 3.15.

(i) $$\begin{aligned} D_{in}(S_p, x) =& D_{in}(K_1 \circ \overline{K_{p-1}}, x) = \sum_{j=2}^{p} \left[\sum_{i=1}^{p-1} \binom{p-1-i}{j-2}\right] x^j \\ =& (p-1)x^2 + \binom{p-1}{2}x^3 + \binom{p-1}{3}x^4 + \cdots + x^p = x(1+x)^{p-1} - x. \end{aligned}$$

(ii) $D_{in}(K_n \circ K_1, x) = \sum_{j=2}^{2n} \left[n\binom{n-1}{j-2} + \sum_{i=2}^{n} \binom{n}{i}\binom{n}{j-i} \right] x^j + x^n.$

(iii) $$D_{in}(K_2 \circ \overline{K_m}, x) = \sum_{j=2}^{2(m+1)} \left[\binom{2m}{j-2} + 2\sum_{i=1}^{m} \binom{2m-i}{j-2} \right] x^j + \sum_{\sum_{i=1}^{m} ib_i=2}^{2m} \left[\sum_{\sum_{i=1}^{m} b_i=2} \binom{2}{b_1, b_2, \ldots, b_m} \prod_{i=1}^{m} \binom{m}{i}^{b_i} \right] x^{\sum_{i=1}^{m} ib_i}$$

References

[1] Akram Alqesmah, Anwar Alwardi and R. Rangarajan, *Connected injective domination of graphs*, Bulletin of the International Mathematical Virtual Institute, **7**, 73–83 (2017).

[2] S. Alikhani, Y. H. Peng, *Dominating sets and domination polynomial of certain graphs II*, Opuscula Mathematica, **30(1)**, 37–51 (2010).

[3] S. Alikhani, Y.H. Peng, *Dominating sets and domination polynomial of cycles*, Global Journal of Pure and Applied Mathematics, **4(2)**, 151–162 (2008).

[4] S. Alikhani, Y.H. Peng, *Dominating sets and domination polynomial of paths*, International Journal of Mathematics and Mathematical Sciences, Article ID 542040, doi:10.1155/2009/542040, (2009).

[5] S. Alikhani, Y.H. Peng, *Introduction to domination polynomial of a graph*, Ars Combin., **114**, 257–266 (2014).

[6] A. Alwardi, B. Arsit'c, I. Gutman, N. D. Soner, *The common neighborhood graph and its energy*, Iran. J. Math. Sci. Inf., **7(2)**, 1–8 (2012).

[7] Anwar Alwardi, Akram Alqesmah and R. Rangarajan, *Independent injective domination of graphs*, Int. J. Adv. Appl. Math. and Mech. **3(4)**, 142Ű-151 (2016).

[8] Anwar Alwardi, R. Rangarajan and Akram Alqesmah, *On the Injective domination of graphs*, In communication.

[9] Anwar Alwardi and P.M. Shivaswamy, *On the neighborhood polynomial of graphs*, Bulletin of the International Mathematical Virtual Institute, **6**, 13–24 (2016).

[10] Anwar Alwardi, N. D. Soner and Karam Ebadi, *On the common neighbourhood domination number*, Journal of Computer and Mathematical Sciences, **2(3)**, 547–556 (2011).

[11] J. A. Bondy, U. S. R. Murty, *Graph theory with applications*, The Macmillan Press Ltd., London, Basingstoke (1976).

[12] F. Harary, *Graph theory*, Addison-Wesley, Reading Mass (1969).

[13] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc., New York (1998).

[14] W. X. Hong, L. H. You, *On the eigenvalues of firefly graphs*, Transactions on Combinatorics **3(3)**, 1–9 (2014).

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