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Weakly Additively Regular Rings and Special Families of Prime Ideals

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Abstract A commutative ring R is additively regular if for each pair of elements $f, g \in R$ with f regular (meaning not a zero divisor), there is an element $r \in R$ such that $g + rf$ is regular. Also R is a Marot ring if each regular ideal can be generated by a set of regular elements. Each additively regular ring is Marot, but a Marot ring need not be additively regular. A property that sits properly between these two is the notion of weakly additively regular: for $f, g \in R$ with f regular, there is a pair of elements $s, t \in R$ such that $gs + ft$ is regular and $sR + fR = R$. Each of these properties can be defined with regard to the set of prime ideals that contain only zero divisors. Thus we introduce the following notions for a given nonempty family of primes $\mathcal{P} = \{P_\alpha\}$. First, $\mathcal{P}$ is an additively regular family if for each pair of elements $f, g \in R$ with $f \in R\backslash \bigcup P_\alpha$, there is an element $r \in R$ such that $g + fr \in R\backslash \bigcup P_\alpha$. Similarly, $\mathcal{P}$ is a weakly additively regular family if for $f, g \in R$ with $f \in R\backslash \bigcup P_\alpha$, there is a pair of elements $s, t \in R$ such that $gs + ft \in R\backslash \bigcup P_\alpha$ and $sR + fR = R$. Finally, $\mathcal{P}$ is a Marot family if each ideal I that is not contained in $\bigcup P_\alpha$ can be generated by the set $I \cap S$ where $S = R\backslash \bigcup P_\alpha$.

1 Introduction

Throughout the paper, each ring is commutative with a nonzero identity. For a ring R, $Z(R)$ denotes the set of zero divisors of R, $T(R) = \{r/s \mid r, s \in R,\ s \notin Z(R)\}$ denotes the total quotient ring of R and $\mathrm{Reg}(R) = R\backslash Z(R)$ denotes the set of regular elements of R. Also an ideal is regular if it contains at least one regular element.

The notion of an additively regular ring was introduced by Gilmer and Huckaba in [4] (also see [3, Lemma B]). Specifically, a ring R is said to be *additively regular* if for each element $t \in T(R)$, there is an element $d \in R$ such that $t + d$ is regular (so a unit in $T(R)$). Equivalently, if $b, c \in R$ with b regular, then there is an element $h \in R$ such that $c + bh$ is a regular element of R. A related concept is that of a Marot ring: R is a *Marot ring* if each regular ideal can be generated by a set of regular elements. It is easy to show that R is Marot if and only if for each pair of elements $b, c \in R$ with b regular, the ideal $bR + cR$ can be generated by a (finite) set of regular elements. With this it is easy to see that an additively regular ring is a Marot ring. Examples are known of Marot rings that are not additively regular, we will see others below.

A ring R has the *regular finite union property* if for each regular ideal I and each finite set of regular ideals $\{J_1, J_2, \ldots, J_n\}$, I is contained in $\bigcup J_i$ if and only if the regular elements of I are contained in $\bigcup J_i$. In [13], Portelli and Spangher noted that each additively regular ring satisfies (the then unnamed) regular finite union property. Later, Matsuda gave an example of a Marot ring that does not have the regular finite union property [12, Theorem 10]. (Note that in [12], R is said to have *property (FU)*, we have used "regular finite union property" in order to have a more complete description.) He also showed that a ring with the regular finite union property need not be additively regular [12, Propositions 11 & 12]. Below we show that this ring is weakly additively regular (see Example 5.1). The ring R in [10, Example 2.4] shows that with regard to the finite union property, it is not enough to restrict the finite sets $\{J_1, J_2, \ldots, J_n\}$ to (regular) prime ideals (also see [1, Example 3.6]).

The original intent of this article was to introduce the notion of a *weakly additively regular ring* as a ring R with the property that for each pair of elements $f, g \in R$ with f regular and $g \in Z(R)$, there is a pair of elements $s, t \in R$ such that $gs + ft$ is regular and $sR + fR = R$. However, in constructing examples, it became apparent that all of these notions can be generalized with regard to nonempty sets of primes and the complement of the union of these primes. Before introducing the definitions, we recall that a ring R is said to have *few zero divisors* if $Z(R)$ can be realized as the union of a finite set of prime ideals (equivalently, $T(R)$ has only finitely many maximal ideals). It is known that a ring with few zero divisors is additively regular (see [3, Lemma B]). Every total quotient ring is additively regular, so a ring that is additively regular need not have few zero divisors. Also for each ring R, the polynomial ring $R[X]$ is additively regular (see [13, Proposition 3] and [6, Theorem 7.5]).

Let R be a ring and let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of prime ideals of R. Also let $S = R \backslash \bigcup P_\alpha$. We introduce the following special types of families.

(i) $\mathcal{P}$ is a *FZD family* if there is a finite set of primes $\{Q_1, Q_2, \ldots, Q_n\}$ such that $\bigcup P_\alpha = \bigcup Q_i$.

(ii) $\mathcal{P}$ is an *additively regular family* if for each pair of elements $f, g \in R$ with $g \in \bigcup P_\alpha$ and $f \in S$, there is an element $t \in R$ such that $g + ft \in S$.

(iii) $\mathcal{P}$ is a *weakly additively regular family* if for each pair of elements $f, g \in R$ with $g \in \bigcup P_\alpha$ and $f \in S$, there is a pair of elements $s, t \in R$ such that $gs + ft \in S$ and $sR + fR = R$.

(iv) $\mathcal{P}$ is a *finite union family* if for each finite set of ideals $\{J_1, J_2, \ldots, J_n\}$ where no J_i is contained in $\bigcup P_\alpha$, an ideal $I \nsubseteq \bigcup P_\alpha$ is contained in $\bigcup J_i$ if and only if $I \cap S \subseteq \bigcup J_i$.

(v) $\mathcal{P}$ is a *Marot family* if each ideal $I \nsubseteq \bigcup P_\alpha$ can be generated by the set $I \cap S$.

For $1 \leq n < 5$, statement (n) implies statement $(n+1)$. Also except for weakly additively regular and finite union, there are examples to show that in general $(n+1)$ does not imply (n). Also note that if both $p, q \in S = R \backslash \bigcup P_\alpha$, then certainly $q \cdot 1 + p \cdot 0 \in S$.

When $\bigcup P_\alpha = Z(R)$, these definitions match with the corresponding ring properties. For example, R is additively regular if and only if $\{P_\alpha\}$ is an additively regular family. Note that "FZD" corresponds to the notion of a ring with few zero divisors.

A *regular Bezout ring* is a ring for which each finitely generated regular ideal is principal [there are several other terms for this in the literature, including: *quasi-Bezout*, *almost Bezout* and *Bezout* (see for example [11], [2] and [7], respectively]. The corresponding definition for a "Bezout family" is that $\{P_\alpha\}$ is a *Bezout family* if each finitely generated ideal that is not contained in $\bigcup P_\alpha$ is principal. Below we show that a Bezout family is a finite union family. As noted above, the polynomial ring $R[X]$ is always additively regular, so there are additively regular families that are not Bezout families. The rings R and S in Example 5.2 below are regular Bezout rings that are not additively regular. We do not know of an example of a regular Bezout ring that is not weakly additively regular (nor one of a Bezout family that is not a weakly additively regular family).

If R is an additively regular ring with only finitely many regular maximal ideals, then each invertible ideal is principal [10, Theorem 3.5]. In Theorem 3.1, we extend the conclusion to certain types of weakly additively regular families. Specifically, if $\mathcal{P} = \{P_\alpha\}$ is a weakly additively regular family and there are only finitely many maximal ideals that have nonempty intersection with $S = R \backslash \bigcup P_\alpha$, then each invertible ideal I such that $I \cap S \neq \emptyset$ is principal. As a corollary, if R is a weakly additively regular ring with only finitely many regular maximal ideals, then each invertible ideal is principal (Corollary 3.2).

Obviously, if R is a Bezout domain, then each nonempty set $\mathcal{P} \subsetneq \mathrm{Spec}(R)$ is a Bezout family. Also, if R is a PID, then each nonempty set $\mathcal{P} = \{P_\alpha\} \subsetneq \mathrm{Max}(R)$ is weakly additively regular (see Theorem 3.9). In addition, if $\mathcal{P}$ is finite, then the family is also additively regular (Theorem 3.9). For the integers, if $\mathcal{P}$ contains all but finitely many maximal ideals, then $\mathcal{P}$ is not an additively regular family. However, there do exist infinite sets of primes of $\mathbb{Z}$ that are additively regular families, but in some cases simply adding one prime to the set makes the larger set lose the additively regular property (see Examples 5.5 and 5.6 below).

2 Special Families

We start with the connection between the special families and the original definitions for rings with nonzero zero divisors.

Theorem 2.1. *Let R be a ring and let $\mathcal{P} = \{P_\alpha \in \mathrm{Spec}(R) \mid P_\alpha \subseteq Z(R)\}$.*

(i) *R has few zero divisors if and only if $\mathcal{P}$ is a FZD family.*

(ii) *R is additively regular if and only if $\mathcal{P}$ is an additively regular family.*

(iii) *R is weakly additively regular if and only if $\mathcal{P}$ is a weakly additively regular family.*

(iv) *R has the regular finite union property if and only if $\mathcal{P}$ is a finite union family.*

(v) *R is a Marot ring if and only $\mathcal{P}$ is a Marot family.*

(vi) *R is a regular Bezout ring if and only if $\mathcal{P}$ is a Bezout family.*

Proof. The corresponding set $S = R\backslash\bigcup P_\alpha$ is the set of regular elements of R. With this observation, each equivalence is clear. Also note that the same equivalences hold for a set $\mathcal{Q} = \{Q_\beta\}$ in place of $\mathcal{P}$ provided $\bigcup Q_\beta = Z(R)$. For example, if R is a reduced ring, then $Z(R)$ is the union of the minimal primes of R but there may be primes in $Z(R)$ that are not minimal. Also, for any ring R, $Z(R)$ is the union of the primes that are maximal with respect to missing $R\backslash Z(R)$. □

It is clear that an additively regular family is a weakly additively regular family. Also one can trivially show that a finite union family is a Marot family. Simply start with an ideal I such that I is not contained in $\bigcup P_\alpha$ and let J be the ideal generated by $I\cap S$. Then clearly $J \subseteq I$ and $\emptyset \neq J\cap S = I\cap S \subseteq I$. Hence if $\mathcal{P}$ is a finite union family we have $I = J$. Next we show that a FZD family is also an additively regular family, and both weakly additively regular families and Bezout families are finite union families.

Theorem 2.2. *Let R be a ring and let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of prime ideals of R.*

(i) *If $\mathcal{P}$ is an FZD family, then it is an additively regular family.*

(ii) *If $\mathcal{P}$ is an additively regular family, then it is a weakly additively regular family.*

(iii) *If $\mathcal{P}$ is a weakly additively regular family, then it is a finite union family.*

(iv) *If $\mathcal{P}$ is a Bezout family, then it is a finite union family.*

(v) *If $\mathcal{P}$ is a finite union family, then it is a Marot family.*

Proof. Throughout the proof we let $S = R\backslash\bigcup P_\alpha$. As noted above, it is clear that an additively regular family is weakly additively regular. Also a finite union family is a Marot family.

Assume $\mathcal{P}$ is an FZD family with corresponding finite set $\{Q_1, Q_2, \ldots, Q_n\}$ such that $\bigcup P_\alpha = \bigcup Q_i$. We may further assume there are no containment relations among the Q_is. Next let $f, g \in R$ be such that $g \in \bigcup P_\alpha$ and $f \in S$. Then g is in at least one Q_i. If $g \in \bigcap Q_i$, then $g + f$ is in no Q_i and thus $g + f \in S$. If at least one Q_i does not contain g, then by renumbering if necessary, we may assume there is a positive integer $m < n$ such that $g \in Q_j$ for $1 \leq j \leq m$ and $g \notin Q_k$ for $m+1 \leq k \leq n$. As there are no containment relations among the Q_is, there is an element $t \in \bigcap_{k=m+1}^{n} Q_k\backslash\bigcup_{j=1}^{m} Q_j$. The element $g + ft$ is in no Q_i and thus it is in S. It follows that $\mathcal{P}$ is an additively regular family.

Next we show that a weakly additively regular family is a finite union family.

Suppose $\mathcal{P}$ is a weakly additively regular family. Let $J_1, \ldots, J_n$ be a finite family of ideals with $J_k \cap S \neq \emptyset$ for each k (equivalently $\cap J_k$ has nonempty intersection with S). Also, let I be an ideal that is contained in neither $\bigcup P_\alpha$ nor $\bigcup J_k$. Next, let $g \in I\backslash\bigcup J_k$ and choose an element $f \in I\cap J_1 \cap J_2 \cap \cdots \cap J_n$ that is in S. By weakly additively regular, there are elements $s, t, y, z \in R$ such that $gs + ft \in S$ and $sy + fz = 1$. Then $g = gsy + gfz$. Since $f \in \bigcap J_k$ and $g \notin \bigcup J_k$, $gsy \notin \bigcup J_k$. Hence $gs \notin \bigcup J_k$ but $ft \in \bigcap J_k$, and so $gs + ft \in I\cap S$ but in no J_k. Thus $\mathcal{P}$ is a finite union family.

Finally we show that a Bezout family is also a finite union family. We continue with the ideals I, $J_1, \ldots, J_n$ in the previous paragraph but replace the assumption about I not being contained in $\bigcup J_k$ with the assumption that $\emptyset \neq I\cap S \subseteq \bigcup J_k$. Suppose $b \in I$ and $c \in I\cap S$. Since $\mathcal{P}$ is a Bezout family, the ideal $bR + cR$ is principal, necessarily a generator for this ideal must come from $I\cap S$. So we have $bR + cR = dR$ for some $d \in I\cap S \subseteq \bigcup J_i$. Since d divides b, $b \in \bigcup J_k$ and therefore $I \subseteq \bigcup J_k$. Hence $\mathcal{P}$ is a finite union family. □

Corollary 2.3. *If R is a weakly additively regular ring, then it has the regular finite union property and it is a Marot ring.*

Proof. Let $\mathcal{P} = \{P_\alpha \in \text{Spec}(R) \mid P_\alpha \subseteq Z(R)\}$. Then by Theorem 2.1, R is weakly additively regular if and only if $\mathcal{P}$ is a weakly additively regular family. Also R has the regular finite union property if and only if $\mathcal{P}$ is a finite union family. Thus the result follows from Theorem 2.2 (and [12, Page 134]). □

Theorems 2.1 and 2.2 also combine to establish that a regular Bezout ring has the regular finite union property.

Corollary 2.4. *If R is a regular Bezout ring, then it has the regular finite union property.*

By way of the next five results we show that a ring that has the regular finite union satisfies a strong form of the Marot property. The first of these five is a simple observation about finite unions.

Theorem 2.5. *Let R be a ring and let I, $J_1, J_2, \ldots, J_n$ be regular ideals of R. If the regular elements of I are contained in $\bigcup J_k$, then they are also contained in $\bigcup J'_k$ where each J'_k is the ideal generated by the regular elements in $I \cap J_k$. Hence, if we also have that R is a Marot ring, then $I \subseteq \bigcup J_k$ if and only if $I = \bigcup J'_k$.*

Next we extend this observation to a family of prime ideals.

Theorem 2.6. *Let R be a ring and let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of nonzero prime ideals of R with $S = R \backslash \bigcup P_\alpha$. Also let I, $J_1, J_2, \ldots, J_n$ be ideals of R such that none are contained in $\bigcup P_\alpha$. If $I \cap S$ is contained in $\bigcup J_k$, then $I \cap S$ is also contained in $\bigcup J'_k$ where each J'_k is the ideal generated by the set of elements $I \cap J_k \cap S$. Hence, if we also have that $\mathcal{P}$ is a Marot family, then $I \subseteq \bigcup J_k$ if and only if $I = \bigcup J'_k$.*

We say that R is a *strong Marot* ring if for each regular ideal I and each finite partition $\{X_1, X_2, \ldots, X_n\}$ of $I \cap \text{Reg}(I)$, $I = \bigcup H_k$ where each H_k is the ideal generated by the corresponding set X_k. For a family of prime ideals $\mathcal{P} = \{P_\alpha\}$, $\mathcal{P}$ is a *strong Marot family* if for each ideal I that is not contained in $\bigcup P_\alpha$ and each finite partition $X = \{X_1, X_2, \ldots, X_n\}$ of $I \cap S$ (with $S = R \backslash \bigcup P_\alpha$), $I = \bigcup H_k$ where each H_k is the ideal generated by the corresponding set X_k.

Theorem 2.7. *Let R be a ring and let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of nonzero prime ideals of R with $S = R \backslash \bigcup P_\alpha$. Then $\mathcal{P}$ is a finite union family if and only if it is a strong Marot family.*

Proof. Suppose $\mathcal{P}$ is a finite union family and I is an ideal that is not contained in $\bigcup P_\alpha$. Next let $\{X_1, X_2, \ldots, X_n\}$ be a finite partition of $I \cap S$. For each k, let H_k be the ideal generated by X_k. Then clearly $I \cap S \subseteq \bigcup H_k$. Hence $I = \bigcup H_k$ since $\mathcal{P}$ is a finite union family.

For the converse, let I, $J_1, J_2, \ldots, J_n$ be ideals where none are contained in $\bigcup P_\alpha$ and $I \cap S \subseteq \bigcup J_k$. Then for each k, $Y_k = I \cap J_k \cap S$ is a nonempty set. So let B_k be the ideal generated by Y_k.

If $\mathcal{P}$ is a strong Marot family, then $I = \bigcup B_k$. Clearly $B_k \subseteq J_k$ for each k. Hence $I \subseteq \bigcup J_k$ and therefore a strong Marot family is a finite union family. □

Corollary 2.8. *Let R be a ring. Then R has the regular finite union property if and only if for each regular ideal I and each finite partition $\{X_1, X_2, \ldots, X_n\}$ of $I \cap \text{Reg}(R)$ (the regular elements of I), $I = \bigcup H_k$ where each H_k is the (regular) ideal generated by the corresponding set X_k.*

Corollary 2.9. *Let R be a ring.*

(i) For a nonempty set of nonzero prime ideals $\mathcal{P} = \{P_\alpha\}$, if I is not contained in $\bigcup P_\alpha$ and $\mathcal{P}$ is a finite union family with $S = R \backslash \bigcup P_\alpha$, then for each nonempty set X of $I \cap S$, either X generates I or $Y = (I \cap S) \backslash X$ does.

(ii) If I is a regular ideal of a ring R where R has the regular finite union property, then for each nonempty set $X \subseteq I \cap \text{Reg}(R)$, at least one of X and $(I \cap \text{Reg}(R)) \backslash X$ generates I as an ideal.

In [14], P. Quartararo and H.S. Butts introduced the notion of a u-ring as a ring R with the property that for each ideal I, if I is contained in a finite union of ideals $J_1, J_2, \ldots, J_n$, then I is contained in at least one of the J_ks. They showed that a ring R is a u-ring if and only if for each maximal ideal M, either R/M is infinite or R_M is a Bezout ring (a ring where each ideal is principal) [14, Theorem 2.6].

For a u-ring R, we can give an alternate characterization of when it has the regular finite union property, and also when a nonempty set of nonzero primes $\mathcal{P}$ is a finite union family.

Corollary 2.10. *Let R be a u-ring.*

(i) For a nonempty set of nonzero primes $\mathcal{P} = \{P_\alpha\}$ with $S = R\backslash \bigcup P_\alpha$, $\mathcal{P}$ is a finite union family if and only if for each ideal I that is not contained in $\bigcup P_\alpha$, each finite partition $\{X_1, X_2, \ldots, X_n\}$ contains a set X_k that generates I as an ideal of R.

(ii) R has the regular finite union property if and only if for each regular ideal I, every finite partition $\{X_1, X_2, \ldots, X_n\}$ of $I \cap \mathrm{Reg}(I)$ contains a set X_k that generates I as an ideal.

We have two variations on u-rings, the first restricts to considering only ideals not contained in a particular union of primes $\bigcup P_\alpha$ and the second is a stronger type of strong Marot ring. First, we say that $\mathcal{P} = \{P_\alpha\}$ is a *u-family* if for each finite collection of ideals $I, J_1, J_2, \ldots, J_n$ where none are contained in $\bigcup P_\alpha$, $I \subseteq \bigcup J_i$ if and only if $I \subseteq J_i$ for some i. For a ring R that is not a domain, R is a *regular u-ring* if the set $\mathcal{P} = \{P \in \mathrm{Spec}(R) \mid P \subseteq Z(R)\}$ is a u-family. Also, $\mathcal{P}$ is a *very strong Marot family* if for each ideal I that is not contained in $\bigcup P_\alpha$, each finite partition of $I\backslash \bigcup P_\alpha$ contains a set that generates I as an ideal. For a ring R, it is a *very strong Marot ring* if for each regular ideal I, each finite partition of $I \cap \mathrm{Reg}(R)$ contains a set that generates I as an ideal.

Next we show that a u-family can be characterized in much the same way that the notion of a u-ring can be characterized in terms of the maximal ideals (see [14, Proposition 2.1 and Theorem 2.6]).

Theorem 2.11. *Let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of nonzero primes of a ring R. Then the following are equivalent.*

(i) $\mathcal{P}$ is a u-family.

(ii) For each finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ where no M_i is contained in $\bigcup P_\alpha$, the ring R_T for $T = R\backslash \bigcup M_i$ has the property that if $I, J_1, J_2, \ldots, J_n$ are ideals of R that are not contained in $\bigcup P_\alpha$, $IR_T \subseteq \bigcup J_i R_T$ implies $IR_T \subseteq J_i R_T$ for some i.

(iii) For each maximal ideal M that is not contained in $\bigcup P_\alpha$, either R/M is infinite or BR_M is principal for each finitely generated ideal $B \nsubseteq \bigcup P_\alpha$.

Proof. We start by showing (1) implies (2). Let $T = R\backslash \bigcup M_i$ where $M_1, M_2, \ldots, M_n$ are maximal ideals that are not contained in $\bigcup P_\alpha$. Also let $I, J_1, J_2, \ldots, J_n$ be ideals of R that are not contained in $\bigcup P_\alpha$. Further assume that $IR_T \subseteq \bigcup J_k R_T$. Let J_k' denote the inverse image of J_k in R. Then we have $I \subseteq \bigcup J_k'$ and so $I \subseteq J_k'$ for some k. It follows that $IR_T \subseteq J_k' R_T = J_k R_T$.

Next we show (2) implies (1). We prove the contrapositive. Suppose there are ideals $I, J_1, J_2, \ldots, J_n$ that are not contained in $\bigcup P_\alpha$ such that $I \subseteq \bigcup J_k$ but where no J_k contains I. For each maximal ideal $N \subseteq \bigcup P_\alpha$ (if any), $IR_N = R_N = J_k R_N$ for each k. Thus for each k, there is a maximal ideal M_k that is not contained in $\bigcup P_\alpha$ where IR_{M_k} is not contained in $J_k R_{M_k}$. We have a finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ (with repeats possible). For $T = R\backslash \bigcup M_i$, we have $IR_T \subseteq \bigcup J_k R_T$, but there is no k such that $IR_T \subseteq J_k R_T$ as IR_{M_k} is not contained in $J_k R_{M_k}$.

To see that (1) implies (3), we prove the contrapositive. So suppose there is a maximal ideal M that is not contained in $\bigcup P_\alpha$ where R/M is finite and there is a finitely generated ideal $I \subseteq M$ where I is not contained in $\bigcup P_\alpha$ and IR_M is not principal. We will construct a finite set of ideals, all properly contained in I with none contained in $\bigcup P_\alpha$ such that I is contained (in fact equal to) the union of these ideals. The proof is adapted from the argument Quartararo and Butts used to establish [14, Theorem 1.8].

First, let $X = \{x_1 = 1, x_2, \ldots, x_q = 0\} \subsetneq R$ be a complete set of representatives for R/M. Next let $I = b_1R + b_2R + \cdots + b_nR$. As an index set for our "bad" set of ideals, we let $r = (r_1, r_2, \ldots, r_n)$ be an n-tuple where each $r_i \in \{x_1, x_2, \ldots, x_q\}$ with at least one $r_i \neq 0$. Define an ideal J_r as the sum $J_r = (\sum r_i b_i)R + IM$. As IR_M is not principal, Nakayama's Lemma implies we have proper containment $J_r R_M \subsetneq IR_M$ for each r. Also note that IM is not contained in $\bigcup P_\alpha$. So no J_r is contained in $\bigcup P_\alpha$.

To see that $I \subseteq \bigcup J_r$, consider $f = \sum v_i b_i$. If each v_i is in M, then $f \in IM$ and so it is in $\bigcup J_r$. So assume some v_j is not in M. For each v_i, there is an element $x_{k_i} \in X$ and an element $c_i \in M$ such that $v_i = x_{k_i} + c_i$ (with $x_{k_i} = x_q = 0$ and $c_i = v_i$ if $v_i \in M$). Thus $f = \sum v_i b_i = \sum x_{k_i} + \sum c_i b_i$ with $\sum c_i b_i \in IM$. As some v_j is not in M, the corresponding x_{k_j} is not in M. Hence the n-tuple $s = (x_{k_1}, x_{k_2}, \ldots, x_{k_n})$ is an allowed "r" and therefore $f \in J_s$. It follows that $I = \bigcup J_r$ with proper containment $J_r \subsetneq I$ for each r.

To complete the proof we show (3) implies (2). We start with a finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ where no M_i is contained in $\bigcup P_\alpha$ and consider $T = R \backslash \bigcup M_i$. We have two "pure" cases, (i) R/M_i infinite for each M_i and (ii) each finitely generated I that is not contained in $\bigcup P_\alpha$ is such that IR_{M_i} is principal for each M_i, and then a final "mixed" case where R/M_i is infinite for some M_i and finite for others but all these forming a subset that satisfies (ii).

For all three cases, $\mathrm{Max}(R_T) = \{M_1R_T, M_2R_T, \ldots, M_nR_T\}$. Thus in case (i), R_T is u-ring and so R_T satisfies the statement in (2).

For case (ii), start with ideals $I, J_1, J_2, \ldots, J_m$ that are not contained in $\bigcup P_\alpha$ with $IR_T \subseteq \bigcup J_k R_T$. As the goal is to get $IR_T \subseteq J_k R_T$ for some k and there are only finitely many J_is, it suffices to establish such a containment in the case that I is finitely generated.

With I finitely generated, we have $IR_{M_i} = a_i R_{M_i}$ for some $a_i \in I$. Also for each i, there is an element $m_i \in R \backslash M_i$ that is in $\bigcap_{j \neq i} M_j$. Consider the element $a = \sum m_i a_i$. For a fixed i, there are elements $t_{i,j} \in R_{M_i}$ such that $a_j = t_{i,j} a_i$ for each $j \neq i$. We then have $a = a_i(m_i + \sum_{j \neq i} m_j t_{i,j})$ with $m_j \in M_i R_{M_i}$ for all $j \neq i$ and $m_i \in R \backslash M_i$. Hence $(m_i + \sum_{j \neq i} m_j t_{i,j})$ is unit of R_{M_i} and thus $aR_{M_i} = a_i R_{M_i} = IR_{M_i}$ for each i and so we have that $IR_T = aR_T$ is principal. Since some $J_k R_T$ contains a, we have $IR_T \subseteq J_k R_T$ for each of these J_ks. So again T satisfies the statement in (2).

For the mixed case, we first the split into a problem about two rings, R_V where V is the complement of the union of those maximal ideals $M \in \{M_1, \ldots, M_n\}$ such that R/M is infinite, and R_W where W is the complement of those M where R/M is finite. By case (ii), IR_W is principal. Also note that R_V satisfies (2) by case (i). So we have ideals $J_v, J_w \in \{J_1, J_2, \ldots, J_m\}$ such that $IR_V \subseteq J_v R_V$ and $IR_W = aR_W \subseteq J_w R_W$ some $a \in J_w$. There may be more than one "choice" for J_v, what we need is one that contains a.

For the rest of the proof, we make use of the argument given in the proof of [14, Theorem 2.5]. The notation here is somewhat different.

Let $I = b_1R + b_2R + \cdots + b_mR$ and assume $\{M_1, M_2, \ldots, M_r\}$ is the set of those maximal ideals in the set $\{M_1, M_2, \ldots, M_n\}$ such that R/M_i is infinite. For each M_i in this set, there is a countably infinite set $\{s_{i,1}, s_{i,2}, \ldots\} \subseteq R \backslash M_i$ such that $s_{i,j} - s_{i,h} \in M_i$ implies $j = h$. By the Chinese Remainder Theorem, there are elements $\{y_1, y_2, \ldots\}$ such that $y_j - s_{i,j} \in M_i$ for each pair i, j. It follows that $y_j - y_h \in M_i$ for some i if and only if $j = h$. Thus $y_j - y_h$ is a unit in R_V for all $j \neq h$. Let $q \in \bigcap_{r+1 \leq k \leq n} M_k \backslash \bigcup_{1 \leq i \leq r} M_i$ and let $x_j = qy_j$. Since q is a unit in R_V, $x_j - x_h$ is a unit in R_V for all $j \neq h$. We also have $x_j - x_h \in \bigcap_{r+1 \leq k \leq n} M_k \backslash \bigcup_{1 \leq i \leq r} M_i$.

Next let $c_k = a + \sum x_k^i b_i \in I$ for each $1 \leq k$. We have infinitely many c_ks, so clearly at least one J_i contains at least $m+1$ of these elements. Without loss of generality we may assume J_1 contains $c_1, c_2, \ldots, c_{m+1}$. Let A be the $m+1 \times m+1$ matrix where the entry in the ith row, jth column is x_j^{i-1}. Also let $\bar{b} = \begin{bmatrix} a & b_1 & b_2 & \cdots & b_m \end{bmatrix}$ and $\bar{c} = \begin{bmatrix} c_1 & c_2 & \cdots, & c_{m+1} \end{bmatrix}$. Then we have $\bar{b}A = \bar{c}$. As A is a Vandermonde matrix, $d = det(A) = \prod_{i<j}(y_j - y_i)$ which is a unit in R_V and an element of the Jacobson radical of R_W. Hence by Cramer's Rule, da and each db_i is in J_1. It follows that $J_1 R_V = IR_V$. Moreover, since d is in the Jacobson radical of R_W and $aR_W = IR_W$, $b_i = t_i a$ for some $t_i \in R_W$ and so $c_1 = a + \sum y_k^i b_i = a(1 + q')$ for some q' in the Jacobson radical of R_W. Hence $a \in J_1 R_W$ and thus $J_1 R_W = IR_W$. As $J_1 R_V = IR_V$, we have $IR_T = J_1 R_T$. □

For very strong Marot, there does not seem to be a good local characterization, but we do have a semilocal one.

Theorem 2.12. *Let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of nonzero primes of a ring R. Then the following are equivalent.*

(i) $\mathcal{P}$ is a very strong Marot family.

(ii) For each finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ where no M_i is contained in $\bigcup P_\alpha$, the ring R_T for $T = R\backslash \bigcup M_i$ has the property that if I is an ideal that is not contained in $\bigcup P_\alpha$, then for each finite partition $X = \{X_1, X_2, \ldots, X_m\}$ of $I\backslash \bigcup P_\alpha$, $IR_T = H_k R_T$ for some k where H_k is the ideal of R generated by X_k.

Proof. The proof that (1) implies (2) is trivial as there is a k such that $I = H_k$.

For the converse we prove the contrapositive. For this we start with an ideal I that is not contained in $\bigcup P_\alpha$ where there is a finite partition $X = \{X_1, X_2, \ldots, X_m\}$ of $I\backslash \bigcup P_\alpha$ such that no X_i generates I as an ideal of R. For each i, let H_i be the ideal generated by X_i. As in the proof of Theorem 2.11, $IR_N = R_N = H_i R_N$ for each maximal ideal $N \subseteq \bigcup P_\alpha$ and each i. Hence, for each i, there is a maximal ideal M_i that is not contained in $\bigcup P_\alpha$ such that IR_{M_i} properly contains $H_i R_{M_i}$. Then for $T = R\backslash \bigcup M_i$, we have $IR_T \supsetneq H_i R_T$ for each i. □

To see the difficulty in adapting the local characterization of u-families to obtain one for very strong Marot families (or even strong Marot families), we revisit an example that appeared in [10].

Example 2.13. (cf. [10, Example 2.4]) Let $D = \mathbb{Z}[\sqrt{10}]$. Then $M = 2D + \sqrt{10}D$ and $N = 5D + \sqrt{10}D$ are the only maximal ideals that contain $\sqrt{10}$. Consider the family of maximal ideals $\mathcal{P} = \{P_\alpha\} = \text{Max}(D)\backslash\{M, N\}$.

(i) Neither M nor N is principal.

(ii) D_M and D_N are rank one discrete valuation domains, so for each (finitely generated) ideal B of D, both BD_M and BD_N are principal.

(iii) For M, let X_1 be the set of numbers that are in $M\backslash M^2$ but not in $\bigcup P_\alpha$ (for example, $\sqrt{10} \in X_1$) and let X_2 be the set of numbers that are in $M^2\backslash \bigcup P_\alpha$ (for example, $2 \in X_2$). Since $M^2 = 2D$, M^2 is the ideal generated by X_2. It is clear that X_1 contains each element of $NM\backslash M^2$ that is not contained in $\bigcup P_\alpha$. If $p \in M\backslash M^2$ is not in N, then it must be contained in at least one P_α for otherwise $pR_M = MR_M$ implies $M = pR$. Thus X_1 generates $NM = N \cap M$.

(iv) As $M \supsetneq NM$ and $M \supsetneq M^2$, $M \neq NM \cup M^2$. Hence $\mathcal{P}$ is not a (very) strong Marot family.

Theorem 2.14. *Let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of nonzero prime ideals of a ring R. Then $\mathcal{P}$ is a very strong Marot family if and only it is both a u-family and a strong Marot family.*

Proof. First suppose $\mathcal{P}$ is a very strong Marot family and let I be ideal not in $\bigcup P_\alpha$. Suppose we have ideals $J_1, J_2, \ldots, J_n$ with $I \subseteq \bigcup J_i$ and no J_i in $\bigcup P_\alpha$. Then for each i, $J_i' = I \cap J_i$ is an ideal that is not contained in $\bigcup P_\alpha$ and $I = \bigcup J_i'$. Also, since $\mathcal{P}$ is a Marot family, J_i' is generated by $Y_i = J_i'\backslash \bigcup P_\alpha$. Clearly, each $b \in I\backslash \bigcup P_\alpha$ is contained in at least one J_i'. Using recursion, we can build a finite set of pairwise disjoint sets $\{X_1, X_2, \ldots, X_n\}$ of $I\backslash \bigcup P_\alpha$: start with $X_1 = Y_1$ and then let $X_k = Y_k\backslash \bigcup_{i=1}^{k-1} X_i$ for $1 < k \leq n$. Since $\mathcal{P}$ is a very strong Marot family, some X_k generates I as an ideal and from this it follows that $I = J_k'$.

For the converse, suppose $\mathcal{P}$ is both a u-family and a strong Marot family and let I be an ideal that is not contained in $\bigcup P_\alpha$. Let $\{X_1, X_2, \ldots, X_n\}$ be a finite partition of $I\backslash \bigcup P_\alpha$. Since we have a strong Marot family, $I = \bigcup H_k$ where each H_k is the ideal generated by the corresponding X_k. But then we get $I = H_k$ for some k as $\mathcal{P}$ is also a u-family. □

In the next section we concentrate on weakly additively regular rings, additively regular rings and regular Bezout rings.

3 Weakly additively regular

As mentioned above, if R is an additively regular ring with only finitely many regular maximal ideals, then each invertible ideal is principal. We next consider a case where the same conclusion holds for certain invertible ideals with regard to a weakly additively regular family of primes.

Theorem 3.1. *Let $\mathcal{P} = \{P_\alpha$ be a weakly additively regular family for a ring R and let $S = R\backslash \bigcup P_\alpha$. If there are only finitely many maximal ideals of R that have nonempty intersection with S, then each invertible ideal that is not contained in $\bigcup P_\alpha$ is principal.*

Proof. Assume $M_1, M_2, \ldots, M_n$ are the maximal ideals of R that have nonempty intersection with S. Also let I be an invertible ideal that is not contained in $\bigcup P_\alpha$. Thus there is an element $f \in I \cap S$.

Next let $T = R\backslash \bigcup M_i$ and $J = \bigcap M_i$. Since $M_i \cap S \neq \emptyset$ and S is multiplicatively closed, there is an element $g \in J \cap S$. Since T is the complement of a finite union of maximal ideals, the M_iR_Ts are the only maximal ideals of R_T. Hence IR_T is principal and thus there is an element $b \in I$ such that $bR_T = IR_T$. In addition, since $fg \in S$, $fgR_N = R_N = IR_N$ for each maximal ideal N which misses S.

If $b \in S$, then we have $bR = I$ by checking locally. So for the remainder of the proof assume $b \notin S$. Since $\mathcal{P}$ is a weakly additively regular family, there are elements $p, q \in R$ such that $h = fgp + bq \in S$ and $fgR + qR = R$. We will show that $hR = I$.

First, since $h \in S$, we have $hR_N = IR_N$ for each maximal ideal N that misses S. So all we need is to have $hR_T = bR_T$. For this, since $f \in I$, there is an element $y \in R_T$ such that $f = by$. Thus viewed in R_T, we have $h = fgp + bq = bygp + bq = (ygp + q)b$. As $fg \in J$ and $fgR + qR = R$, $q \in T$ and $ygp \in JR_T$, the Jacobson radical of R_T. Therefore $ygp + q$ is a unit of R_T and we have $hR_T = bR_T = IR_T$ as desired. Hence $I = hR$ is principal. □

Note that for a given weakly additively regular family $\mathcal{P} = \{P_\alpha\}$, if there are infinitely many maximal ideals that are not contained in $\bigcup P_\alpha$, but only finitely many of these are regular, then an invertible ideal that is not contained in $\bigcup P_\alpha$ need not be principal. Later, we make use of the ring in [10, Example 2.4] to provide such an example.

Corollary 3.2. *Let R be a weakly additively regular ring. If R has only finitely many regular maximal ideals, then each invertible ideal is principal.*

Proof. The set $\mathcal{P} = \{P_\alpha \mid P_\alpha \subseteq Z(R)\}$ is a weakly additively regular family. Also $S = R\backslash \bigcup P_\alpha$ is simply the set of regular elements of R. As an invertible ideal is regular, it has nonempty intersection with S. Thus if R has only finitely many regular maximal ideals, then each invertible ideal is principal by Theorem 3.1. □

For a pair of elements $f, g \in R$, we say the ordered pair of elements (f, g) is an *additively regular pair with respect to the family of primes* $\mathcal{P} = \{P_\alpha\}$ if $f \in S = R\backslash \bigcup P_\alpha$ and there is an element $t \in R$ such that $g + ft \in S$. Similarly (f, g) is a *weakly additively regular pair* if (again) $f \in S$ and there are elements $s, t \in R$ such that $gs + ft \in S$ with $sR + fR = R$. For each of these properties, we drop the reference to the family when $\bigcup P_\alpha = Z(R)$. Note that certain pairs are always additively regular.

Lemma 3.3. *Let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of primes of a ring R and let $S = R\backslash \bigcup P_\alpha$.*

(i) If both $f, g \in S$, then (f, g) is an additively regular pair with respect to $\mathcal{P}$.

(ii) If f is a unit of R, then (f, g) is an additively regular pair with respect to $\mathcal{P}$.

(iii) If f divides g and $f \in S$, then (f, g) is an additively regular pair with respect to $\mathcal{P}$.

(iv) If $f \in S$ and g is nilpotent, then (f, g) is an additively regular pair with respect to $\mathcal{P}$.

Proof. If both $f, g \in S$, then clearly $g + f \cdot 0$ meets the requirement for an additively regular pair. So for the rest of the proof we assume $g \in \bigcup P_\alpha$ and $f \in S$.

Every unit is contained in S, thus for (2) and (3), it suffices to prove the statement in (3). Assume f divides g and write $g = fb$ for some $b \in R$. Since $f \in S$, $b \in \bigcup P_\alpha$. Thus for a sum

$g + ft$, we may factor out the f to get $g + ft = f(b + t)$. Now simply let $t = 1 - b$ so that $g + ft = f \in S$.

For the case g is nilpotent, we have $g \in \bigcap P_\alpha$ and thus $g + f$ is in no P_α, which puts it in S. □

By Lemma 3.3, if $\mathcal{P} = \{P_\alpha\}$ is such that $R \backslash \bigcup P_\alpha$ is the set of units of R, then rather trivially $\mathcal{P}$ is an additively regular family. Also, if $\mathcal{P} = \{P_\alpha\}$ is a finite set of primes of a ring R, then trivially $\mathcal{P}$ has the FZD property. Hence it is an additively regular family.

As stated above, we do not know if each regular Bezout ring is weakly additively regular. At least for certain Bezout domains we can show that each nonempty set of (nonzero) primes is a weakly additively regular family. The basis for this claim starts with noticing that if d, f, g, s, t, y and z are elements of a ring R such that $d = gs + ft$, $f = dy$ and $g = dz$, then $d = g(s \pm jy) + f(t \mp jz)$ for each $j \in R$. For at least some Bezout domains, one can show that if d, f and g are nonzero nonunits of one of these particular Bezout domains R with $d = gs + ft$, $f = dy$ and $g = dz$ for some elements $s, t, y, z \in R$, then there are elements $h, k \in R$ such that $fR + (s + hy)R = R = gR + (t + kz)R$. In Theorem 3.6, we show if R is a Bezout domain with this property, then each nonempty set of prime ideals is a weakly additively regular family. Before presenting this result, we show that each R of the form $R = D(X)$ for some Prüfer domain D has this property as does each “adequate" Bezout domain. Recall that a Bezout domain R is said to be *adequate* if for each pair of nonzero elements $k, m \in R$, there is a pair of elements $b, c \in R$ such that $k = bc$ and $gcd(b, m) = 1 \neq gcd(j, m)$ for each nonunit j that divides c. All PIDS are adequate as is the ring of entire functions [5]. For PIDs, if each prime that divides k also divides m, then $b = 1$ and $c = k$ work. Otherwise, factor k into powers of primes $k = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, the element b is the product of those $p_i^{r_i}$ where p_i does not divide m. For the ring of entire function, we set $b = 1$ if each zero of k is also a zero of m. Otherwise, b is an entire function whose zero set is $Z(b) = Z(k) \backslash Z(m)$ and for each $\beta \in Z(b)$, the multiplicity of β as a zero of b is the same as its multiplicity as a zero of k.

Theorem 3.4. *Let R be a Bezout domain and let d, f and g be nonzero nonunits of R with $d = gs + ft$, $f = dy$ and $g = dz$ for some elements $s, t, y, z \in R$. If R is either adequate or of the form $R = D(X)$ for some Prüfer domain D, then there is an $h \in R$ such that $fR + (s + hy)R = R$.*

Proof. We have $1 = zs + yt$. There are two possibilities for y, y a unit, and y a nonunit. The simplest case is y a unit. In this case, simply note that setting $h = (1 - s)y^{-1}$ yields $s + hy = s + (1 - s) = 1$, so trivially $fR + (s + hy)R = R$.

For the remainder of the proof we suppose y is a nonunit.

We start with the case that R is adequate. For this case, we have a pair of elements $b, c \in R$ such that $f = bc$, $gcd(b, s) = 1 \neq gcd(e, s)$ for each nonunit e that divides c. Consider f and $s + by$. We have $gcd(b, s) = 1 = gcd(y, s)$. So $gcd(by, s) = 1$. Suppose p is a nonzero element of R that divides both f and $s + by$. If some nonunit factor of p divides s, then it also divides by which is impossible. Thus $gcd(p, s) = 1$. Similarly, if some nonunit factor of p divides by, then it also divides s, also impossible. Thus $gcd(p, b) = gcd(p, by) = 1$. As p divides $f = bc$, p divides c, but then each nonunit factor of p, if any, divides s. Hence p is a unit and we have $fR + (s + by)R = R$.

Next we consider the case that $R = D(X)$ for some Prüfer domain D. In this case there are polynomials $a(X), b(X), m(X), n(X), p(X), q(X), u(X), v(X)$ and $w(X)$ in $D[X]$ with $C(u) = C(v) = C(w) = D$, $f = a(X)/u(X)$, $g = b(X)/u(X)$, $t = m(x)/v(X)$, $s = n(X)/v(X)$, $y = p(X)/w(X)$ and $z = q(X)/w(X)$. Since $1 = zs + yt$, $D = C(wv) = C(qn + pm) \subseteq C(n) + C(p) \subseteq D$. Let $j > deg(n) + deg(w)$. Then $C(wn + pvX^j) = C(wn) + C(pv) = C(n) + C(p) = D$, the second equality follows from knowing $C(w) = D = C(v)$. So in this case, we may set $h = X^j$ to have $s + yh = s + yX^j$ a unit of R. □

Corollary 3.5. *If R is either a PID or the ring of entire functions and $d, f, g, s, t, y, z \in R$ are such that $d = gs + ft$, $f = dy$ and $g = dz$ with d, f and g nonzero, then there is an element $h \in R$ such that $fR + (s + hy)R = R$.*

Theorem 3.6. *Let R be a Bezout domain. If for all nonzero $a, b, c \in R$ with $a = bp + cq$, $c = am$ and $b = an$ for some $p, q, m, n \in R$, there are elements i and j in R such that $cR + (p + im)R = R = bR + (q + jn)R$, then each nonempty set of nonzero prime ideals is a weakly additively regular family.*

Proof. Let $\mathcal{P} = \{P_\alpha\}$ and suppose $f \in S = R\backslash \bigcup P_\alpha$ and $g \in \bigcup P_\alpha$. Since R is Bezout, there are elements $d, s, t, y, z \in R$ such that $d = gs + ft$, $f = dy$ and $g = dz$. By assumption, we have an $h \in R$ such that $fR + (s + hy)R = R$. We also have $d = g(s + hy) + f(t - hz)$. As $d = fy$ with $f \in S$ and S is saturated, we also have $d \in S$. Therefore $\mathcal{P}$ is a weakly additively regular family. □

An immediate corollary is the following.

Corollary 3.7. *If R is the ring of entire function and $\mathcal{P} = \{P_\alpha\}$ is a nonempty set of prime ideals of R, then $\mathcal{P}$ is a weakly additively regular family.*

We also have the following for Bezout domains of the form $R = D(X)$ for some Prüfer domain D.

Theorem 3.8. *If $R = D(X)$ where D is a Prüfer domain, then each nonempty set of prime ideals $\mathcal{P} = \{P_\alpha\}$ is additively regular.*

Proof. Let $f(X), g(X) \in R$ be such that $g(X) \in \bigcup P_\alpha$, $f(X) \in S = R\backslash \bigcup P_\alpha$. Then there are polynomials $a(X), b(X), u(X) \in D[X]$ such that $f(X) = a(X)/u(X)$ and $g(X) = b(X)/u(X)$. Since D is a Prüfer domain each P_α has the form $Q_\alpha R$ for some prime ideal Q_α of D. For $a(X)$ and $b(X)$ we have $C(a) \nsubseteq \bigcup Q_\alpha$ and $C(b) \subseteq \bigcup Q_\alpha$. Let $h(X) = b(X) + a(X)X^n$ where $n > deg(b(X))$. Then $C(h) = C(a) + C(b)$ which is not contained in $\bigcup Q_\alpha$. It follows that $h(X) \in S$ and therefore $\mathcal{P}$ is additively regular. □

Theorem 3.9. *Let $\mathcal{P} = \{P_\alpha\} \subsetneq \mathrm{Max}(R)$ be a nonempty set of maximal ideals of a PID R.*

(i) $\mathcal{P}$ is a weakly additively regular family.

(ii) If $\mathcal{P}$ is finite, then it is an additively regular family.

(iii) If $R = \mathbb{Z}$ and $\mathrm{Max}(R)\backslash\mathcal{P}$ is finite (and nonemtpy), then $\mathcal{P}$ is not an additively regular family.

Proof. Theorems 3.4 and 3.6 combine to establish that $\mathcal{P}$ is a weakly additively regular family. In the event $\mathcal{P}$ is finite, it is a FZD family and thus an additively regular family.

Finally we consider the case that $R = \mathbb{Z}$ and there are only finitely many maximal ideals that are not contained in $\mathcal{P}$. As we have assumed $\mathcal{P}$ is a proper subset of $\mathrm{Max}(R) = \mathrm{Max}(\mathbb{Z})$, $\mathrm{Max}(R)\backslash\mathcal{P} = \{M_1, M_2, \ldots, M_n\}$. For each $1 \leq i \leq n$, let $p_i > 1$ be the prime that generates M_i and set $f = \prod p_i$. Next choose a nonzero nonunit $g \in \bigcup P_\alpha$ that is in no M_i. Then for a sufficiently large positive integer k, there is no integer t such that $g + f^k t$ is a unit of $\mathbb{Z}$. Moreover, no p_i divides $g + f^k t$. It follows that there is no integer t such that $g + f^k t$ is in S. Therefore $\mathcal{P}$ is not an additively regular family. □

With regard to the question of whether or not each regular Bezout ring is weakly additively regular, we can show that a regular Bezout ring that is “regularly adequate" is weakly additively regular where by *regularly adequate* we mean that for pair of elements $f, s \in R$ with f regular (and with R a regular Bezout ring), there is a pair of elements $b, c \in R$ (necessarily, regular) such that $f = bc$, $bR + sR = R$ and for each nonunit divisor e of c, $eR + sR \subsetneq R$.

Theorem 3.10. *Let R be a regular Bezout ring. If R is regularly adequate, then R is weakly additively regular.*

Proof. Assume R is regularly adequate and let $d, f, g, s, t, y, z \in R$ be such that $d = gs + ft$, $f = dy$, $g = dz$ with f regular. Since the goal is to find an $h \in R$ such that $fR + (s + hy) = R$ (in order to have regular $d = g(s + hy) + f(t - hz)$), we may assume $fR \subsetneq dR \subsetneq dR$ (by Lemma 3.3). Thus we have that y is not a unit. We are done if $fR + sR = R$, so assume proper containment.

Since R is regularly adequate, there is a pair of elements $b, c \in R$ such that $f = bc$, $bR + sR = R$ and $eR + sR \subsetneq R$ for each nonunit e that divides c. As in the proof of Theorem 3.4, $fR + (s + by)R = R$. Hence R is weakly additively regular. □

4 Rings of the form A+B rings and A+zB[[z]]

Let D be an integral domain that is not a field and let $\mathcal{P} = \{P_\alpha\}_{\alpha\in\mathcal{A}}$ be a nonempty set of nonzero primes of D. Next let $\mathcal{I} = \mathcal{A} \times \mathbb{N}$ and for each $i = (\alpha, n)$, let K_i denote the quotient field of D/P_α. Also for each i, let z_i be an indeterminate over the field K_i. For $B = \sum K_i$ and $C = \sum \mathrm{z}_i K_i[[\mathrm{z}_i]]$ we form a pair of rings $R = D + B$ and $S = D + C$ from $D \times B$ and $D \times C$, respectively with addition defined as $(r, b) + (s, c) = (r + s, b + c)$ and multiplication defined as $(r, b)(s, c) = (rs, rc + sb + bc)$. Both R and S are reduced rings. We say that R is the *ring of the form $A + B$ corresponding to D and $\mathcal{P}$* and S is the *ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$*. For both R and S, the regular ideals have the form $IR = I + B$ and $IS = I + C$, respectively, where I is an ideal of D that is not contained in $\bigcup P_\alpha$. In particular, each regular prime ideal has the form $PR = P + B$ and $PS = P + S$ for some prime P that is not contained in $\bigcup P_\alpha$. For S, the regular elements and zero divisors are easy to distinguish, (r, b) is a zero divisor if and only if $r \in \bigcup P_\alpha$. On the other hand for R, if $r \in \bigcup P_\alpha$, then (r, b) is a zero divisor for each $b \in B$. But consider a nonzero idempotent e of B (so $e_i = 1$ for finitely many i and $e_i = 0$ for all other i). Both $(0, e)$ and $(1, -e)$ are nonzero zero divisors with $(0, e)(1, -e) = (0, 0)$. For R, an element (s, c) is a zero divisor if and only if there is an $i \in \mathcal{I}$ such that $s_i + c_i = 0$ (where s_i is the image of s in K_i and c_i is the ith component of c). In both rings, if (t, d) is regular, then $(t, d)R = (t, 0)R = tD + B$ (for $d \in B$) and $(t, d)S = (t, 0)S = tD + C$ (for $d \in C$). Both constructions have been used extensively to create examples of reduced rings with various desired properties (not always “positive").

We start by collecting a few useful properties of rings formed using these techniques. First we consider the rings of the form $A + B$.

Theorem 4.1. [9, Theorems 8.3 & 8.4] *Let $\mathcal{P}$ be a nonempty set of prime ideals of a domain D and let $R = D + B$ be the $A + B$ ring corresponding to D and $\mathcal{P}$.*

(i) For each $i \in \mathcal{I}$, the set $M_i = \{(r, b) \in R \mid r_i = -b_i\}$ is both a maximal ideal and a minimal prime ideal of R. All other prime ideals of R are of the form $P + B$ for some prime ideal P of D.

(ii) The total quotient ring of R can be identified with the ring $D_\mathcal{S} + B$ where $\mathcal{S} = D\backslash \bigcup\{P_\alpha \mid P_\alpha \in \mathcal{P}\}$.

(iii) If I is an ideal of D such that $I\cap\mathcal{S} \neq \emptyset$, then $IR = I+B$ is a regular ideal of R. Conversely, if J is a regular ideal of R, then $J = I + B = IR$ for some I of D such that $I \cap \mathcal{S} \neq \emptyset$.

(iv) If I is an ideal of D, then IR is an invertible of R if and only if I is an invertible ideal of D and $I \cap \mathcal{S} \neq \emptyset$.

For rings of the form $A + \mathrm{z}B[[\mathrm{z}]]$ we have the following from [8].

Theorem 4.2. [8, Theorem 3.7] *Let D be a domain (that is not a field) and let $\mathcal{P} = \{P_\alpha\}_{\alpha\in\mathcal{A}}$ be a nonempty set of nonzero prime ideals of D. Next, let $R = D + C$ be the ring of form $A + ZB[[Z]]$ corresponding to D and $\mathcal{P}$.*

(i) The element $r = (u, b) \in R$ is a unit of R if and only if u is a unit of D.

(ii) $Z(R) = \{(a, b) \mid a \in \bigcup P_\alpha,\ b \in B\}$.

(iii) The total quotient ring of R can be identified with the ring $D_\mathcal{S} + C$ where $\mathcal{S} = D\backslash \bigcup\{P_\alpha \mid P_\alpha \in \mathcal{P}\}$.

(iv) Each regular ideal of R has the form $IR = I+C$ for some ideal I of D that is not contained in $\bigcup P_\alpha$. Also if I is an ideal of D that is not contained in $\bigcup P_\alpha$, then $IR = I+B$ is regular.

(v) If I is an ideal of D, then IR is invertible if and only if I is an invertible ideal of D and I is not contained in $\bigcup P_\alpha$.

(vi) For each prime ideal P of D, the ideal $P + C$ is a prime of R.

(vii) For each $i = (\alpha, n)$, let $C_i = \{b \in C \mid b_i = 0\}$. The ideal $P_\alpha + C_i$ is a minimal prime ideal of R properly contained in the prime $P_\alpha + C$.

(viii) $\mathrm{Spec}(R) = \{P+C \mid P \in \mathrm{Spec}(D)\} \bigcup \{P_\alpha + C_i \mid i = (\alpha, n) \in \mathcal{I}\}$ *and* $\mathrm{Max}(R) = \{M+C \mid M \in \mathrm{Max}(D)\}$.

As $R = D + B$ has infinitely many idempotents, there are infinitely many primes that are maximal with respect to containing only zero divisors. Hence R is never a ring with few zero divisors. On the other hand, the ring $S = D + \mathrm{z}B[[\mathrm{z}]]$ has only two idempotents, 1 and 0. It turns out that $S = D + \mathrm{z}B[[\mathrm{z}]]$ has few zero divisors if and only if $\mathcal{P}$ is an FZD family.

Theorem 4.3. *Let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of prime ideals of an integral domain D and let $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. Then S has few zero divisors if and only if $\mathcal{P}$ is an FZD family.*

Proof. As noted above, an element (s, b) is a zero divisor if and only if $s \in \bigcup P_\alpha$. Thus for a prime ideal Q of D, $Q + C \subseteq Z(S)$ if and only if $Q \subseteq \bigcup P_\alpha$. It follows that if primes $Q_1, Q_2, \ldots, Q_n \in \mathrm{Spec}(D)$ are such that $\bigcup Q_k = \bigcup P_\alpha$, then $Z(S) = \bigcup Q_k + C$ is a finite union of prime ideals.

For the converse we make use of the fact that an ideal J of S is contained in $Z(S)$ if and only if $J \subseteq I + C$ for some ideal I of D that is contained in $\bigcup P_\alpha$. Hence if S has few zero divisors, then $\bigcup P_\alpha$ can be realized as a finite union of prime ideals. □

Let $\mathcal{P} = \{P_\alpha\}$ be the set of height one primes of the UFD $D = K[\mathrm{x}, \mathrm{y}]$ that are contained in the maximal ideal $M = \mathrm{x}D + \mathrm{y}D$. Then clearly $M = \bigcup P_\alpha$ and so the corresponding ring $S = D + C$ has few zero divisors even though the family $\mathcal{P}$ is an infinite set of incomparable primes.

We next look at the possibility that the created rings $D + B$ and $D + C$ are additively regular. After that we characterize the weakly additively regular rings of these two forms plus Marot rings, and those with the regular finite union property. We also consider regular Bezout rings.

Theorem 4.4. *Let $R = D + B$ be the ring of the form $A + B$ corresponding to a domain D and a nonempty set of nonzero prime ideals $\mathcal{P} = \{P_\alpha\}_{\alpha \in \mathcal{A}}$ of D also let $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. The following are equivalent.*

(i) *R is additively regular.*

(ii) *$\mathcal{P}$ is an additively regular family.*

(iii) *S is additively regular.*

Proof. For both R and S, an element (r, b) is regular if and only if there is no $i \in \mathcal{A} \times \mathbb{N}$ such that $r_i + b_i = 0$. In both rings, a necessary condition for (r, b) to be regular is that $r \in D \backslash \bigcup P_\alpha$. In S, this condition is also sufficient, but in R it is not (even if r is a unit of D).

We start by showing (1) and (3) imply (2). Let $f \in D \backslash \bigcup P_\alpha$ and $g \in \bigcup P_\alpha$. Then for each $b \in B$ and $c \in C$, $(g, b) \in Z(R)$ and $(g, c) \in Z(S)$. It is also the case that $(f, 0)$ is regular in both R and S. To have $(g, c) + (f, 0)(h, d)$ regular for some (h, d), a necessary condition is that $g + fh \in D \backslash \bigcup P_\alpha$. Hence (2) holds if at least one of R and S is additively regular.

Next assume $\mathcal{P}$ is an additively regular family. We first show that S is additively regular.

For this let (f, a) be a regular element of S and let (g, b) be a zero divisor of S. Then from above, $f \in D \backslash \bigcup P_\alpha$ and $g \in \bigcup P_\alpha$. Then there is an element $h \in D$ such that $g + fh \in D \backslash \bigcup P_\alpha$. It follows that the element $(g, b) + (f, a)(h, 0) = (g + fh, b + ha)$ is regular (since $g + fh \in D \backslash \bigcup P_\alpha$).

The proof that R is additively regular requires a more careful analysis since even for a unit $u \in D$, there are elements $k \in B$ such that $(u, k) \in Z(R)$ (simply choose an $i = (\alpha, n)$ and define k by $k_i = -u_i$ and $k_j = 0$ for $j \neq i$).

As above, let (f, a) be a regular element of R and let (g, b) be a zero divisor of R. Then $f \in D \backslash \bigcup P_\alpha$ but all we know about g is that there is an i such that $g_i + b_i = 0$. We split the proof into two cases. The first case is when $g \in D \backslash \bigcup P_\alpha$.

In this case, there are only finitely many $i \in \mathcal{I}$ where $g_i + b_i = 0$. We will construct an element $c \in B$ such that $f_i c_i + a_i c_i \neq 0$ for these i and $c_j = 0$ for all other js. Since (f, a) is regular, there is no i such that $f_i + a_i = 0$. So for c, we simply set $c_i = 1$ when $g_i + b_i = 0$ and $c_j = 0$ for all other j. It follows that $(g, b) + (f, a)(0, c) = (g, b + fc + ac)$ is regular.

Next suppose $g \in \bigcup P_\alpha$. Then there is an element $h \in D$ such that $g+fh \in D\backslash \bigcup P_\alpha$. We will construct a $c \in B$ such that $(g,b)+(f,a)(h,c) = (g+fh, b+fc+ha+ac)$ is regular. There is no i such that $g_i + f_ih_i$ is 0. On the other hand, there are infinitely many i such that $g_i + b_i = 0$, but only finitely many where $b_i \neq 0$, and only finitely many where $a_i \neq 0$. Similarly, we may select at most finitely many $i \in \mathcal{I}$ where $c_i \neq 0$. Hence $x_i = g_i + f_ih_i + b_i + f_ic_i + h_ia_i + a_ic_i$ reduces to $x_i = g_i + f_ih_i \neq 0$ except for at most finitely many choices of i. When both $b_i = 0 = a_i$, we set $c_i = 0$.

Assume i is such that at least one of b_i and a_i is nonzero. Regrouping, we have $x_i = (g_i + f_ih_i + b_i + h_ia_i) + (f_i + a_i)c_i$. If $g_i + f_ih_i + b_i + h_ia_i = 0$, we set $c_i = 1$; otherwise, we set $c_i = 0$. This completes the definition of c and the result is that there is no i such that $x_i = 0$. Hence $(g,b) + (f,a)(h,c)$ is regular.

Therefore R is additively regular. □

Corollary 4.5. *Let $R = D + B$ be the ring of the form $A + B$ corresponding to a domain D and a nonempty set of nonzero prime ideals $\mathcal{P} = \{P_\alpha\}_{\alpha\in\mathcal{A}}$ of D also let $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. If $\mathcal{P}$ is a finite set, then both R and S are additively regular.*

Proof. Simply apply (in order) Theorems 4.3, 2.2 and 4.4. □

Next we consider weakly additively regular rings and families.

Theorem 4.6. *Let $R = D + B$ be the ring of the form $A + B$ corresponding to a domain D and a nonempty set of nonzero prime ideals $\mathcal{P} = \{P_\alpha\}_{\alpha\in\mathcal{A}}$ of D also let $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. The following are equivalent.*

(i) R is weakly additively regular.

(ii) $\mathcal{P}$ is a weakly additively regular family.

(iii) S is weakly additively regular.

Proof. As noted in the proof of Theorem 4.4, for both R and S, an element (r,b) is regular if and only if there is no $i \in \mathcal{A}\times\mathbb{N}$ such that $r_i + b_i = 0$. Also, a necessary condition for (r,b) to be regular is that $r \in D\backslash \bigcup P_\alpha$.

We start by showing (1) and (3) imply (2). Let $f \in D\backslash \bigcup P_\alpha$ and $g \in \bigcup P_\alpha$. Then $(f,0)$ is regular in both R and S and $(g,0)$ is a zero divisor in both rings. If at least one of R and S is weakly additively regular, there are elements (p,b) and (q,c) such that $(g,0)(p,b) + (f,0)(q,c)$ is regular with (p,b) and $(f,0)$ comaximal. It follows that p and f are comaximal in D and we have $gp + fq \in D\backslash \bigcup P_\alpha$.

Next assume $\mathcal{P}$ is a weakly additively regular family. For both R and S, each regular ideal is the extension to that ring of an ideal of D that is not contained in $\bigcup P_\alpha$. In particular, if (f,a) is a regular element in R (or S), then $(f,a)R = (f,0)R$ ($(f,a)S = (f,0)S$). Thus if $pD + fD = D$, then $(p,0)R + (f,a)R = (p,0)R + (f,0)R$ and $(p,0)S + (f,a)S = (p,0)S + (f,0)S$.

As above, the easier case is to show that S is weakly additively regular.

For this let (f,a) be a regular element of S and let (g,b) be a zero divisor of S. Then from above, $f \in D\backslash \bigcup P_\alpha$ and $g \in \bigcup P_\alpha$. Also, there is are elements $p,q \in D$ such that $gp + fq \in D\backslash \bigcup P_\alpha$ and $pD + fD = D$. It follows that the element $(g,b)(p,0) + (f,a)(q,0) = (gp + fq, pb + qa)$ is regular (since $gp + fq \in D\backslash \bigcup P_\alpha$). There are elements $x,y \in D$ such that $px + fy = 1$. In S, we get $(p,0)(x,0) + (f,0)(y,0) = (1,0)$. Hence $S = (p,0)S + (f,0)S = (p,0)S + (f,a)S$.

Continue with the assumption that (f,a) is regular and (g,b) is a zero divisor, but now of the ring R. Then $f \in D\backslash \bigcup P_\alpha$ and there is no i such that $f_i + a_i = 0$, but all we know about g is that there is an i such that $g_i + b_i = 0$. We split the proof into two cases.

The first case is when $g \in D\backslash \bigcup P_\alpha$. For this case, we start by setting $p = 1$ and $q = 0$ (as elements of D). There are there are only finitely many $i \in \mathcal{I}$ where $g_i + b_i = 0$. As in the proof of Theorem 4.4, we define $c \in B$ by setting $c_i = 1$ when $g_i + b_i = 0$ and set $c_i = 0$ for all other i. Then $(g,b)(1,0) + (f,a)(0,c) = (g, b + fc + ac)$ is such that $g \in D\backslash \bigcup P_\alpha$ and there is no i such that $g_i + b_i + f_ic_i + a_ic_i = 0$. Hence $(g,b)(1,0) + (f,a)(0,c)$ is regular with $(1,0)R + (f,a)R = R$.

Next suppose $g \in \bigcup P_\alpha$. Then there are elements $p, q \in D$ such that $gp + fq \in D \backslash \bigcup P_\alpha$ with $pD + fD = D$. From above, we have $(p,0)R + (f,a)R = R$. So all we need to do is to define a $c \in B$ such that $(g,b)(p,0) + (f,a)(q,c) = (gp + fq, pb + fc + qa + ac)$ is regular. There is no i such that $g_ip_i + f_ih_i$ is 0. But there are infinitely many i such that $g_ip_i + p_ib_i = 0$. However, there are only finitely many $i \in \mathcal{I}$ where $p_ib_i \neq 0$, and only finitely many where $a_i \neq 0$. Similarly, we may select at most finitely many $i \in \mathcal{I}$ where $c_i \neq 0$. As in the proof of Theorem 4.4, $x_i = g_ip_i + f_iq_i + p_ib_i + f_ic_i + q_ia_i + a_ic_i$ reduces to $x_i = g_ip_i + f_iq_i \neq 0$ for all but finitely many $i \in \mathcal{I}$. Thus, when both $p_ib_i = 0 = a_i$, we set $c_i = 0$.

Assume i is such that at least one of p_ib_i and a_i is nonzero. Regrouping, we have $x_i = (g_ip_i + f_iq_i + p_ib_i + q_ia_i) + (f_i + a_i)c_i$. If $g_ip_i + f_iq_i + p_ib_i + q_ia_i = 0$, we set $c_i = 1$ and then have $x_i = (f_i + a_i) \neq 0$; otherwise, we set $c_i = 0$ to have $x_i = g_ip_i + f_iq_i + p_ib_i + q_ia_i \neq 0$. This completes the definition of c and the result is that there is no i such that $x_i = 0$. Hence $(g,b)(p,0) + (f,a)(q,c)$ is regular.

Therefore R is weakly additively regular. □

Recall from above that we showed that each nonempty set of primes of a PID is a weakly additively regular family (Theorem 3.9). Thus when the base domain is a PID, the corresponding rings $R = D + B$ and $S + D + C$ are weakly additively regular.

Corollary 4.7. *Let D be a PID and let $\mathcal{P} = \{P_\alpha\}_{\alpha \in \mathcal{A}}$ be a nonempty set of nonzero prime ideals of D and let $R = D + B$ be the ring of form $A + B$ and $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. Both R and S are weakly additively regular. In addition, if $\mathcal{P}$ is a finite set, then both R and S are additively regular*

Proof. By Theorem 3.9, $\mathcal{P}$ is a weakly additively regular family. Hence both R and S are weakly additively regular by Theorem 4.6. Also, by Corollary 4.5 both R and S are additively regular when $\mathcal{P}$ is finite. □

Theorem 4.8. *Let $R = D + B$ be the ring of the form $A + B$ corresponding to a domain D and a nonempty set of nonzero prime ideals $\mathcal{P} = \{P_\alpha\}_{\alpha \in \mathcal{A}}$ of D also let $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. Finally, let $\mathcal{S} = D \backslash \bigcup P_\alpha$. The following are equivalent.*

(i) R has the regular finite union property.

(ii) $\mathcal{P}$ is a finite union family.

(iii) S has the regular finite union property.

Proof. For an ideal I of D such that $I \cap \mathcal{S} \neq \emptyset$, $IR = I + B$ is a regular ideal of R and $IS = I + C$ is a regular ideal of S. These are the only types of regular ideals in the two rings. Thus if we have finitely many regular ideals in R (or S), we may assume these ideals have the form IR, $J_1R, J_2R, \ldots, J_nR$ (IS, $J_1S, \ldots, J_nS$) for ideals I, $J_1, \ldots, J_n$ of D, each with nonempty intersection with $\mathcal{S}$.

Suppose R has the regular finite union property. As in the proof that weakly additively regular rings have the regular finite union property, we start with the assumption that I is not contained in $\bigcup J_k$ with the goal of finding an element in $I \cap \mathcal{S}$ that is not in $\bigcup J_k$. In R, we have $IR = I + B$ is not contained in $\bigcup J_kR$. Hence by the regular finite union property, there is a regular element $(f,b) \in IR$ that is not in $\bigcup J_kR$. Since $B \subseteq J_kR$ for each k, it must be that $f \notin \bigcup J_k$. We also have $f \in \mathcal{S}$. Thus (1) implies (2). The proof that (3) implies (2) is essentially the same, simply substitute C for B.

Next assume $\mathcal{P}$ is a finite union family. We will show that (2) implies (3). As in the proof that (1) implies (2), we start with regular ideals IS, J_1S, $J_2S, \ldots, J_nS$ such that IS is not contained in $\bigcup J_kS$. Since each J_kS contains C, there must be an element $f \in I$ that is not in $\bigcup J_k$. But then by assumption, we may assume $f \in \mathcal{S}$. In S, the element $(f,0)$ is a regular element contained in IS but not $\bigcup J_kS$. □

Theorem 4.9. *Let $R = D + B$ be the ring of the form $A + B$ corresponding to a domain D and a nonempty set of nonzero prime ideals $\mathcal{P} = \{P_\alpha\}_{\alpha \in \mathcal{A}}$ of D also let $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. Finally, let $\mathcal{S} = D \backslash \bigcup P_\alpha$. The following are equivalent.*

(i) R is a Marot ring.

(ii) $\mathcal{P}$ is a Marot family.

(iii) S is a Marot ring.

Proof. For an ideal I of D with $I \cap \mathcal{S} \neq \emptyset$, a necessary condition for $(r,b) \in I + B$ $(\in (I+C))$ to be regular is that $r \in \mathcal{S}$. Also, for both R and S, each regular ideal is extended from an ideal of D that is not contained $\bigcup P_\alpha$. Hence $IR = I + B$ and $IS = I + C$ are generated by regular elements if and only if I has a generating set Y such that $Y \subseteq \mathcal{S}$. □

Theorem 4.10. *Let $R = D + B$ be the ring of the form $A + B$ corresponding to a domain D and a nonempty set of nonzero prime ideals $\mathcal{P} = \{P_\alpha\}_{\alpha\in\mathcal{A}}$ of D also let $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. The following are equivalent.*

(i) R is a regular Bezout ring.

(ii) $\mathcal{P}$ is a Bezout family.

(iii) S is a regular Bezout ring.

Proof. The proof follows easily from the fact that the regular ideals of R and S all have the form $IR = I + B$ and $IS = I + C$ for those ideals I of D that are not contained in $\bigcup P_\alpha$. □

Corollary 4.11. *Let D be a Bezout domain and let $\mathcal{P} = \{P_\alpha\}_{\alpha\in\mathcal{A}}$ be a nonempty set of nonzero prime ideals of D and let $R = D + B$ be the ring of form $A + B$ and $S = D + C$ be the ring of the form $A + \mathrm{z}B[[\mathrm{z}]]$ corresponding to D and $\mathcal{P}$. Both R and S are regular Bezout domains.*

Proof. Since D is a Bezout domain, $\mathcal{P}$ is (trivially) a Bezout family. It follows that both R and S are regular Bezout rings. □

5 Examples

Our first example is formed using Nagata's principle of idealization: for a ring D and D-module B, one can form a ring $R = D(+)B$ from $D \times B$ with $(r,b) + (s,c) = (r+s, b+c)$ and $(r,b)(s,c) = (rs, rc+sb)$. When D is an integral domain, an element $(r,b) \in R$ is a zero divisor if and only if $rc = 0$ for some nonzero $c \in B$ (see for example [6, Theorem 25.3]). In addition, (r,b) is a unit if and only if r is a unit of D. We revisit Matsuda's example of a ring with the regular finite union property that is not additively regular.

Example 5.1. (cf. [12, Propositions 11 & 12]) Let $\mathcal{P} = \{p\mathbb{Z} \mid p \text{ odd prime}\}$ and for each odd prime p_n, let $B_n = \mathbb{Z}/p_n\mathbb{Z}$. Form the ring $R = \mathbb{Z}(+)B$ using idealization where $B = \sum B_n$. Matsuda showed that R is not additively regular, but does have the regular finite union property. We will show that R is weakly additively regular.

Proof. If m is divisible by an odd prime, then (m,b) is a zero divisor for each $b \in B$. On the other hand if f is a power of 2, then (f,c) is a regular element for each $c \in B$.

Since f is a power of 2, $fB_n = B_n$ for each n. Hence (f,c) divides $(0,d)$ for each $d \in B$. Thus by Lemma 3.3, we may start with a zero divisor (m,b) where $m \neq 0$ and a regular element (f,c) that is not a unit with f positive (so f is a positive power of 2). Factor m as $m = s2^k$ with s odd (and neither 1 nor -1) and $k \geq 0$. It does no harm to start by multiplying (f,c) by $(2^k,0)$ to get $(2^k f, 2^k c)$. Since s is odd, there are integers x and y such that $sx + fy = 1$. Clearly $(x,0)R + (f,c)R = R$. We also have $(m,b)(x,0) + (2^k f, 2^k c)(y,0) = (mx + 2^k fy, xb + 2^k yc) = (2^k, xb + 2^k yc)$ a regular element in R. Thus R is weakly additively regular. □

Here are the $A + B$ and $A + \mathrm{z}B[[\mathrm{z}]]$ versions of Example 5.1.

Example 5.2. Let $R = \mathbb{Z} + B$ be the $A + B$ ring corresponding to $\mathbb{Z}$ and $\mathcal{P}$ where $\mathcal{P} = \{p_n\mathbb{Z} \mid p_n$ an odd prime$\}$. Also let $S = \mathbb{Z} + C$ be the $A + \mathrm{z}B[[\mathrm{z}]]$ ring corresponding to $\mathbb{Z}$ and $\mathcal{P}$. By Corollaries 4.11 and 4.7, both R and S are regular Bezout rings that are also weakly additively regular. But neither is additively regular (Theorems 3.9 and 4.4).

Recall that Theorem 3.1 shows that if $\mathcal{P} = \{P_\alpha\}$ is a weakly additively regular family such that there are only finitely many maximal ideals that are not contained in $\bigcup P_\alpha$, then each invertible ideal that is not contained in $\bigcup P_\alpha$ is principal. In the next example, we show that it is not enough to know that there are only finitely many regular ideals that are not contained in $\bigcup P_\alpha$. In this case the family (called $\mathcal{Q}$) is actually an FZD family. We make use of the domain and family of primes from Example 2.13 to build the ring R.

Example 5.3. Let $D = \mathbb{Z}[\sqrt{10}]$ and let $\mathcal{P} = \mathrm{Max}(D)\backslash\{M, N\}$ where $M = 2D + \sqrt{10}D$ and $N = 5D + \sqrt{10}D$. For $R = D + B$, the $A + B$ ring corresponding to D and $\mathcal{P}$, choose any finite set of prime ideals $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_n\}$ that includes neither $MR = M + B$ nor $NR = N + B$. Since $\mathcal{Q}$ is a FZD family, it is also a weakly additively regular family. The maximal ideals MR and NR are the only regular maximal ideals of R, so trivially only finitely many regular maximal ideals of R are not contained in $\bigcup Q_i$. Both MR and NR are invertible, but neither is principal.

For the next example, we construct a partitioning of the prime numbers into a pair of infinite sets X and Y in such a way that for each pair of integers g, f such that some prime in X divides g and no prime in X divides f, there is an integer h such that no prime in X divides $g + fh$. This will allow the construction of a ring of the form $A + B$ and one of the form $A + \mathrm{z}B[[\mathrm{z}]]$ where each is additively regular. The ring $R = D + B$ has infinitely many regular maximal ideals of the form $M + B$, and the ring $S = D + C$ has infinitely many regular maximal ideals of the form $M + C$. Also, R has infinitely many maximal ideals of the form $N + B$ that are not regular, and S has infinitely many maximal ideals of the form $N + C$ that are not regular. The particular partitioning here is based on a general scheme for partitioning the primes of $\mathbb{Z}$ developed by Jim Coykendall. In the appendix, we present one of two general schemes by Coykendall that satisfy the sufficient condition in Theorem 4.4. We also show that such a partitioning can be fragile in the sense that if some particular prime is shifted from the set X to the set Y, then the new partitioning $\{X', Y'\}$ yields rings $R' = \mathbb{Z} + B'$ and $S' = \mathbb{Z} + C'$ that are not additively regular.

To start let $\mathcal{Q}_r$ be the set of odd primes that are congruent to r mod 8 for $r \in \{1, 3, 5, 7\}$. Next let $X = \{2\} \cup \mathcal{Q}_3 \cup \mathcal{Q}_5$ and $Y = \mathcal{Q}_1 \cup \mathcal{Q}_3$.

Lemma 5.4. *For $X = \{2\} \cup Q_3 \cup Q_5$ and $Y = \mathbb{Q}_1 \cup Q_7$, if n is a positive integer with a prime factor in X and $r = q_1 q_2 \cdots q_m$ with each $q_i \in Y$, then there are infinitely many positive integers k such that each prime factor of $n + rk$ is in Y.*

Proof. Factor $n = st$ into positive integers s and t where each prime factor of s is in X and no prime factor of t is in X. It suffices to show there are infinitely many $k > 1$ such that each prime factor of $n + rtk = t(s + rk)$ is in Y. We already know each prime factor of t (if any) is in Y. What we will show is that there are infinitely many k such that $s + rk$ is a prime in the set Y.

Consider the system of congruences $x \equiv s \pmod{r}$, $x \equiv 1 \pmod{8}$. Since 8 and r are relatively prime, there is a positive integer y that satisfies both congruences. Next consider the integers of the form $8rz + y$. Since $gcd(8, r) = 1 = gcd(s, r)$, $gcd(y, 8r) = 1$. Hence by Dirichlet's Theorem, there are infinitely many primes of the form $8rz + y$. Each of these primes is in Y since $y \equiv 1 \pmod{8}$. Also, for such a prime we have $8rz + y = s + rk$ for some k since $y \equiv s \pmod{r}$. □

Example 5.5. Let $\mathcal{P} = \{pZ \mid p \in X\}$ where $X = \{2\} \cup \mathcal{Q}_3 \cup \mathcal{Q}_5$, also let $Y = \mathcal{Q}_1 \cup \mathcal{Q}_7$. Then the ring $R = \mathbb{Z} + B$ corresponding to $\mathbb{Z}$ and $\mathcal{P}$ is additively regular with infinitely many regular maximal ideals $Q + B$ where $Q = q\mathbb{Z}$ for some $q \in Y$ and infinitely many maximal ideals of the form $P + B$ for some $P \in \mathcal{P}$ that are not regular. Also the ring $S = \mathbb{Z} + C$ corresponding to $\mathbb{Z}$ and $\mathcal{P}$ is additively regular with infinitely many regular maximal ideals $Q + C$ where $Q = q\mathbb{Z}$ for some $q \in Y$ and infinitely many maximal ideals of the form $P + C$ for some $P \in \mathcal{P}$ that are not regular.

Proof. As in the proof of Lemma 5.4, suppose n is a nonzero integer with at least one prime factor in X and let r be a (finite) product of primes in the set Y. Here we no longer assume that n is positive.

If n is positive, then by Lemma 5.4, there is an integer z such that each prime factor of $n + rz$ is in Y. On the other hand, if n is a negative integer, then by Lemma 5.4 there is an integer z such that each prime factor of $(-n) + rz$ is in Y. In this case each prime factor of $n + r(-z)$ is in Y.

For the case f is a negative integer with each prime factor in Y, $-f$ can play the role of r to get an integer $n + rk = n + f(-k)$ with each prime factor in Y. That R and S are additively regular now follows by Theorem 4.4 (and Lemma 5.4). □

With a very small change in the partitioning using X and Y, it possible to get corresponding rings $R = \mathbb{Z} + B$ and $S = \mathbb{Z} + C$ that are not additively regular. For the new partitioning, we simply move 2 to get $X' = \mathcal{Q}_3 \cup \mathcal{Q}_5$ and $Y' = \{2\} \cup \mathcal{Q}_1 \cup \mathcal{Q}_7$.

Example 5.6. Let $P = \{p\mathbb{Z} \mid p \in X'\}$ where $X' = \mathcal{Q}_3 \cup \mathcal{Q}_5$ and let $Y' = \{2\} \cup \mathcal{Q}_1 \cup \mathcal{Q}_7$. Then neither the ring $R' = \mathbb{Z} + B'$ nor the ring $S' = \mathbb{Z} + C'$ corresponding to $\mathbb{Z}$ and $\mathcal{P}$ is additively regular. Since both X' and Y' are infinite sets, there are infinitely many regular maximal ideals of the form $M + B'$ and $M + C'$ and infinitely many maximal ideals of the form $P + B'$ and $P + C'$ that are not regular.

Proof. By Theorem 4.4, it suffices to find a pair of positive integers f, g such that some prime in X' divides g and each prime divisor of f is in Y' with the property that for each k, $g + fk$ has at least one prime divisor in X'.

Consider $f = 8$ and $g = 3$. From elementary number theory, each integer of the form $8k + 3$ has at least one prime divisor in X'. Therefore by Theorem 4.4, neither R' nor S' is additively regular. □

6 Appendix

Below is one of the two general schemes developed by Jim Coykendall for constructing a partition of the (positive) primes into two infinite sets so that the corresponding rings of the form $A + B$ and $A + zB[[z]]$ are additively regular. Note that a key feature is that the prime divisors of the selected integer N are placed in the set X.

Let $N > 2$ be an integer and let $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_s$ be a list of the integers between 1 and N that are relatively prime to N (not necessarily in order) but with both t and s positive. For X we set $X = \{p \in \mathbb{Z}^+ \mid p \text{ prime with } p \equiv a_i \pmod{N} \text{ for some } 1 \leq i \leq t\} \cup \{p \in \mathbb{Z}^+ \mid p \text{ a prime divisor of } N\}$ and then $Y = \{q \in \mathbb{Z}^+ \mid q \text{ prime with } q \equiv b_j \pmod{N} \text{ for some } 1 \leq j \leq s\}$. By Dirichlet's Theorem, both X and Y are infinite. For a prime that does not divide N, it is clear that the prime is in exactly one of the sets X and Y.

As above, if n is a positive integer with at least one prime divisor in X and r is an integer with each prime divisor in Y, then we may factor $n = fg$ where each prime divisor of f is in X and each prime divisor of g (if any) is in Y. We shift to considering $n + rgx$ for some integer x. We have $n + rgx = g(f + rx)$, so to have no prime divisor of $n + rgx$ in X it suffices to make $f + rx$ a prime in the set Y. As above, we consider the system of congruences $y \equiv f \pmod{r}$ and $y \equiv b_1 \pmod{N}$. For a solution z, we consider the set of positive integers of the form $Nrk + z$. We have $gcd(Nr, z) = 1$ since $gcd(r, f) = 1 = gcd(N, r)$ and $gcd(N, z) = 1$. Hence by Dirichlet's Theorem, there are infinitely many primes of the form $Nrk + z$ for positive k when $r > 0$, negative k when $r < 0$. Each such prime is in the set Y and congruent to f mod r. Thus there are infinitely many primes of the form $rh + f \in Y$.

As in the proof of Example 5.5, if n is negative, we simply consider the positive integer $n' = -n$. We have infinitely many integers h' such that $rh' + w$ is a prime in Y. It follows that the only prime divisor of $r(-h) + n$ is in Y.

Therefore, for $P = \{p\mathbb{Z} \mid p \in X\}$, the corresponding rings $R = \mathbb{Z} + B$ and $S = \mathbb{Z} + C$ are additively regular with infinitely many regular maximal ideals of the form $Q + B$ and $Q + C$ where $Q = q\mathbb{Z}$ for some $q \in Y$ and infinitely many maximal ideals of the form $P + B$ ($\subsetneq Z(R)$) and $P + C$ ($\subsetneq Z(S)$) where $P = p\mathbb{Z}$ for some $p \in X$.

While the "bad" partitioning in Example 5.6 places 2 in the set Y, simply start with an odd integer $N > 1$ and when partitioning the primes that do not divide N, place 2 (and those congruent to 2 mod N) in the set Y. For an alternate partitioning scheme (with 2 in Y), start with $p_1 = 3, p_2 = 7 \in X$, then recursively define p_n to be the smallest prime of the form $2^{m_n}(2k_n + 1) - 1$ for some nonnegative integer k_n and some integer $m_n > m_{n-1}$ with $2p_{n-1} < 2^{m_n}$. We have $p_n \equiv -1 \pmod{2^{m_n}}$ and $p_n \equiv 2^{m_n} - 1 \pmod{2^{m_n+1}}$. In addition, note that p_n is the only prime in X that is congruent to $2^{m_n} - 1$ mod 2^{m_n+1}. All larger primes in X are congruent to -1 mod 2^{m_n+1}, and each smaller prime in X is congruent mod 2^{m_n+1} to a positive integer that is strictly smaller than $2^{m_n}-1$. In terms of pairs (p_n, m_n), the first five are $(3,2),(7,3),(31,5),(127,7),(1279,8)$ [note that while $2\cdot 7<16$ and $47=16\cdot 3-1$ is prime, 47 is larger than $31=2^5-1$]. The set Y simply contains each prime that is not in X.

Assume this partitioning and suppose $g>1$ and $f>1$ are such that each prime divisor of g is in X and each prime divisor of f is in Y. Next let m be a positive integer such that $2^m>4g$ and consider positive integers of the form $g+2^m fk$ with $k>0$. By Dirichlet's Theorem there are infinitely many choices of k such that $g+2^m fk$ is prime. By the argument above, at most one such prime is in X as all are congruent to g mod 2^m (since $4g<2^m$).

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