Palestine Journal of Mathematics

Vol. 7(1)(2018) , 257–261

MINIMAL AFFINE TRANSLATION SURFACES IN HYPERBOLIC SPACE

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 53A10; Secondary 53A35.

Keywords and phrases: Affine translation surface, polynomial affine translation surface, Hyperbolic geometry, Gaussian curvature, mean curvature.

Abstract. In this paper we study the minimal polynomial affine translation surfaces in the 3-dimensional hyperbolic space $\mathbb{H}^3$. We suppose that the affine translation surface is minimal in $\mathbb{E}^3$ and then we prove that there are not any minimal polynomial affine translation surfaces in $\mathbb{H}^3$.

1 Introduction

The surfaces in the 3-dimensional Euclidean space E^3 which can be written as a sum of two real curves are called *translation surface*, i.e., $r(x,y) = \alpha(x) + \beta(y)$, where α and β are real curves. Such surfaces are obtained by translating one curve along another and are well-studied in Euclidean and Lorentzian space [1, 2, 6, 11].

The half-space model of the hyperbolic space $\mathbb{H}^3$ is $\mathbb{R}^3_+ = \left\{(x,y,z) \in \mathbb{R}^3; z > 0\right\}$ equipped with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

In this model, the surfaces can be considered as a sum of planar curves, in other words the curves in $\mathbb{R}^3_+$ involved in Euclidean space. A classification for complete umbilical surfaces in the hyperbolic space with constant mean curvature is as follows

Totally geodesic surfaces or geodesic planes. In our model of hyperbolic space, they may be identified with vertical Euclidean planes and Euclidean hemispheres that orthogonally intersect the xy- plane. For such a surface, the mean curvature is zero. After an isometry of $\mathbb{H}^3$, any geodesic plane can be considered as

$$P(a) = \left\{(x,y,z) \in \mathbb{R}^3_+; x^2 + y^2 + z^2 = a^2\right\},$$

where $a > 0$ [7]. For the other models: *equidistant spheres*, *horospheres*, *hyperbolic spheres* see also [7].

A translation surface in the half-space model of the 3-dimensional hyperbolic space $\mathbb{H}^3$ is a graph surface given by

$$r(x,y) = (x, y, f(x) + g(y)) \qquad \text{(type I)}$$

or

$$r(x,z) = (x, f(x) + g(z), z) \qquad \text{(type II)}$$

where f and g are smooth functions [8].

A surface with $H = 0$ at every point then called *minimal.* Examples of minimal surfaces in $\mathbb{H}^3$ are totally geodesic planes. Such minimal translation surfaces are studied by R. Lopez [8], by proving that there are no minimal translation surfaces in $\mathbb{H}^3$ of type I and the only minimal translation surfaces in $\mathbb{H}^3$ of type II are totally geodesic planes.

Most recently H. Liu and Y. Yu introduced a new translation surface so-called affine translation surfaces. The *affine translation surface* in E^3 is defined as a parameter surface $r(u,v)$

which can be written as

$$r(u,v) = (u, v, f(u) + g(v + au))$$

for some non zero constant a and smooth functions $f(u)$ and $g(v+au)$. Such surfaces are investigated in [5]. The authors classified minimal affine translation surfaces in three dimensional Euclidean space. They proved that if $r(x,y) = (x, y, z(x,y))$ is a minimal affine translation surface, then either $z(x,y)$ is linear or can be written as

$$z(x,y) = \frac{1}{c} \log \frac{\cos\left(c\sqrt{1+a^2}x\right)}{\cos\left[c(y+ax)\right]}. \tag{1.1}$$

The minimal translation surface given by (1.1) is called *generalized Sherk surface* or *affine Sherk surface* in E^3. For more details of affine translation surfaces, we refer the reader to [3, 9, 13].

In this paper we study the affine translation surfaces in $\mathbb{H}^3$, then we provide a result for polynomial minimal affine translation surfaces in $\mathbb{H}^3$.

2 Polynomial Affine Translation Surfaces in E^3

Let M an affine translation surface in E^3 parametrized by

$$r : U \subseteq E^2 \to E^3, (x,y) \mapsto r(x,y) = (x, y, f(x) + g(y+ax)), a \neq 0 \tag{2.1}$$

where f and g are real-valued and smooth functions on U. Then the Gauss and mean curvatures of M are given, respectively,

$$K = \frac{LN - M^2}{EG - F^2} = f''g''D^{-4} \tag{2.2}$$

and

$$H = \frac{LG - 2FM + NE}{2(EG - F^2)} = \frac{1}{2}\left[f''\left(1 + g'^2\right) + g''\left(1 + a^2 + f'^2\right)\right] D^{-3} \tag{2.3}$$

where $f' = \dfrac{df(x)}{dx}$, $g' = \dfrac{dg(v)}{dv} = \dfrac{dg(y+ax)}{d(y+ax)}$ and $D^2 = EG - F^2$ for $v = y + ax$. Note that the affine translation surface given by (2.1) is flat, i.e. $K \equiv 0$, if and only if at least one of f or g is a linear function.

A *polynomial translation surface* is parametrized by

$$r : U \subseteq E^2 \to E^3, (x,y) \mapsto r(x,y) = (x, y, f(x) + g(y)),$$

where f and g are polynomial functions on U [10, 12]. We suppose that the polynomials f and g are given by

$$f = b_m u^m + b_{m-1}u^{m-1} + \ldots + b_1 u + b_0$$

and

$$g = c_n v^n + c_{n-1}v^{n-1} + \ldots + c_1 v + c_0$$

where b_m and c_n are non-zero constants. Then in [4] we considered the polynomial affine translation surfaces in E^3 with constant curvature. So we proved the following two non-existence results:

Theorem 2.1. *There does not exist a polynomial affine translation surface with non-zero constant Gaussian curvature in E^3.*

Theorem 2.2. *There does not exist a polynomial affine translation surface with constant mean curvature in E^3.*

3 Minimal Polynomial Affine Translation Surfaces in $\mathbb{H}^3$

Let $X : M \subset \mathbb{R}^2 \to \mathbb{R}^3_+$ be an immersion in $\mathbb{R}^3_+$. Suppose that n is a unit normal vector field with respect to the hyperbolic metric and N is a unit normal vector field with respect to the Euclidean metric, thus the relation between the unit normal vector fields on M is given by $N = n/z$. If κ_i are the hyperbolic principal curvatures and κ_i^e are the Euclidean principal curvatures, the equation related with κ_i and κ_i^e can be written as follows

$$\kappa_i = z\kappa_i^e + N_3, \tag{3.1}$$

where N_3 is the third component of the unit normal vector N. If we put by H and H_e the hyperbolic and Euclidean mean curvature on M, respectively, we get

$$H(x,y,z) = zH_e(x,y,z) + N_3(x,y,z). \tag{3.2}$$

By considering (3.2) we have the following result for the polynomial affine translation surfaces corresponding the graphs of $z(x,y) = f(x) + g(y+ax)$.

Theorem 3.1. *Assume that the given affine translation surfaces in $\mathbb{E}^3$ is minimal, then there are not minimal polynomial affine translation surfaces in $\mathbb{H}^3$.*

Proof. Let M be a polynomial affine translation surface in $\mathbb{H}^3$. The H_e and N_3 can be written, respectively,

$$H_e = \frac{1}{2}\left[f''\left(1+g'^2\right) + g''\left(1+a^2+f'^2\right)\right] D^{-3}$$

and

$$N_3 = \frac{1}{\left(1+\left(f'+ag'\right)^2+g'^2\right)^{1/2}}$$

Substituting the above equation in equation (3.2) we obtain the mean curvature on M as follows

$$H = (f+g)\frac{\left(f''\left(1+g'^2\right)+g''\left(1+a^2+f'^2\right)\right)}{\left(1+\left(f'+ag'\right)^2+g'^2\right)^{3/2}} + \frac{2}{\left(1+\left(f'+ag'\right)^2+g'^2\right)^{1/2}}. \tag{3.3}$$

If the surface is minimal, we get

$$(f+g)\frac{\left(f''\left(1+g'^2\right)+g''\left(1+a^2+f'^2\right)\right)}{\left(1+\left(f'+ag'\right)^2+g'^2\right)^{3/2}} + \frac{2}{\left(1+\left(f'+ag'\right)^2+g'^2\right)^{1/2}} = 0$$

We can rewrite this equation as

$$(f+g)\left(f''\left(1+g'^2\right)+g''\left(1+a^2+f'^2\right)\right) + 2\left(1+\left(f'+ag'\right)^2+g'^2\right) = 0. \tag{3.4}$$

Differentiating (3.4) with respect to y, we have

$$\begin{gathered} g'\left[f''\left(1+g'^2\right)+g''\left(1+a^2+f'^2\right)\right] + (f+g)\left[2f''g'g''+g'''\left(1+a^2+f'^2\right)\right] \\ +2\left(2\left(f'+ag'\right)ag''+2g'g''\right) = 0 \end{gathered} \tag{3.5}$$

On the other hand suppose that the polynomials f and g are given by

$$f = b_m u^m + b_{m-1}u^{m-1} + \ldots + b_1 u + b_0$$

and

$$g = c_n v^n + c_{n-1}v^{n-1} + \ldots + c_1 v + c_0$$

where b_m and c_n are non-zero constants. Replacing f and g in (3.5) we get a polynomial expression in u and v vanishing, i.e., all the coefficients are zero. Let us consider some cases of equation (3.5)

Case 1. $m, n \geq 2$

i. Suppose that $m > n\,(\geq 2)$ The dominant term according to $u^{3m-2}v^{n-3}$ which comes from $ff'^2g''' + 2f''gg'g'' + gg'''$ having the coefficient $b_m^3 c_n m^2 n\,(n-1)\,(n-2)$. This cannot vanish since $b_m, c_n \neq 0$ and $m > n \geq 2$.

ii. Suppose that $n > m\,(\geq 2)$ or $m = n\,(\geq 2)$ Using similar way, this case cannot occur.

Case 2. $m, n \geq 1$

i. $m > n = 1$. We get $g = cv + d$ with real constants c, d and $c \neq 0$. If we consider this situation in equation (3.5), we obtain

$$f''\left(c^3 + c\right) = 0$$

Since $c \neq 0$ we have $f'' = 0$, i.e., f is a linear function. However in this case the degree of f must be 1. This contradicts our initial assumption, so the equation (3.5) is not satisfying.

ii. $n > m = 1$. In this case $f = bu + d$ (and suppose that $g'' \neq 0$) with real constants b, d and $b \neq 0$. The coefficient of highest degree v^{2n-3}comes from $g'g''\left(1 + a^2 + f'^2\right) + fg'''\left(1 + a^2 + f'^2\right)$ having the coefficient
$c_n^2 n^2\,(n-1)\left(1 + a^2 + b^2\right)$. Then this expression cannot occur since $c_n \neq 0$.

iii. $n = m = 1$. From equation (3.4) this case con not occur.

Case 3. $m, n \geq 0$

i. $m \geq n = 0$. Then g is a constant, so the equation (3.5) is satisfied. But if g is constant, from the equation (3.4) f is not be a polynomial function. It is a contradiction, so this situation cannot occur.

ii. $n \geq m = 0$. Then $f(f = b)$ is constant, so the equation (3.5) can rewrite with this case in the following way

$$g'\left[g''\left(1 + a^2\right)\right] + (f + g)\left[g''\left(1 + a^2\right)\right]' + 2\left(2\left(ag'\right)ag'' + 2g'g''\right) = 0$$

Corresponding to the same idea like in case1, 2 we can say that this situation cannot occur since $c_n \neq 0$. Therefore the proof is completed. □

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Received: June 29, 2016.

Accepted: April 11, 2017.