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B-Y. Chen Inequalities for Semi-Slant Submanifolds in Normal Paracontact Metric Manifolds

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Abstract The aim of the present paper is to study B.Y. Chen inequalities for semi-slant submanifolds in a normal paracontact metric space form.

1 Introduction

The Riemannian invariants of a Riemannian manifold are the intrinsic characteristic of the Riemannian manifold. B-Y. Chen recalls one of the basic problems in submanifolds theory as to find simple relationship between the main extrinsic invariants and the main intrinsic invariants of a submanifolds[2]. A sharp inequality was established for the sectional curvature of a submanifold in a real space form in terms of the scalar curvature and squared mean curvature[3]. After then many geometers obtained similar inequalities for submanifolds in different space forms. Contact version of these equalities were studied by many geometers[1, 5, 8, 9].

Motivated by the studies of the above authors, we have established Chen inequalities for semi-slant submanifolds in a normal paracontact space form which has not been attempted so far.

2 Preliminaries

A m-dimensional Riemannian manifold $(\bar{M}, g)$ is said to be almost contact metric manifold if it admits an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ, and a 1-form η satisfying

$$\begin{aligned} \phi^2 &= I - \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \phi\xi = 0, \ \ \eta \circ \phi = 0 \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \ \ \eta(X) = g(X, \xi), \end{aligned} \tag{2.1}$$

for any vector fields X and Y on $\bar{M}$. An almost paracontact metric manifold $\bar{M}$ is said to be normal if

$$(\bar{\nabla}_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \ \ \bar{\nabla}_X \xi = \phi X, \tag{2.2}$$

for any vector fields X, Y on $\bar{M}$[6], where $\bar{\nabla}$ denotes the Riemannian connection with respect to g. If a normal paracontact metric manifold $\bar{M}$ is of a constant sectional curvature c, denoted by $\bar{M}(c)$, then its the Riemannian curvature tensor $\bar{R}$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{1}{4}(c+3)\{g(Y, Z)X - g(X, Z)Y\} + \frac{1}{4}(c-1)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned} \tag{2.3}$$

for any vector fields X, Y, Z on $\bar{M}$[6].

Let (M, g) be a Riemannian manifold and we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_M(p),\ \ p \in M$. For any orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_M(p)$, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j}^{n} K(e_i \Lambda e_j). \tag{2.4}$$

We denote by

$$(\inf K)(p) = \inf\{K(\pi) | \pi \subset T_M(p), dim\pi = 2\}. \tag{2.5}$$

The first Chen invariant $\delta_M(p)$ is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p). \tag{2.6}$$

We recall the following Lemma of Chen [4] for later use.

Lemma 2.1. *Let $n \geq 2$ and $a_1, a_2, ..., a_n, c$ be $(n+1)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + c\right). \tag{2.7}$$

Then $2a_1a_2 \geq c$ and the equality holds if and only if

$$a_1 + a_2 = a_3 = . \ . \ . = a_n.$$

Now, let M be an n-dimensional isometrically immersed submanifold of a normal paracontact metric manifold $\bar{M}$ with induced by g. Denoting by h, ∇ and $\nabla^{\perp}$ the second fundamental form of M, the induced connection on M and $T^{\perp}M$, respectively. Then the Gauss and Weingarten formulae are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.8}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.9}$$

for all vector fields X, Y tangent to M and V normal to M, where A_V is the shape operator of M in the direction of V. The second fundamental form h and shape operator are related A_V by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.10}$$

By R we denote the Riemannian curvature tensor of M, then the equation of Gauss is given by

$$\begin{aligned} g(\bar{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) - g(h(X, Z), h(Y, W)) \\ &+ g(h(X, W), h(Y, Z)), \end{aligned} \tag{2.11}$$

for any vector fields X, Y, Z, W on M.

Let $p \in M$ and $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space $T_M(p)$. Then the mean curvature tensor $H(p)$ is defined by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{2.12}$$

Also we put

$$h_{ij}^r = g(h(e_i, e_j), e_r), \ \ 1 \leq i, j \leq n, \ \ n+1 \leq r \leq 2m+1, \tag{2.13}$$

where $\{e_{n+1}, e_{n+2}, ..., e_{2m+1}\}$ are orthonormal basis the normal space $T_M^{\perp}(p)$ and

$$\| h \|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.14}$$

On the other hand, for an orthonormal basis $\{e_1, e_2, ..., e_{n-1}, e_n = \xi\}$ of $T_M(p)$, the scalar curvature τ at p of M assumes the form

$$2\tau = \sum_{i \neq j}^{n-1} K(e_i \wedge e_j) + 2\sum_{i=1}^{n-1} K(e_i \wedge \xi). \tag{2.15}$$

Now, Let M be a submanifold of a normal paracontact metric manifold $\bar{M}$. For $X \in \Gamma(TM)$, we put

$$\phi X = TX + NX, \tag{2.16}$$

where TX and NX denote the tangential and normal components of ϕX. In this case, let us denote

$$\| T \|^2 = \sum_{i,j=1}^{n} g^2(Te_i, e_j) \ \ and \ \ tr(T) = \sum_{i=1}^{n} g(Te_i, e_i). \tag{2.17}$$

Definition 2.2. Let M be a submanifold of a normal paracontact metric manifold $\bar{M}$. A differentiable distribution D on M is called slant distribution if for each $p \in M$ and each non-zero vector field $X \in D_p$, the angle $\theta_D(X)$ between ϕX and the vector subspace D_p is constant, independent of the choice of $p \in M$ and $X \in D_p$. In this case, the constant angle θ_D is called the slant angle D.

A submanifold M is said to be a slant if for any $p \in M$ and $X \in T_M(p)$, the angle between ϕX and $T_M(p)$ is constant, that is, it doesn't dependent of the choice of $p \in M$ and $X \in T_M(p)$. The angle $\theta \in (0, \frac{\pi}{2})$ is called the slant angle of M in $\bar{M}$.

Invariant and anti-invariant submanifolds are special slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

Definition 2.3. Let M be a submanifold of a normal paracontact metric manifold $\bar{M}$. M is said to be semi-slant submanifold if there exist two orthogonal distributions D_T and D_θ on M such that
i.) TM admits the orthogonal direct sum $TM = D^T \oplus D^\theta$,
ii.) the distribution D^T is an invariant, that is, $\phi(D^T) = D^T$
iii.) the distribution D^θ is a slant with slant angle $\theta \neq 0, \frac{\pi}{2}$.

Theorem 2.4. *Let M be a slant submanifold of a normal paracontact metric manifold $\bar{M}$ with slant angle θ. Then we have*

$$g(TX, TY) = \cos^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.18}$$

$$g(NX, NY) = \sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{2.19}$$

for any $X, Y \in \Gamma(TM)$.

3 B-Y. Chen Inequalities for Semi-Slant Submanifolds in a Normal Paracontact Metric Space Form

In this section, we will establish B.Y. Chen inequalities for proper semi-slant submanifolds in a normal paracontact metric space form. We will consider a plane sections π-invariant and π-slant by T

Theorem 3.1. *Let M be an $n = 2(p+q)+1$-dimensional proper semi-slant submanifold of a $2m+1$-dimensional paracontact metric space form $\bar{M}(c)$. Then we have*

$$K(\pi) \geq \tau + \left(1 - \frac{n}{2}\right)\left(\frac{n^2}{n-1}\|H\|^2 - \frac{(n+1)(c+3)}{4}\right) + \left(\frac{c-1}{4}\right)\left((p-1)\cos^2\theta + q - (n-1)\right), \tag{3.1}$$

for any plane π invariant by T and tangent to slant distribution D^θ and

$$K(\pi) \geq \tau + \left(1 - \frac{n}{2}\right)\left(\frac{n^2}{n-1}\|H\|^2 - \frac{(n+1)(c+3)}{4}\right) + \left(\frac{c-1}{4}\right)\left(p\cos^2\theta + q - n\right), \tag{3.2}$$

for any plane π invariant by T and tangent to invariant distribution D^T, where $dim(D^\theta) = 2p$ and $dim(D^T) = 2q+1$.

The equality case in (3.1) and (3.2) hold at a point $p \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, ..., e_{n-1}, e_n = \xi\}$ of $T_M(p)$ and an orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m+1}\}$ of $T_M^{\perp}p$ such that the shape operator of M in $\bar{M}(c)$, at a point p take the following forms;

$$A_{n+1} = \begin{pmatrix} a & 0 & 0...0 \\ 0 & b & 0...0 \\ 0 & 0 & ...\mu I_{n-2} \end{pmatrix}, \quad a + b = \mu \tag{3.3}$$

and

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0...0 \\ h_{12}^r & -h_{11}^r & 0...0 \\ 0 & 0 & ...0_{n-2} \end{pmatrix}, \quad r \in \{n+2, ..., 2m+1\}. \tag{3.4}$$

Proof. By using (2.3) and (2.11), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{1}{4}(c+3)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ \frac{1}{4}(c-1)\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ &+ g(X,Z)\eta(Y)\eta(W) - \eta(X)\eta(W)g(Y,Z) + g(\phi Y,Z)g(\phi X,W) \\ &- g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\ &- g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)), \end{aligned} \tag{3.5}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Now let

$$\{e_1, e_2 = \sec\theta T e_1, e_3, e_4 = \sec\theta T e_3, ..., e_{2p} = \sec\theta T e_{2p-1}, e_{2p+1}, e_{2p+2} = Te_{2p+1}$$
$$, ..., e_{2(p+q)-1}, e_{2(p+q)} = Te_{2(p+q)-1}, \xi = e_{2(p+q)+1)}\}$$

be an orthonormal basis of $T_M(p)$ such that e_i, $1 \leq i \leq 2p$, are tangent to D^θ and e_j, $2p+1 \leq j \leq 2(p+q)+1 = n$, are tangent to D^T. Making use of (2.15) and (3.5), we have

$$2\tau = \sum_{i,j=1}^{n-1} g(R(e_i,e_j)e_i,e_j) + 2\sum_{i=1}^{n-1} g(R(e_i,\xi)e_i,\xi) = -\frac{1}{4}(c+3)n(n-1) + \frac{1}{4}(c-1)\{2(n-1) - \sum_{i,j=1}^{n} g^2(\phi e_i, e_j)\} + n^2\|H\|^2 - \|h\|^2. \tag{3.6}$$

We note that in general, $g(\phi X, X) \neq 0$, for any unit vector $X \in \Gamma(T\bar{M})$, in almost paracontact metric manifolds contrary to complex manifolds. Whereas, we can construct that the paracontact metric structure ϕ such that $g(\phi X, X) = 0$. If $\bar{M}$ is $2m+1$-dimensional almost paracontact metric manifold with an orthonormal basis $\{e_1, e_2, ..., e_m, e_{m+1}, ..., e_{2m}, e_{2m+1} = \xi\}$, then we can define ϕ by

$$\phi(e_i) = e_{m+i} \;\; and \;\; \phi(e_{m+i}) = e_i, \;\; i \in \{1, 2, ..., m\}.$$

In this case, we can easily to see that $g(\phi e_i, e_i) = 0$. Obviously, we observe

$$g(\phi e_1, e_2) = g(Te_1, \sec\theta Te_1) = \cos\theta$$

and

$$g(\phi e_{2p+1}, e_{2p+2}) = g(Te_{2p+1}, Te_{2p+1}) = 1,$$

from which

$$g^2(\phi e_i, e_{i+1}) = \left\{ \begin{array}{lll} \cos^2\theta & ; & for \;\; i = 1, 2, ..., 2p-1 \\ 1 & ; & for \;\; i = 2p+1, ..., 2(p+q)-1 \end{array} \right\}$$

Thus we have

$$\sum_{i,j=1}^{n} g^2(\phi e_i, e_j) = 2(p\cos^2\theta + q). \tag{3.7}$$

From (3.6) and (3.7), we reach at

$$2\tau \;\; = \;\; -\frac{1}{4}(c+3)n(n-1) + \frac{1}{4}(c-1)\{2(n-1) - 2(p\cos^2\theta + q)\} + n^2\|H\|^2 - \|h\|^2,$$

or

$$n^2\|H\|^2 = 2\tau + \|h\|^2 + \frac{1}{4}(c+3)n(n-1) + \frac{1}{2}(c-1)\{p\cos^2\theta + q - (n-1)\}.$$

If we put

$$\begin{aligned} \epsilon \;\; &= \;\; 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 + \left(\frac{c+3}{4}\right)n(n-1) \\ &+ \;\; \left(\frac{c-1}{2}\right)\{p\cos^2\theta + q - (n-1)\}, \end{aligned} \tag{3.8}$$

we obtain

$$n^2\|H\|^2 = (n-1)(\|h\|^2 + \epsilon). \tag{3.9}$$

Let $p \in M$, $\pi \subset T_M(p)$, dimπ=2 and π orthogonal to ξ, and invariant by T. We discusses two cases.
(i) The plane section π is tangent to D^θ. We suppose that $\pi = sp\{e_1, e_2\}$ and we take $e_{n+1} = H/\|H\|$. Relation (2.7) becomes

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left\{\sum_{i,j=1}^{n}\sum_{r=n+1}^{2m+1}(h_{ij}^r)^2 + \epsilon\right\}$$

or,

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon\right\}.$$

By using Lemma 2.1, we can infer

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \epsilon. \tag{3.10}$$

From the Gauss equation for $X = W = e_1$ and $Y = Z = e_2$, we reach

$$\begin{aligned} K(\pi) &= g(R(e_1,e_2)e_1,e_2) = \left(\frac{c+3}{4}\right)\{g(e_2,e_1)g(e_2,e_1) - g(e_1,e_1)g(e_2,e_2)\} \\ &+ \left(\frac{c-1}{4}\right)\{g(\phi e_2,e_1)g(\phi e_1,e_2) - g(\phi e_1,e_1)g(\phi e_2,e_2) \\ &- 2g(\phi e_1,e_2)g(\phi e_1,e_2)\} + g(h(e_1,e_1),h(e_2,e_2)) - g(h(e_1,e_2),h(e_1,e_2)) \\ &= -\left(\frac{c+3}{4}\right) - \left(\frac{c-1}{4}\right)g^2(\phi e_2,e_1) + \sum_{r=n+1}^{2m+1}\left[h_{11}^r h_{22}^r - (h_{12}^r)^2\right] \\ &= -\left(\frac{c+3}{4}\right) - \left(\frac{c-1}{4}\right)\cos^2\theta + \sum_{r=n+1}^{2m+1}\left[h_{11}^r h_{22}^r - (h_{12}^r)^2\right]. \end{aligned} \tag{3.11}$$

Taking account of (3.10) and (3.11), we obtain

$$\begin{aligned} K(\pi) &\geq -\left(\frac{c+3}{4}\right) - \left(\frac{c-1}{4}\right)\cos^2\theta + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \frac{\epsilon}{2} \\ &+ \sum_{r=n+2}^{2m+1}h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 \\ &= -\left(\frac{c+3}{4}\right) - \left(\frac{c-1}{4}\right)\cos^2\theta + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2 \\ &+ \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{\epsilon}{2}, \end{aligned}$$

that is,

$$K(\pi) \geq -\left(\frac{c+3}{4}\right) - \left(\frac{c-1}{4}\right)\cos^2\theta + \frac{\epsilon}{2}. \tag{3.12}$$

Substituting (3.8) into (3.12), we obtain (3.1).

In the same way, if the subspace π is tangent to D^T, we obtain (3.2).

If at any point $p \in M$, equalities in (3.1) and (3.2) hold, then inequalities in (3.10) and (3.12) become equalities. Thus we have

$$\begin{aligned} &h_{ij}^{n+1} = 0, \quad i \neq j, \quad i,j > 2 \\ &h_{ij}^r = 0, \quad i \neq j, \quad r = n+1,...,2m+1 \\ &h_{11}^r + h_{22}^r = 0, \quad r = n+2,...,2m+1 \\ &h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad j > 2, \\ &h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ...h_{nn}^{n+1}. \end{aligned}$$

If we choose $\{e_1,e_2\}$ such that $h_{12}^{n+1} = 0$ and we denoting $a = h_{11}^r$, $b = h_{22}^r$, $\mu = h_{33}^{n+1} = h_{44}^{n+1} = ... = h_{nn}^{n+1}$, then the shape operators take the desired (3.3) and (3.4) forms. □

Theorem 3.2. *Let M be an $n = 2(p+q)+1$-dimensional semi-slant submanifold of a $2m+1$-dimensional paracontact metric space form $\bar{M}$. Then*
i.) For any plane section π-invariant by T and tangent to D^θ,

$$\begin{aligned}\delta_M &\leq \left(\frac{n-2}{2}\right)\left(\frac{n^2}{n-1}\|H\|^2 - \frac{(n+1)(c+3)}{4}\right) \\ &+ \left(\frac{c-1}{4}\right)\left(n-1-(p-1)\cos^2\theta - q\right)\end{aligned} \tag{3.13}$$

and for any plane section π-invariant by T and tangent to D^T

$$\begin{aligned}\delta_M &\leq \left(\frac{n-2}{2}\right)\left(\frac{n^2}{n-1}\|H\|^2 - \frac{(n+1)(c+3)}{4}\right) \\ &+ \left(\frac{c-1}{4}\right)\left(n-p\cos^2\theta - q\right).\end{aligned} \tag{3.14}$$

The equality case of inequalities (3.13) and (3.14) hold at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_M(p)$ and an orthonormal basis $\{e_{n+1}, e_{n+2}, ..., e_{2m+1}\}$ of $T_M^\perp(p)$ such that the shape operators of M in $\bar{M}(c)$ at p have the (3.15) and (3.16) following forms;

$$A_{n+1} = \begin{pmatrix} a & 0 & 0...0 \\ 0 & b & 0...0 \\ 0 & 0 & ...\mu I_{n-2} \end{pmatrix}, \quad a+b=\mu \tag{3.15}$$

and

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0...0 \\ h_{12}^r & -h_{11}^r & 0...0 \\ 0 & 0 & ...0_{n-2} \end{pmatrix}, \quad r \in \{n+2, ..., 2m+1\}. \tag{3.16}$$

Example 3.3. Let M be a submanifold of $\mathbb{R}^9$ with coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, t)$ given by

$$\psi(u, w, v, s, t) = (u, o, w, o, v\cos\theta, v\sin\theta, s, 0, t).$$

It is easy to see that the tangent bundle of M is spanned by the vectors

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_3}, \quad e_3 = \cos\theta\frac{\partial}{\partial y_1} + \sin\theta\frac{\partial}{\partial y_2}, \quad e_4 = \frac{\partial}{\partial y_3}, \quad e_5 = \frac{\partial}{\partial t}.$$

On the other hand, we can define the almost paracontact metric structure ϕ of $\mathbb{R}^9$ by

$$\phi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad \phi(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}, \quad \phi(\frac{\partial}{\partial t}) = 0, 1 \leq i \leq 4, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt.$$

Then for any vector field $X = \lambda_i \frac{\partial}{\partial x_i} + \mu_i \frac{\partial}{\partial y_i} + \nu \frac{\partial}{\partial t} \in \Gamma(\mathbb{R}^9)$, we can easily to see that ϕ satisfies (2.1). Thus $\phi(TM)$ is spanned by

$$\phi e_1 = \frac{\partial}{\partial y_1}, \quad \phi e_2 = \frac{\partial}{\partial y_3}, \quad \phi e_3 = \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2}, \quad \phi e_4 = \frac{\partial}{\partial x_3}, \quad \phi e_5 = 0.$$

Since $g(\phi e_1, e_3) = \cos\theta$ and $\phi e_2 = e_4$, we can define $D^\theta = sp\{e_1, e_3\}$ and $D^T = sp\{e_2, e_4, e_5\}$. Thus M defines a 5-dimensional semi-slant submanifold of $\mathbb{R}^9$ with usual paracontact metric structure.

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