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SOME RESULTS ON λ - SPIRALLIKE GENERALIZED SAKAGUCHI TYPE FUNCTIONS

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Abstract.The aim of the present paper is to introduce a new subclass $\mathcal{L}_s(\alpha, \beta, \lambda, t)$ using Sakaguchi type functions and λ -Spirallike functions and to investigate Characterization and Subordination results for functions in this class. We discuss several consequences of our results.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disc $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$.
Let $\mathcal{K}$ be the familiar class of functions that are convex in $\mathcal{U}$ and let $\mathcal{S}^\lambda(\alpha)$ denote the class of λ-spirallike functions of order α. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^\lambda(\alpha)$ [4] if

$$\Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \alpha cos\lambda, \quad (z \in \mathcal{U},\, |\lambda| < \pi/2,\, 0 \le \alpha < 1)$$

Note that $\mathcal{S}^\lambda(0) = \mathcal{S}^\lambda$ is the class of λ-spirallike functions introduced by Spacek [10].
Further, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}^\lambda(\alpha)$ if

$$\Re\left\{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha cos\lambda, \quad (z \in \mathcal{U},\, |\lambda| < \pi/2,\, 0 \le \alpha < 1)$$

Note that $\mathcal{C}^\lambda(0) = \mathcal{C}^\lambda$ is the class of functions for which $zf'(z)$ is λ-spirallike in $\mathcal{U}$ introduced by Roberstson [7] and the class $\mathcal{C}^\lambda(\alpha)$ was introduced and studied by Chichra [1]. A function $f(z) \in \mathcal{C}^\lambda(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^\lambda(\alpha)$.
Now we introduce a new subclass $\mathcal{L}_S(\alpha, \beta, \lambda, t)$ defined using Sakaguchi type functions and λ-Spirallike functions as follows.

Definition 1.1. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{L}_S(\alpha, \beta, \lambda, t)$ if it satisfies

$$\Re\left\{e^{i\lambda}\frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]'}\right\} > \alpha cos\lambda, \tag{1.2}$$

for some $0 \le \alpha < 1$, $0 \le \beta < 1$, $t \ne 1$, $|t| \le 1$, $|\lambda| < \pi/2$, and $z \in \mathcal{U}$.

By giving specific values to λ, β, t we obtain the following subclasses studied by various researchers in earlier works

- For $\beta = 0$ and $\beta = 1$, we obtain the subclass $\mathcal{P}^\lambda(\alpha, t)$ and $\mathcal{M}^\lambda(\alpha, t)$ introduced and studied by Goyal and Goswami [3].
- For $\lambda = 0$, $\beta = 0$ and $\lambda = 0$, $\beta = 1$, we obtain the subclass $\mathcal{S}(\alpha, t)$ and $\mathcal{T}(\alpha, t)$ introduced and studied by Owa et. al. [5].

- For $\beta = 0$ and $t = 0$ we obtain the subclass $\mathcal{S}_P^{\alpha}(\lambda)$ studied in [6].
- For $\lambda = 0$ we obtain the subclass $\mathcal{L}_S(\alpha, \beta, t)$ studied in [9].

In our present investigation we need the following definitions and also a related result due to Wilf [11].

Definition 1.2. (Convolution) Given two functions f and g in the class $\mathcal{A}$, where f is given by (1.1) and g is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product (or convolution) $f * g$ is defined by the power series $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathcal{U})$.

Definition 1.3. (Subordination Principle) For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ and write $f \prec g$, if there exists a Schwarz function ω, which is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z)), \quad z \in \mathcal{U}$.

Definition 1.4. (Subordinating factor sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in $\mathcal{U}$, we have the subordination given by

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \quad (z \in \mathcal{U}, a_1 = 1).$$

Lemma 1.5. *[11] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0, \quad (z \in \mathcal{U}).$$

2 Main results

In this section first we prove the Characterization results for the functions in the class $\mathcal{L}_S(\alpha, \beta, \lambda, t)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in the class $\mathcal{L}_S(\alpha, \beta, \lambda, t)$ if*

$$\left|\frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]'} - 1\right| < 1 - \gamma \tag{2.1}$$

where $0 \le \gamma < 1$, $0 \le \beta < 1$, $|t| \le 1$, $t \ne 1$, , *provided that*

$$|\lambda| \le cos^{-1}(\frac{1-\gamma}{1-\alpha}) \tag{2.2}$$

for some α, $0 \le \alpha < 1$ and $z \in \mathcal{U}$.

Proof. Suppose that

$$\frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]'} - 1 < (1-\gamma)\omega(z),$$

$|\omega(z)| < 1$ for all $z \in \mathcal{U}$. Now

$$\Re\left\{e^{i\lambda}\frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]'}\right\} = cos\lambda + (1-\gamma)\Re\{e^{i\lambda}\omega(z)\}$$

$$\ge cos\lambda - (1-\gamma)|e^{i\lambda}\omega(z)|$$

$$\ge cos\lambda - (1-\gamma) \ \ge \alpha cos\lambda$$

provided that $|\lambda| \le cos^{-1}(\frac{1-\gamma}{1-\alpha})$.
This completes the proof. □

Theorem 2.2. *If*

$$\left|\frac{(1-t)zf^{'}(z)+\beta(1-t)z^2f^{''}(z)}{(1-\beta)[f(z)-f(tz)]+\beta z[f(z)-f(tz)]'}-1\right| < (1-\alpha)cos\lambda \tag{2.3}$$

for some $0 \le \alpha < 1$, $0 \le \beta < 1$, $t \ne 1$, $|t| \le 1$, $|\lambda| < \pi/2$, *and* $z \in \mathcal{U}$. *Then* $f(z)$ *belongs to the class* $\mathcal{L}_S(\alpha, \beta, \lambda, t)$

Proof. Set $\gamma = 1-(1-\alpha)cos\lambda$, in the above Theorem. □

Theorem 2.3. *If the function* $f(z) \in \mathcal{A}$, *satisfies the inequality*

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|sec\lambda+(1-\alpha)|u_n|]|a_n| \le (1-\alpha), \tag{2.4}$$

where $u_n = \sum_{k=0}^{n-1} t^k$, $(t \ne 1, |t| \le 1, 0 \le \alpha < 1, 0 \le \beta < 1, |\lambda| < \pi/2)$, *then* $f(z) \in \mathcal{L}_s(\alpha, \beta, \lambda, t)$.

Proof. By Theorem (2.2) it suffices to show that

$$\left|\frac{(1-t)zf^{'}(z)+\beta(1-t)z^2f^{''}(z)}{(1-\beta)[f(z)-f(tz)]+\beta z[f(z)-f(tz)]'}-1\right| < (1-\alpha)cos\lambda$$

Since

$$\begin{aligned}
&\left|\frac{(1-t)zf^{'}(z)+\beta(1-t)z^2f^{''}(z)}{(1-\beta)[f(z)-f(tz)]+\beta z[f(z)-f(tz)]'}-1\right| \\
&= \left|\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta](n-u_n)a_nz^n}{z+\sum_{n=2}^{\infty}[1+(n-1)\beta]u_na_nz^n}\right| \\
&< \frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n||a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n||a_n||z|^{n-1}} \\
&< \frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n||a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n||a_n|}.
\end{aligned}$$

The last expression is bounded above by $(1-\alpha)cos\lambda$, if

$$\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n||a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n||a_n|} < (1-\alpha)cos\lambda$$

which is equivalent to

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|sec\lambda+(1-\alpha)|u_n|]|a_n| < (1-\alpha)$$

□

Remark: Suitable choices of λ, β, t yield the characterization results derived in [3], [5], [6] and [9].

3 Subordination results

In this section we prove the Subordination results for the functions in the class $\mathcal{L}_s(\alpha,\beta,\lambda,t)$.

Theorem 3.1. *Let $f \in \mathcal{A}$ satisfies the inequality (2.4) and suppose that $g \in \mathcal{K}$. Then*

$$\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}(f*g)(z) \prec g(z) \tag{3.1}$$

$z \in \mathcal{U}$, $|t| \leq 1, t \neq 1, 0 \leq \beta < 1, 0 \leq \alpha < 1$, $|\lambda| < \pi/2$ and

$$\Re\{f(z)\} > -\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)}{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}, \quad (z \in \mathcal{U}). \tag{3.2}$$

The constant factor $\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}$ *in the subordination result (3.1) cannot be replaced by any larger one.*

Proof. Let $f \in \mathcal{A}$ satisfy the inequality (2.4) and suppose that $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then we have

$$\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}(f*g)(z) \tag{3.3}$$

$$= \frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}(z+\sum_{n=2}^{\infty} a_n c_n z^n)$$

By definition (1.4) the subordination result (3.1) holds true if the sequence

$$\left\{\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}a_n\right\}_{n=1}^{\infty} \tag{3.4}$$

is a subordinating factor sequence with $a_1 = 1$.
In view of lemma (1.5) it is enough to prove the inequality:

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}a_n z^n\right\} > 0, \quad (z \in \mathcal{U}). \tag{3.5}$$

Now,

$$\Re\left\{1+\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}\sum_{n=1}^{\infty} a_n z^n\right\}$$
$$= \Re\left\{1+\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}z\right.$$
$$+\frac{1}{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)}$$
$$\left.\sum_{n=2}^{\infty}(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]a_n z^n\right\}.$$

when $|z| = r, (0 < r < 1)$,

$$\geq 1-\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}r$$
$$-\frac{1}{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)}$$
$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|sec\lambda+(1-\alpha)|u_n|]|a_n|r^n$$

$$> 1-\frac{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}r$$
$$-\frac{1-\alpha}{(1+\beta)[|1-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)}r$$
$$> 0$$

Then (3.5) holds in $\mathcal{U}$. This proves the inequality (3.1). The inequality (3.2) follows from (3.1), by taking convex function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$. To prove the sharpness of the constant $\frac{(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]}$ we consider the function $f_0(z) \in \mathcal{L}_S(\alpha, \beta, \lambda, t)$ given by

$$f_0(z) = z - \frac{(1-\alpha)sec\lambda}{(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|]} z^2 \tag{3.6}$$

From (3.1),

$$\frac{(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]} f_0(z) \prec \frac{z}{1-z}, \quad (z \in \mathcal{U}) \tag{3.7}$$

For the function f_0, it is easy to verify that

$$\min \left\{ \Re \left\{ \frac{(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]} f_0(z) \right\} \right\} = -\frac{1}{2}. \quad (|z| \leq 1)$$

This shows that the constant $\frac{(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]}$ is the best possible, which completes the proof. □

Remark: Suitable choices of λ, β, t yield the subordination results derived in [2], [3], [6] and [9].

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