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HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR MT$_{(g,m,\varphi)}$-PREINVEX FUNCTIONS

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Abstract In the present paper, the notion of MT$_{(g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left hand side of Gauss-Jacobi type quadrature formula involving MT$_{(g,m,\varphi)}$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for MT$_{(g,m,\varphi)}$-preinvex functions that are twice differentiable via Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

1 Introduction

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In (see [12], [15]) and the references cited therein, Tunç and Yidirim defined the following so-called MT-convex function:

Definition 1.2. A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to the class of MT(I), if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.2}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [11], [13], [14]) and the references cited therein, also (see [7]) and the references cited therein.

Fractional calculus (see [11]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [4]–[11]) and the references cited therein.

Definition 1.4. (see [2]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 1.5. (see [3]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please (see [3], [5]) and the references therein.

Definition 1.6. (see [6]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R^{\star}_m|f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R^{\star}_m|f|$ (see [8]).
Recently, Liu (see [9]) obtained several integral inequalities for the left hand side of (1.3) under the Definition 1.4 of P-function.
Also in (see [10]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of $\mathrm{MT}_{(g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left hand side of (1.3) involving $\mathrm{MT}_{(g,m,\varphi)}$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for $\mathrm{MT}_{(g,m,\varphi)}$-preinvex functions that are twice differentiable via fractional integrals are given. In Section 4, some applications to special means are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

2 New integral inequalities for $\mathrm{MT}_{(g,m,\varphi)}$-preinvex functions

Definition 2.1. (see [1]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$.

We next give new definition, to be referred as $MT_{(g,m,\varphi)}$-preinvex function.

Definition 2.3. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$, $g : [0,1] \longrightarrow (0,1)$ be a differentiable function and $\varphi : I \longrightarrow \mathbb{R}$ is a continuous increasing function. For $f : K \longrightarrow \mathbb{R}$ and $m \in (0,1]$, if

$$f(m\varphi(y) + g(t)\eta(\varphi(x),\varphi(y),m)) \leq \frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f(\varphi(x)) + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f(\varphi(y)), \tag{2.1}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, then we say that f belong to the class of $MT_{(g,m,\varphi)}(K)$ with respect to η.

Remark 2.4. In Definition 2.3, it is worthwhile to note that the class $MT_{(g,m,\varphi)}(K)$ is a generalization of the class MT(I) given in Definition 1.2 on $K = I$ with respect to $\eta(\varphi(x),\varphi(y),m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x, y \in K$, $g(t) = t$, $\forall t \in (0,1)$ and $m = 1$.

Example 2.5. Let $f(x) = -|x|$, $g(t) = t$, $\varphi(x) = x$ and

$$\eta(x,y,m) = \begin{cases} y - mx, & \text{if } x \geq 0,\ y \geq 0; \\ y - mx, & \text{if } x \leq 0,\ y \leq 0; \\ mx - y, & \text{if } x \geq 0,\ y \leq 0; \\ mx - y, & \text{if } x \leq 0,\ y \geq 0. \end{cases}$$

Then $f(x)$ is a MT$_{(t,m,x)}$-preinvex function with respect to $\eta : \mathbb{R} \times \mathbb{R} \times (0,1] \longrightarrow \mathbb{R}$ and any fixed $m \in (0,1]$. However, it is obvious that $f(x) = -|x|$ is not a convex function on $\mathbb{R}$.

In this section, in order to prove our main results regarding some new integral inequalities involving $MT_{(g,m,\varphi)}$-preinvex functions, we need the following new lemma:

Lemma 2.6. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with $\varphi(a), \varphi(b) \in K$, $a < b$ and $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Then for any fixed $m \in (0,1]$ and $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1} \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)].$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$\begin{aligned} &= \eta(\varphi(b),\varphi(a),m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m) - m\varphi(a))^p \\ &\quad \times (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - m\varphi(a) - g(t)\eta(\varphi(b),\varphi(a),m))^q \\ &\quad \times f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)] \\ &= \eta(\varphi(b),\varphi(a),m)^{p+q+1} \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]. \end{aligned}$$

□

The following definition will be used in the sequel.

Definition 2.7. The Euler Beta function is defined for $x, y > 0$ as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.8. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with $\varphi(a), \varphi(b) \in K$, $a < b$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Let $k > 1$. If $|f|^{\frac{k}{k-1}}$ is a $MT_{(g,m,\varphi)}$-preinvex function on an open m-invex set K with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$, then for any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \left(\frac{m}{2}\right)^{\frac{k-1}{k}} |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q)$$

$$\times \Bigg\{ |f(\varphi(a))|^{\frac{k}{k-1}} \Big[\left(\sqrt{g(1)(1-g(1))} - \sqrt{g(0)(1-g(0))}\right) + \left(\arcsin\sqrt{g(1)} - \arcsin\sqrt{g(0)}\right) \Big]$$

$$+ |f(\varphi(b))|^{\frac{k}{k-1}} \Big[\left(\sqrt{g(1)(1-g(1))} - \sqrt{g(0)(1-g(0))}\right)$$

$$+ \left(\arcsin\sqrt{1-g(1)} - \arcsin\sqrt{1-g(0)}\right) \Big] \Bigg\}^{\frac{k-1}{k}},$$

where $B(g(t); k, p, q) = \displaystyle\int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)].$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a $MT_{(g,m,\varphi)}$-preinvex function on K, combining with Lemma 2.6 and Hölder inequality for all $t \in [0,1]$ and for any fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)] \right]^{\frac{1}{k}}$$

$$\times \left[\int_0^1 \left| f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \right|^{\frac{k}{k-1}} d[g(t)] \right]^{\frac{k-1}{k}}$$

$$\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q)$$

$$\times \left[\int_0^1 \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} |f(\varphi(b))|^{\frac{k}{k-1}} + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} |f(\varphi(a))|^{\frac{k}{k-1}} \right) d[g(t)] \right]^{\frac{k-1}{k}}$$

$$= \left(\frac{m}{2}\right)^{\frac{k-1}{k}} |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q)$$

$$\times \Bigg\{ |f(\varphi(a))|^{\frac{k}{k-1}} \Big[\left(\sqrt{g(1)(1-g(1))} - \sqrt{g(0)(1-g(0))}\right) + \left(\arcsin\sqrt{g(1)} - \arcsin\sqrt{g(0)}\right) \Big]$$

$$+ |f(\varphi(b))|^{\frac{k}{k-1}} \Big[\left(\sqrt{g(1)(1-g(1))} - \sqrt{g(0)(1-g(0))}\right)$$

$$+ \left(\arcsin\sqrt{1-g(1)} - \arcsin\sqrt{1-g(0)}\right) \Big] \Bigg\}^{\frac{k-1}{k}}.$$

□

Corollary 2.9. *Under the conditions of Theorem 2.8 for $g(t)=t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq\left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}}|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Theorem 2.10. *Let $\varphi: I\longrightarrow\mathbb{R}$ be a continuous increasing function and $g:[0,1]\longrightarrow(0,1)$ is a differentiable function. Assume that $f: K=[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ is a continuous function on the interval of real numbers K° with $\varphi(a),\varphi(b)\in K$, $a<b$ with $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $l\geq 1$. If $|f|^l$ is a $MT_{(m,\varphi)}$-preinvex function on an open m-invex set K with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $m\in(0,1]$, then for any fixed $p,q>0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{l}}|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);p,q)$$

$$\times\left[|f(\varphi(a))|^l B\left(g(t);p-\frac{1}{2},q+\frac{1}{2}\right)+|f(\varphi(b))|^l B\left(g(t);p+\frac{1}{2},q-\frac{1}{2}\right)\right]^{\frac{1}{l}},$$

where $B(g(t);p,q)=\int_0^1 g^p(t)(1-g(t))^q d[g(t)]$.

Proof. Since $|f|^l$ is a $MT_{(g,m,\varphi)}$-preinvex function on K, combining with Lemma 2.6 and Hölder inequality for all $t\in[0,1]$ and for any fixed $m\in(0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}$$

$$\times\int_0^1\Big[g^p(t)(1-g(t))^q\Big]^{\frac{l-1}{l}}\Big[g^p(t)(1-g(t))^q\Big]^{\frac{1}{l}}f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)]\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 g^p(t)(1-g(t))^q\Big|f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))\Big|^l d[g(t)\right]^{\frac{1}{l}}$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);p,q)$$

$$\times\left[\int_0^1 g^p(t)(1-g(t))^q\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}|f(\varphi(b))|^l+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}|f(\varphi(a))|^l\right)d[g(t)]\right]^{\frac{1}{l}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{l}}|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);p,q)$$

$$\times\left[|f(\varphi(a))|^l B\left(g(t);p-\frac{1}{2},q+\frac{1}{2}\right)+|f(\varphi(b))|^l B\left(g(t);p+\frac{1}{2},q-\frac{1}{2}\right)\right]^{\frac{1}{l}}.$$

□

Corollary 2.11. *Under the conditions of Theorem 2.10 for $g(t) = t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{l}} |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \Big[\beta(p+1, q+1)\Big]^{\frac{l-1}{l}}$$

$$\times \left[|f(\varphi(a))|^l \beta\left(p + \frac{1}{2}, q + \frac{3}{2}\right) + |f(\varphi(b))|^l \beta\left(p + \frac{3}{2}, q + \frac{1}{2}\right)\right]^{\frac{1}{l}}.$$

3 Hermite-Hadamard type fractional integral inequalities for $MT_{(g,m,\varphi)}$-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for $MT_{(g,m,\varphi)}$-preinvex functions via fractional integrals, we need the following new fractional integral identity:

Lemma 3.1. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $\varphi(a), \varphi(b) \in K$, $a < b$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f : K \longrightarrow \mathbb{R}$ be a twice differentiable function on K° and f'' is integrable on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$. Then for $\alpha > 0$, we have*

$$\frac{\eta(\varphi(x), \varphi(a), m)^{\alpha+1}}{(\alpha+1)\eta(\varphi(b), \varphi(a), m)}$$

$$\times\Big[(1 - g^{\alpha+1}(1))f'(m\varphi(a) + g(1)\eta(\varphi(x), \varphi(a), m))$$

$$-(1 - g^{\alpha+1}(0))f'(m\varphi(a) + g(0)\eta(\varphi(x), \varphi(a), m))\Big]$$

$$+\frac{\eta(\varphi(x), \varphi(b), m)^{\alpha+1}}{(\alpha+1)\eta(\varphi(b), \varphi(a), m)}$$

$$\times\Big[(1 - g^{\alpha+1}(1))f'(m\varphi(b) + g(1)\eta(\varphi(x), \varphi(b), m))$$

$$-(1 - g^{\alpha+1}(0))f'(m\varphi(b) + g(0)\eta(\varphi(x), \varphi(b), m))\Big]$$

$$+\frac{\eta(\varphi(x), \varphi(a), m)^{\alpha}}{\eta(\varphi(b), \varphi(a), m)}$$

$$\times\Big[g^\alpha(1)f(m\varphi(a) + g(1)\eta(\varphi(x), \varphi(a), m)) - g^\alpha(0)f(m\varphi(a) + g(0)\eta(\varphi(x), \varphi(a), m))\Big]$$

$$+\frac{\eta(\varphi(x), \varphi(b), m)^{\alpha}}{\eta(\varphi(b), \varphi(a), m)}$$

$$\times\Big[g^\alpha(1)f(m\varphi(b) + g(1)\eta(\varphi(x), \varphi(b), m)) - g^\alpha(0)f(m\varphi(b) + g(0)\eta(\varphi(x), \varphi(b), m))\Big]$$

$$-\frac{\alpha}{\eta(\varphi(b), \varphi(a), m)}$$

$$\times\left[\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)} (t - m\varphi(a))^{\alpha-1} f(t)dt\right.$$

$$\left.+\int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)} (t - m\varphi(b))^{\alpha-1} f(t)dt\right]$$

$$= \frac{\eta(\varphi(x),\varphi(a),m)^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 (1-g^{\alpha+1}(t))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 (1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \quad (3.1)$$

Proof. A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Let denote

$$I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)$$

$$= \frac{\eta(\varphi(x),\varphi(a),m)^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 (1-g^{\alpha+1}(t))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)} \int_0^1 (1-g^{\alpha+1}(t))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \quad (3.2)$$

Using Lemma 3.1 and the relation (3.2), the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

Theorem 3.2. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $\varphi(a),\varphi(b) \in A$, $a < b$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Assume that $f : A \longrightarrow \mathbb{R}$ be a twice differentiable function on A°. If $|f''|^q$ is a $MT_{(g,m,\varphi)}$-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q > 1$, $p^{-1}+q^{-1} = 1$, then for $\alpha > 0$, we have*

$$|I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)|$$

$$\le \left(\frac{m}{2}\right)^{\frac{1}{q}} \frac{B^{\frac{1}{p}}(g(t);p,\alpha)}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Bigg[|f''(\varphi(a))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)\Big]+|f''(\varphi(x))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)\Big]\Bigg]^{\frac{1}{q}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Bigg[|f''(\varphi(b))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)\Big]+|f''(\varphi(x))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)\Big]\Bigg]^{\frac{1}{q}}\Bigg\}, \quad (3.3)$$

where $B(g(t);p,\alpha) = \displaystyle\int_0^1 (1-g^{\alpha+1}(t))^p d[g(t)]$.

Proof. Suppose that $q > 1$. Using Lemma 3.1, $MT_{(g,m,\varphi)}$-preinvexity of $|f''|^q$, Hölder inequality and taking the modulus, we have

$$|I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)|$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-g^{\alpha+1}(t)||f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|d[g(t)]$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-g^{\alpha+1}(t)||f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|d[g(t)]$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 |f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|^q d[g(t)]\right)^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 |f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|^q d[g(t)]\right)^{\frac{1}{q}}$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$
$$\times\left[\int_0^1\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}|f''(\varphi(a))|^q\right)d[g(t)]\right]^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))^p d[g(t)]\right)^{\frac{1}{p}}$$
$$\times\left[\int_0^1\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}|f''(\varphi(b))|^q\right)d[g(t)]\right]^{\frac{1}{q}}$$
$$=\left(\frac{m}{2}\right)^{\frac{1}{q}}\frac{B^{\frac{1}{p}}(g(t);p,\alpha)}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}$$
$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Bigg[|f''(\varphi(a))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$
$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)\Big]+|f''(\varphi(x))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$
$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)\Big]\Bigg]^{\frac{1}{q}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Bigg[|f''(\varphi(b))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$
$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)\Big]+|f''(\varphi(x))|^q\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$
$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)\Big]\Bigg]^{\frac{1}{q}}\Bigg\}.$$

$\square$

Corollary 3.3. *Under the conditions of Theorem 3.2 for $g(t)=t$ and $|f''|\leq K$, we get*

$$\Bigg|\frac{-\eta(\varphi(x),\varphi(a),m)^{\alpha+1}f'(m\varphi(a))-\eta(\varphi(x),\varphi(b),m)^{\alpha+1}f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$+\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))+\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$

$$\leq\frac{K}{(1+\alpha)^{1+\frac{1}{p}}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

Theorem 3.4. *Let $\varphi:I\longrightarrow\mathbb{R}$ be a continuous increasing function and $g:[0,1]\longrightarrow(0,1)$ is a differentiable function. Suppose $A\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:A\times A\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $m\in(0,1]$ and let $\varphi(a),\varphi(b)\in A$, $a<b$ with $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Assume that $f:A\longrightarrow\mathbb{R}$ be a twice differentiable function on A°. If $|f''|^q$ is a $MT_{(g,m,\varphi)}$-preinvex function on $[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q\geq 1$, then for $\alpha>0$, we have*

$$|I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)|$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{q}}\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[|f''(\varphi(a))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)-B\left(g(t);\alpha+\frac{1}{2},\frac{1}{2}\right)\Bigg]$$

$$+|f''(\varphi(x))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)-B\left(g(t);\alpha+\frac{3}{2},-\frac{1}{2}\right)\Bigg]\Bigg]^{\frac{1}{q}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{q}}\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[|f''(\varphi(b))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)-B\left(g(t);\alpha+\frac{1}{2},\frac{1}{2}\right)\Bigg]$$

$$+|f''(\varphi(x))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)-B\left(g(t);\alpha+\frac{3}{2},-\frac{1}{2}\right)\Bigg]\Bigg]^{\frac{1}{q}},\qquad(3.4)$$

where $B(g(t);p,q)=\int_0^1 g^p(t)(1-g(t))^q d[g(t)]$.

Proof. Suppose that $q \geq 1$. Using Lemma 3.1, $MT_{(g,m,\varphi)}$-preinvexity of $|f''|^q$, the well-known power mean inequality and taking the modulus, we have

$$|I_{f,g,\eta,\varphi}(x;\alpha,m,a,b)|$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-g^{\alpha+1}(t)||f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|d[g(t)]$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-g^{\alpha+1}(t)||f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|d[g(t)]$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 (1-g^{\alpha+1}(t))|f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))|^q d[g(t)]\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 (1-g^{\alpha+1}(t))|f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))|^q d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 (1-g^{\alpha+1}(t))\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}|f''(\varphi(a))|^q\right)d[g(t)]\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 (1-g^{\alpha+1}(t))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 (1-g^{\alpha+1}(t))\left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}|f''(\varphi(b))|^q\right)d[g(t)]\right]^{\frac{1}{q}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{q}}\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[|f''(\varphi(a))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)-B\left(g(t);\alpha+\frac{1}{2},\frac{1}{2}\right)\Bigg]$$

$$+|f''(\varphi(x))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)-B\left(g(t);\alpha+\frac{3}{2},-\frac{1}{2}\right)\Bigg]\Bigg]^{\frac{1}{q}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{q}}\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[|f''(\varphi(b))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)-B\left(g(t);\alpha+\frac{1}{2},\frac{1}{2}\right)\Bigg]$$

$$+|f''(\varphi(x))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)-B\left(g(t);\alpha+\frac{3}{2},-\frac{1}{2}\right)\Bigg]\Bigg]^{\frac{1}{q}}.$$

□

Corollary 3.5. *Under the conditions of Theorem 3.4 for $g(t)=t$ and $|f''|\leq K$, we get*

$$\Bigg|\frac{-\eta(\varphi(x),\varphi(a),m)^{\alpha+1}f'(m\varphi(a))-\eta(\varphi(x),\varphi(b),m)^{\alpha+1}f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$+\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))+\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$

$$\leq\frac{K}{\alpha+1}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\pi-\frac{\sqrt{\pi}(\alpha+1)\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha+3)}\right)^{\frac{1}{q}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

Remark 3.6. For a particular choices of a differentiable function $g:[0,1]\longrightarrow(0,1)$, for example: $e^{-(t+1)}$, $\sin\left(\frac{\pi(t+1)}{5}\right)$, $\cos\left(\frac{\pi(t+1)}{5}\right)$, etc., by our theorems mentioned in this paper we can get some special kinds of Hermite-Hadamard type fractional inequalities.

4 Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 4.1. (see [16]) A function $M:\mathbb{R}^2_+\longrightarrow\mathbb{R}_+$, is called a Mean function if it has the following properties:

(i) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,

(ii) Symmetry: $M(x,y)=M(y,x)$,

(iii) Reflexivity: $M(x,x)=x$,

(iv) Monotonicity: If $x\leq x'$ and $y\leq y'$, then $M(x,y)\leq M(x',y')$,

(v) Internality: $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α,β $(\alpha\neq\beta)$.

(i) The arithmetic mean:

$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}$$

(ii) The geometric mean:

$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}$$

(iii) The harmonic mean:

$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$

(iv) The power mean:

$$P_r := P_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

(v) The identric mean:

$$I := I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(vi) The logarithmic mean:

$$L := L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}; \quad |\alpha| \neq |\beta|, \quad \alpha\beta \neq 0.$$

(vii) The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R}\setminus\{-1,0\}, \quad \alpha \neq \beta.$$

(viii) The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0\,(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a),\varphi(b)) : [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \times [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Therefore one can obtain various inequalities using the results of Section 3 for these means as follows: Replace $\eta(\varphi(x),\varphi(y),m)$ with $\eta(\varphi(x),\varphi(y))$ and setting $\eta(\varphi(x),\varphi(y)) = M(\varphi(x),\varphi(y))$, $\forall x,y \in A$ for value $m = 1$ in (3.3) and (3.4), one can obtain the following interesting inequalities involving means:

$$\Bigg| \frac{M(\varphi(a),\varphi(x))^{\alpha+1}}{(\alpha+1)M(\varphi(a),\varphi(b))}$$

$$\times\Big[(1-g^{\alpha+1}(1))f'(\varphi(a)+g(1)M(\varphi(a),\varphi(x))) - (1-g^{\alpha+1}(0))f'(\varphi(a)+g(0)M(\varphi(a),\varphi(x)))\Big]$$

$$+\frac{M(\varphi(b),\varphi(x))^{\alpha+1}}{(\alpha+1)M(\varphi(a),\varphi(b))}$$

$$\times\Big[(1-g^{\alpha+1}(1))f'(\varphi(b)+g(1)M(\varphi(b),\varphi(x))) - (1-g^{\alpha+1}(0))f'(\varphi(b)+g(0)M(\varphi(b),\varphi(x)))\Big]$$

$$+\frac{M(\varphi(a),\varphi(x))^{\alpha}}{M(\varphi(a),\varphi(b))}\Big[g^{\alpha}(1)f(\varphi(a)+g(1)M(\varphi(a),\varphi(x))) - g^{\alpha}(0)f(\varphi(a)+g(0)M(\varphi(a),\varphi(x)))\Big]$$

$$+\frac{M(\varphi(b),\varphi(x))^{\alpha}}{M(\varphi(a),\varphi(b))}\times\Big[g^{\alpha}(1)f(\varphi(b)+g(1)M(\varphi(b),\varphi(x))) - g^{\alpha}(0)f(\varphi(b)+g(0)M(\varphi(b),\varphi(x)))\Big]$$

$$-\frac{\alpha}{M(\varphi(a),\varphi(b))}$$

$$\times\left[\int_{\varphi(a)+g(0)M(\varphi(a),\varphi(x))}^{\varphi(a)+g(1)M(\varphi(a),\varphi(x))}(t-\varphi(a))^{\alpha-1}f(t)dt+\int_{\varphi(b)+g(0)M(\varphi(b),\varphi(x))}^{\varphi(b)+g(1)M(\varphi(b),\varphi(x))}(t-\varphi(b))^{\alpha-1}f(t)dt\right]\Bigg|$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{q}}\frac{B^{\frac{1}{p}}(g(t);p,\alpha)}{(\alpha+1)M(\varphi(a),\varphi(b))}$$

$$\times\Bigg\{M(\varphi(a),\varphi(x))^{\alpha+2}\Bigg[|f''(\varphi(a))|^{q}\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)\Big]+|f''(\varphi(x))|^{q}\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)\Big]\Bigg]^{\frac{1}{q}}$$

$$+M(\varphi(b),\varphi(x))^{\alpha+2}\Bigg[|f''(\varphi(b))|^{q}\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)\Big]+|f''(\varphi(x))|^{q}\Big[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)\Big]\Bigg]^{\frac{1}{q}}\Bigg\}, \tag{4.1}$$

$$\Bigg|\frac{M(\varphi(a),\varphi(x))^{\alpha+1}}{(\alpha+1)M(\varphi(a),\varphi(b))}$$

$$\times\Big[(1-g^{\alpha+1}(1))f'(\varphi(a)+g(1)M(\varphi(a),\varphi(x)))-(1-g^{\alpha+1}(0))f'(\varphi(a)+g(0)M(\varphi(a),\varphi(x)))\Big]$$

$$+\frac{M(\varphi(b),\varphi(x))^{\alpha+1}}{(\alpha+1)M(\varphi(a),\varphi(b))}$$

$$\times\Big[(1-g^{\alpha+1}(1))f'(\varphi(b)+g(1)M(\varphi(b),\varphi(x)))-(1-g^{\alpha+1}(0))f'(\varphi(b)+g(0)M(\varphi(b),\varphi(x)))\Big]$$

$$+\frac{M(\varphi(a),\varphi(x))^{\alpha}}{M(\varphi(a),\varphi(b))}\Big[g^{\alpha}(1)f(\varphi(a)+g(1)M(\varphi(a),\varphi(x)))-g^{\alpha}(0)f(\varphi(a)+g(0)M(\varphi(a),\varphi(x)))\Big]$$

$$+\frac{M(\varphi(b),\varphi(x))^{\alpha}}{M(\varphi(a),\varphi(b))}\times\Big[g^{\alpha}(1)f(\varphi(b)+g(1)M(\varphi(b),\varphi(x)))-g^{\alpha}(0)f(\varphi(b)+g(0)M(\varphi(b),\varphi(x)))\Big]$$

$$-\frac{\alpha}{M(\varphi(a),\varphi(b))}$$

$$\times\left[\int_{\varphi(a)+g(0)M(\varphi(a),\varphi(x))}^{\varphi(a)+g(1)M(\varphi(a),\varphi(x))}(t-\varphi(a))^{\alpha-1}f(t)dt+\int_{\varphi(b)+g(0)M(\varphi(b),\varphi(x))}^{\varphi(b)+g(1)M(\varphi(b),\varphi(x))}(t-\varphi(b))^{\alpha-1}f(t)dt\right]\Bigg|$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{q}}\frac{M(\varphi(a),\varphi(x))^{\alpha+2}}{(\alpha+1)M(\varphi(a),\varphi(b))}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[|f''(\varphi(a))|^{q}\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)-B\left(g(t);\alpha+\frac{1}{2},\frac{1}{2}\right)\Bigg]$$

$$+|f''(\varphi(x))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)-B\left(g(t);\alpha+\frac{3}{2},-\frac{1}{2}\right)\Bigg]\Bigg]^{\frac{1}{q}}$$

$$+\left(\frac{1}{2}\right)^{\frac{1}{q}}\frac{M(\varphi(b),\varphi(x))^{\alpha+2}}{(\alpha+1)M(\varphi(a),\varphi(b))}\left(g(1)-g(0)-\frac{g^{\alpha+2}(1)-g^{\alpha+2}(0)}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[|f''(\varphi(b))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{g(1)}-\arcsin\sqrt{g(0)}\right)-B\left(g(t);\alpha+\frac{1}{2},\frac{1}{2}\right)\Bigg]$$

$$+|f''(\varphi(x))|^q\Bigg[\left(\sqrt{g(1)(1-g(1))}-\sqrt{g(0)(1-g(0))}\right)$$

$$+\left(\arcsin\sqrt{1-g(1)}-\arcsin\sqrt{1-g(0)}\right)-B\left(g(t);\alpha+\frac{3}{2},-\frac{1}{2}\right)\Bigg]\Bigg]^{\frac{1}{q}}. \tag{4.2}$$

Letting $M(\varphi(a),\varphi(b)) = A, G, H, P_r, I, L, L_p, M_p$ in (4.1) and (4.2), we get the inequalities involving means for a particular choices of a twice differentiable $MT_{(g,1,\varphi)}$-preinvex function f. The details are left to the interested reader.

References

[1] T. S. Du, J. G. Liao and Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s,m)-preinvex functions, *J. Nonlinear Sci. Appl.* **9**, 3112–3126 (2016).

[2] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.* **21**, 335–341 (1995).

[3] T. Antczak, Mean value in invexity analysis, *Nonlinear Anal.* **60**, 1473–1484 (2005).

[4] F. Qi and B. Y. Xi, Some integral inequalities of Simpson type for $GA-\epsilon$-convex functions, *Georgian Math. J.* **20**, 775–788 (2013).

[5] X. M. Yang, X. Q. Yang and K. L. Teo, Generalized invexity and generalized invariant monotonicity, *J. Optim. Theory Appl.* **117**, 607–625 (2003).

[6] R. Pini, Invexity and generalized convexity, *Optimization.* **22**, 513–525 (1991).

[7] H. Kavurmaci, M. Avci and M. E. Özdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, *arXiv:1006.1593v1 [math. CA]*, 1–10 (2010).

[8] D. D. Stancu, G. Coman and P. Blaga, Analiză numerică şi teoria aproximării, *Cluj-Napoca: Presa Universitară Clujeană.* **2**, (2002).

[9] W. Liu, New integral inequalities involving beta function via P-convexity, *Miskolc Math Notes.* **15**, 2, 585–591 (2014).

[10] M. E. Özdemir, E. Set and M. Alomari, Integral inequalities via several kinds of convexity, *Creat. Math. Inform.* **20**, 1, 62–73 (2011).

[11] W. Liu, W. Wen and J. Park, Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals, *J. Nonlinear Sci. Appl.* **9**, 766–777 (2016).

[12] W. Liu, W. Wen and J. Park, Ostrowski type fractional integral inequalities for MT-convex functions, *Miskolc Mathematical Notes.* **16**, 1, 249–256 (2015).

[13] Y. M. Chu, G. D. Wang and X. H. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.* **13**, 4, 725–731 (2010).

[14] X. M. Zhang, Y. M. Chu and X. H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, *J. Inequal. Appl.* (2010).

[15] Y. M. Chu, M. A. Khan, T. U. Khan and T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.* **9**, 5, 4305–4316 (2016).

[16] P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht (2003).

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