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Fixed Point Theorems for $\alpha - \psi -$Contractive Type Mappings in Fuzzy Metric Space

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Abstract Samet et al [3] introduced the concept of $\alpha - \psi -$contractive type mapping and utilized the same concept to prove several interesting fixed point theorems in setting of metric spaces. We are extended the results of Seong-Hoon Cho [15] in fuzzy metric space. In this paper we introduced some new fixed point theorems for $\alpha - \psi -$contractive type mappings are an initiated and some example in fuzzy metric space.

1 Introduction

The interest in fuzzy sets has been constantly growing from the starting paper of Zadeh [10] in 1965. Consequently, a large amount of theoretical and applied results is achieved in the direction of logic, mathematical analysis and general topology with many applications in economy and engineering. The notion of fuzzy metric space was introduced by Kramosil and Michalek [8] in 1975. Now, an important theoretical development is the way of defining the concept of contractive mapping in fuzzy metric space. In 1988 Grabiec [11] introduced the Banach contraction in a fuzzy metric space and extended the fixed point theorems of Banach and Edelstein to fuzzy metric space. Later on, in 1994 George and Veeramani [1] modified the concept of fuzzy metric spaces due to Kramosil and Michalek [8]. In 2002, Gregori and Sapena [17] introduced the notion of fuzzy contraction mapping and proved some fixed point theorems in various classes of complete fuzzy metric space.
Samet et. al [3] introduced the concept of $\alpha - \psi -$ contractive type mapping and utilized the same concept to prove several interesting fixed point theorems in setting of metric spaces. In [9], the authors extended the result of [3] to the case of multifunctions and they generalized the result of [3] as follow.

Theorem 1.1. *[15] Let (X, d) be a complete metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$. Suppose that a mapping $T : X \to X$ satisfies the following conditions:*

(i) $\alpha(x, y)d(Tx, Ty) \le \psi(m(x, y))$ *for all* $x, y \in X$,
where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$;

(ii) for each $x, y \in X$, $\alpha(x, y) \ge 1 \Rightarrow \alpha(Tx, Ty) \ge 1$;

(iii) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1$;

(iv) if $\{x_n\}$ *is a sequence with* $\alpha(x_n, x_{n+1}) \ge 1$ *for all* $n \in N \cup \{0\}$ *and* $\lim_{n\to\infty} x_n = x \in X$, *then* $\alpha(x_n, x) \ge 1$ *for all* $n \in N \cup \{0\}$.

Then T has a fixed point in X.

Theorem 1.2. *[15] Let (X, d) be a complete metric space, and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\psi(t) < t$ for each $t > 0$. Suppose that a mapping $T : X \to X$ such that $\alpha(x, y)d(Tx, Ty) \le \psi(m(x, y))$ for all $x, y \in X$. Assume that there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0) \ge 1$ for all $i, j \in N \cup \{0\}$ with $i \ne j$. Suppose that either T is continuous or $\alpha(T^i x_0, x) \ge 1$ for all $i \in N \cup \{0\}$ whenever $\lim_{n\to\infty} T^i x_0 = x$. Then T has a fixed point of X.*

Based on the same idea, we give some generalizations of the previous concepts of fuzzy contractive mapping in the setting of fuzzy metric spaces. We are extended the results of Seong-Hoon Cho [15] in fuzzy metric space with use the results of Arora and Kumar [12]. The presented theorems extend, generalize and improve many results in literature specially the Banach contraction principle.
The main purpose of this paper is to prove a new type fixed point theorems for $\alpha - \psi -$contractive type mappings and initiate some examples in complete fuzzy metric spaces.

2 Preliminaries

Definition 2.1(Schweizer and Sklar [14]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions **[B.1]** $*$ is commutative and associative
[B.2] $*$ is continuous
[B.3] $a * 1 = a \;\; \forall \; a \in [0,1]$
[B.4] $a * b \leq c * d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.2 (George and Veeramani [1]) The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy metric in $X^2 \times [0,\infty] \to [0,1]$, satisfying the following conditions: for all $x, y, z \in X$, and $t, s > 0$.
[FM.1] $M(x, y, 0) = 0$
[FM.2] $M\,(x, y, t) = 1 \;\forall\;\; t > 0$ if and only if $x = y$.
[FM.3] $M(x, y, t) = M(y, x, t)$
[FM.4] $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
[FM.5] $M(x, y, \bullet) : [0, \infty] \to [0,1]$, is left continuous
[FM.6] $\lim\limits_{t\to\infty} M(x, y, t) = 1.$

Definition 2.3 (George and Veeramani [1]) Let $(X, M, *)$ be a fuzzy metric space and let a sequence $\{x_n\}$ in X is said to be converge to $x \in X$ if $\lim\limits_{n\to\infty} M(x_n, x, t) = 1$, for each $t > 0$.

Definition 2.4 (George and Veeramani [1]) A sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim\limits_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for each $t > 0$, and $p = 1, 2, 3, \cdots$

Definition 2.5 (George and Veeramani [1]) A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in X is convergent in X.

A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.

Definition 2.6 (George and Veeramani [1]) A self mapping $T : X \to X$ is called fuzzy contractive mapping if $M(Tx, Ty, t) > M(x, y, t)$ for each $x \neq y \in X$ and $t > 0$.

Let Ψ the family of functions $\psi : [0, \infty) \to [0,1]$ such that $\sum_{n=1}^{\infty} \psi^n(t) = 1$ for each $t > 0,$ where ψ^n is the n^{th} iteration of ψ.

Lemma 2.1. *For every function $\psi : [0,\infty) \to [0,1]$ the following hold: if ψ is decreasing, then for each $t > 0$, $\lim\limits_{n\to\infty} \psi^n(t) = 1$ implies $\psi(t) > t$.*

Definition 2.7 (Samet, Vetro and Vetro [3]) Let(X, d) be a metric space and $T : X \to X$ be a given mapping. We say that $T$is an $\alpha - \psi -$contractive mapping if there exists two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 2.8 (Arora and Kumar [12])Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi -$contractive mapping if there exists two functions

$\alpha : X\times X\times[0,\infty)\to[0,1]$ and $\psi\in\Psi$ such that

$$\alpha(x,y,t)M(Tx,Ty,t)\geq\psi(M(x,y,t))$$

for all $x,y\in X$.

Remark 2.2. If $T : X\to X$ satisfies the Banach contraction principle, then T is an $\alpha-\psi-$contractive mapping, where $\alpha(x,y,t)=1$ for all $x,y\in X$ and $\psi(t)=kt$ for all $t\geq 0$ and some $k\in[0,1]$.

Definition 2.9 (Samet, Vetro and Vetro [3]) Let $T : X\to X$ and $\alpha : X\times X\to[0,+\infty)$, we say that T is $\alpha-$admissible if

$$x,y\in X,\ \ \alpha(x,y)\geq 1\ \ \Rightarrow\alpha(Tx,Ty)\geq 1.$$

Definition 2.10 (Arora and Kumar [12]) Let $T : X\to X$ and $\alpha : X\times X\times[0,\infty)\to[0,1]$, we say that $T$is $\alpha-$admissible if

$$x,y\in X,\ \ \alpha(x,y,t)\leq 1\ \ \Rightarrow\alpha(Tx,Ty,t)\leq 1.$$

3 Main Results

Theorem 3.1. *Let $(X,M,*)$ be a complete fuzzy metric space $\alpha\in\Lambda$, and let $\psi\in\Psi$. Suppose that a mapping $T : X\to X$ satisfies*

$$\alpha(x,y,t)M(Tx,Ty,t)\geq\psi(m(x,y,t)) \tag{3.1}$$

for all $x,y\in X$. Assume that there exists $x_0\in X$ such that

$$\alpha(T^ix_0,T^jx_0,t)\leq 1 \tag{3.2}$$

for all $i,j\in N\cup\{0\}$ with $i\neq j$.

$$Suppose\ that\ either\ T\ is\ continuous \tag{3.3}$$

or

$$\text{lim sup}\,\alpha(T^nx_0,x,t)\leq 1 \tag{3.4}$$

for any cluster point x of $\{T^nx_0\}$. *Then T has a fixed point in X.*

Proof. Let $x_0\in X$ such that $\alpha(T^ix_0,T^jx_0,t)\leq 1$ for all $i,j\in N\cup\{0\}$ with $i\neq j$. Define a sequence $\{x_n\}\subset X$ by $x_{n+1}=Tx_n$ for all $n\in N\cup\{0\}$.
If $x_n=x_{n+1}$ for some $n\in N\cup\{0\}$. By assumption, $\alpha(x_i,x_j,t)\leq 1$ for all $i,j\in N$ with $i\neq j$.
From (3.1) with $x=x_{n-1}$ and $y=x_n$ we obtain

$$M(x_n,x_{n+1},t)=M(Tx_{n-1},Tx_n,t)\geq\alpha(x_{n-1},x_n,t)M(Tx_{n-1},Tx_n,t)\geq\psi(m(x_{n-1},x_n,t)) \tag{3.5}$$

where $m(x_{n-1},x_n,t)=\max\{M(x_n,x_{n-1},t),\ M(x_n,Tx_n,t),\ M(x_{n-1},Tx_{n-1},t),\ \frac{1}{2}[M(x_n,Tx_{n-1},t)* M(x_{n-1},Tx_n,t)]\}$
we have
$m(x_{n-1},x_n,t)=\max\{M(x_n,x_{n-1},t),\ M(x_n,x_{n+1},t),\ \frac{1}{2}M(x_{n-1},x_{n+1},t)\}$
$\geq\max\{M(x_n,x_{n-1},t),\ M(x_n,x_{n+1},t),\ \frac{1}{2}[M(x_{n-1},x_n,\frac{t}{2})*M(x_n,x_{n+1},\frac{t}{2})]\}$
$=\max\{M(x_n,x_{n-1},t),\ M(x_n,x_{n+1},t)\}$.
Hence from (3.5) we have $M(x_n,x_{n+1},t)\geq\psi(\max\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\})$, because ψ is decreasing. If $\max\{M(x_{n-1},x_n,t),\ M(x_n,x_{n+1},t)\}=M(x_n,x_{n+1},t)$, then $M(x_n,x_{n+1},t)\geq$

$\psi(M(x_n, x_{n+1}, t)) > M(x_n, x_{n+1}, t)$, which is a contradiction. Thus, $\max\{M(x_{n-1}, x_n, t),\ M(x_n, x_{n+1}, t)\} = M(x_{n-1}, x_n, t)$, and so

$$M(x_n, x_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t)) \tag{3.6}$$

for all $n \in N$. Hence we have
$M(x_n, x_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t)) * \psi^2(M(x_{n-2}, x_{n-1}, t)) * \cdots * \psi^n(M(x_0, x_1, t))$, for all $n \in N$.

Thus we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1.$$

We now show that $\{x_n\}$ is a Cauchy sequence. Now

$$M(x_{n+1}, x_{n+2}, t) = M(Tx_n, Tx_{n+1}, t) \geq \alpha(x_n, x_{n+1}, t)M(Tx_n, Tx_{n+1}, t) \geq \psi(m(x_n, x_{n+1}, t)) \tag{3.7}$$

where $m(x_n, x_{n+1}, t) = \max\{M(x_n, x_{n+1}, t),\ M(x_n, Tx_n, t),\ M(x_{n+1}, Tx_{n+1}, t),$
$\frac{1}{2}[M(x_n, Tx_{n+1}, t) * M(x_{n+1}, Tx_n, t)]\}$
we have

$$\begin{aligned} m(x_n, x_{n+1}, t) &= \max\{M(x_n, x_{n+1}, t),\ M(x_n, x_{n+1}, t),\ M(x_{n+1}, x_{n+2}, t),\ \tfrac{1}{2}M(x_n, x_{n+2}, t)\} \\ &\geq \max\{M(x_n, x_{n+1}, t),\ M(x_{n+1}, x_{n+2}, t),\ \tfrac{1}{2}[M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t)]\} \\ &= \max\{M(x_n, x_{n+1}, t),\ M(x_{n+1}, x_{n+2}, t)\}. \end{aligned}$$

Hence from (3.7) we have
$M(x_{n+1}, x_{n+2}, t) \geq \psi(\max\{M(x_n, x_{n+1}, t),\ M(x_{n+1}, x_{n+2}, t)\})$, because ψ is decreasing.
If $\max\{M(x_n, x_{n+1}, t),\ M(x_{n+1}, x_{n+2}, t)\} = M(x_{n+1}, x_{n+2}, t)$ then $M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_{n+1}, x_{n+2}, t)) > M(x_{n+1}, x_{n+2}, t)$, which is a contradiction. Thus, $\max\{M(x_n, x_{n+1}, t),$ $M(x_{n+1}, x_{n+2}, t)\} = M(x_n, x_{n+1}, t)$, and so

$$M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t)) \geq \psi^2 M(x_{n-1}, x_n, t) \tag{3.8}$$

for all $n \in N$. Hence similarly

$$M(x_{n+2}, x_{n+3}, t) \geq \psi(M(x_{n+1}, x_{n+2}, t)) \geq \psi(\psi^2 M(x_{n-1}, x_n, t)) \geq \psi^3(M(x_{n-1}, x_n, t)) \tag{3.9}$$

and
$M(x_{n+3}, x_{n+4}, t) \geq \psi(M(x_{n+2}, x_{n+3}, t)) \geq \psi(\psi^3(M(x_{n-1}, x_n, t)) \geq \psi^4(M(x_{n-1}, x_n, t))$

$$M(x_{n+p-1}, x_{n+p}, t) \geq \psi(M(x_{n+p-2}, x_{n+p-1}, t))\psi(\psi^{p-1}(M(x_{n-1}, x_n, t)) \geq \psi^p(M(x_{n-1}, x_n, t)) \tag{3.10}$$

for all $n \in N$. Hence we have
$M(x_n, x_{n+p}) \geq \psi(M(x_0, x_1, t)) * \psi^2(M(x_0, x_1, t)) * \cdots * \psi^{n+p}(M(x_0, x_1, t))$
for all $n \in N$. Thus we have

$$\lim_{n\to\infty} M(x_n, x_{n+p}) = 1, \text{ for all } n \in N \text{ and } p > 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x_* \in X$ such that $x_* = \lim_{n\to\infty} x_n$. If T is continuous, then $\lim_{n\to\infty} x_n = Tx_*$. So x_* is a fixed point.
Assume that $\lim\sup \alpha(T^n x_0, x, t) \leq 1$ for any cluster point $x$of$\{T^n x\}$.Then$\lim\sup \alpha(x_n, x_*, t) \leq 1$. Hence there exists a sub-sequence $\{x_{n_{(k)}}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} \alpha(x_{n_{(k)}}, x_*, t) \leq 1$. Thus we have

$$M(x_*, Tx_*, t) = \lim_{k\to\infty} M(x_{n_{(k)}}, Tx_*, t) \geq \lim_{k\to\infty} \alpha(x_{n_{(k)}}, x_*, t)M(Tx_{n_{(k)}}, Tx_*, t) \geq \lim_{k\to\infty} \psi(m(x_{n_{(k)}}, x_*, t)) \tag{3.11}$$

where
$m(x_{n_{(k)}}, x_*, t) = \max\{M(x_{n_{(k)}}, x_*, t), M(x_{n_{(k)}}, x_{n_{(k)+1}}, t), M(x_*, Tx_*, t), \frac{1}{2}[M(x_{n_{(k)}}, Tx_*, t) * M(x_*, x_{n_{(k)+1}}, t)]\}$.

Suppose that $M(x_*, Tx_*, t) = a,$ where $0 < a \le 1$ since $\lim_{n\to\infty} x_{n_{(k)}} = x_*$ such that $M(x_*, x_{n_{(k)}}, t) \ge \frac{a}{2}$ for all $n \in N$. Then we have

$$\begin{aligned} M(x_{n_{(k)}}, x_{n_{(k)+1}}, t) &\ge M(x_*, x_{n_{(k)}}, \tfrac{t}{2}) * M(x_*, x_{n_{(k)+1}}, \tfrac{t}{2}) \\ &\ge \tfrac{1}{2}[M(x_{n_{(k)}}, Tx_*, \tfrac{t}{2}) * M(x_*, x_{n_{(k)+1}}, \tfrac{t}{2})] \\ &\ge \tfrac{1}{2}[\tfrac{a}{2} * M(x_{n_{(k)}}, x_*, \tfrac{t}{4}) * M(x_*, Tx_*, \tfrac{t}{4})] \\ &\ge \tfrac{1}{2}[\tfrac{a}{2} * \tfrac{a}{2} * a] = a. \end{aligned}$$

Thus, we obtain $m(x_{n_{(k)}}, x_*, t) = M(x_*, Tx_*, t)$ for all $n \in N$, and solim $\psi(m(x_{n_{(k)}}, x_*, t)) = \psi(M(x_*, Tx_*, t))$.
Hence from (3.11)we have
$M(x_*, Tx_*, t) \ge \lim \psi(m(x_{n_{(k)}}, x_*, t)) = \psi(M(x_*, Tx_*, t)) \ge M(x_*, Tx_*, t)$, which is a contradiction. Thus $M(x_*, Tx_*, t) = 1$, and so x_* is a fixed point of T. □

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy metric space,$\alpha \in \Lambda$ and let $\psi : [0, \infty) \to [0, 1]$ be an upper semi-continuous with $\psi(t) \ge t$ for all $t > 0$. Suppose that a mapping $T : X \to X$ satisfies*
$\alpha(x, y, t)M(Tx, Ty, t) \ge \psi(\max\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\})$
for all $x, y \in X$. Assume that (3.2) is satisfied. If either T is continuous or (3.4) holds, then T has a fixed point in X.

Proof. By (3.2), there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0, t) \le 1$ for all $i, j \in N \cup \{0\}$ with$i \ne j$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for all $n \in N \cup \{0\}$.
If $x_n = x_{n+1}$ for some $n \in N$, then T has a fixed point. Assume that $x_n \ne x_{n+1}$ for all $n \in N \cup \{0\}$. By assumption $\alpha(x_i, x_j, t) \le 1$ for all $i, j \in N$ with $i \ne j$. Thus, we have

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Tx_{n-1}, Tx_n, t) \\ &\ge \alpha(x_{n-1}, x_n, t)M(Tx_{n-1}, Tx_n, t) \\ &\ge \psi(\max\{M(x_{n-1}, x_n, t),\ M(x_{n-1}, x_n, t),\ M(x_n, x_{n+1}, t)\}) \\ &= \psi(\max\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}) \end{aligned}$$

for all $n \in N$.
Suppose that $M(x_n, x_{n+1}, t) \ge \psi(M(x_{n-1}, x_n, t))$. Then we have $M(x_n, x_{n+1}, t) \ge \psi(M(x_n, x_{n+1}, t))$, which is contradiction. Thus we obtain $M(x_n, x_{n+1}, t) \ge \psi(\max\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}) = M(x_{n-1}, x_n, t)$ for all $n \in N$.
By induction, we obtain
$M(x_n, x_{n+1}, t) \ge \psi^n(M(x_0, x_1, t))$ for all $n \in N$. Thus we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \tag{3.12}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Thus for any positive integer P and (3.12) we have

$$\begin{aligned} M(x_n, x_{n+P}, t) &\ge M(x_n, x_{n+1}, \tfrac{t}{P}) * M(x_{n+1}, x_{n+2}, \tfrac{t}{P}) * \cdots * M(x_{n+P-1}, x_{n+P}, \tfrac{t}{P}) \\ &\ge \psi^n(M(x_0, x_1, \tfrac{t}{P})) * \cdots * \psi^{n+P}(M(x_0, x_1, \tfrac{t}{P})) \end{aligned}$$

by (3.12)

$$\lim_{n\to\infty} M(x_n, x_{n+P}, t) \ge 1 * \cdots * 1 = 1$$

i.e. $\{x_n\}$ is a Cauchy sequence, hence convergent. It follows from the completeness of X that there exists $x_* = \lim_{n\to\infty} x_n \in X$.
If T is continuous, then $\lim_{n\to\infty} x_n = Tx_*$ and so $x_* = Tx_*$.
Assume that (3.4) is satisfied.Then, lim sup $\alpha(x_n, x_*, t) \le 1$. Hence, there exists a subsequence $\{x_{n_{(k)}}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \alpha(x_{n_{(k)}}, x_*, t) \le 1$. We have

$$\lim_{k\to\infty} \max\{M(x_{n_{(k)}}, x_*, t),\ M(x_{n_{(k)}}, x_{n_{(k)}+1}, t),\ M(x_*, Tx_*, t)\} = M(x_*, Tx_*, t)$$

.
By using upper semi continuity of ψ, we obtain

$$\begin{aligned} M(x_*, Tx_*, t) &= \lim_{k\to\infty} M(x_{n_{(k)}+1}, Tx_*, t) \\ &\geq \lim_{k\to\infty} \alpha(x_{n_{(k)}}, x_*, t) M(Tx_{n_{(k)}}, Tx_*, t) \\ &\geq \lim\sup \psi(\max\{M(x_{n_{(k)}}, x_*, t),\ M(x_{n_{(k)}}, x_{n_{(k)}+1}, t),\ M(x_*, Tx_*, t)\}) \\ &\geq \psi(M(x_*, Tx_*, t)). \end{aligned}$$

If $M(x_*, Tx_*, t) > 0$, then $M(x_*, Tx_*, t) \geq \psi(M(x_*, Tx_*, t)) > M(x_*, Tx_*, t)$, which is a contradiction. Hence $M(x_*, Tx_*, t) = 1$ and hence x_* is a fixed point of T. □

Example 3.1 Let $X = [0, 1]$ with the standard fuzzy metric, define $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and for all $t > 0$. Let

$$\psi(t) = \begin{cases} \frac{1}{t^2+1} \text{ if } t \in [0, 1] \\ 0, \text{ otherwise} \end{cases}$$

then $\psi \in \Psi$.
Note that ψ is continuous at $t = 1$, and $\sum_{n=1}^{\infty} \psi^n(1) = \frac{1}{2}$.
Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{x^2+1} \text{ if } x \in [0, 1] \\ 0, \text{ otherwise} \end{cases}$$

obviously, T is continuous.
We define $\alpha : X \times X \times [0, \infty) \to [0, 1]$ by

$$\alpha(x, y, t) = \begin{cases} 1 \text{ if } x, y \in [0, 1] \\ 0 \text{ if } x \notin [0, 1] \text{ or } y \notin [0, 1]. \end{cases}$$

Clearly (3.1) of Theorem 3.1 is satisfied. Condition (3.2) of Theorem 3.1 hold with $x_0 = 1$. By applying Theorem 3.1, T has a fixed point. Consider the general contractive condition:

(G) $M(Tx, Ty, t) \geq \psi(\max\{M(x, y, t),\ M(x, Tx, t),\ M(y, Ty, t), \frac{1}{2}[M(x, Ty, t)*M(y, Tx, t)]\})$
for all $x, y \in X$.
It is well known that if T is a self mapping of a complete fuzzy metric space and T satisfied that general contractive condition (G), then Thas a fixed point.

Example 3.2 In Example (3.1), let

$$Tx = \begin{cases} \frac{1}{x^3+1} \text{ if } x \in [0, 1] \\ 0, \text{ otherwise.} \end{cases}$$

Then it is easy to see that condition (3.1) of Theorem (3.1) holds. Obviously condition (3.2) of Theorem (3.1) is satisfied with $x_0 = 1$. For all $n \in N \cup \{0\}$, $T^n x_0 = T^n 1 = \frac{1}{2^n} \in [0, 1]$. Hence (3.4) of Theorem 3.1 is satisfied. By applying Theorem 3.1, T has a fixed point.

Remark 3.3. Let $(X, M, *)$ be a fuzzy metric space and let $\alpha \in \Lambda$. Consider the following condition:

(i) For each $x, y, z \in X$, $\alpha(x, y, t) \leq 1$ and $\alpha(y, z, t) \leq 1$ implies $\alpha(x, z, t) \leq 1$;

(ii) For each $x, y \in X$, $\alpha(x, y, t) \leq 1$ implies $\alpha(Tx, Ty, t) \leq 1$;

(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \leq 1$;

(iv) If $\{x_n\}$is a sequence with $\alpha(x_n, x_{n+1}, t) \leq 1$ for all $n \in N \cup \{0\}$ and $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x, t) \leq 1$ for all $n \in N \cup \{0\}$;

(v) There exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0, t) \leq 1$ for all $i, j \in N \cup \{0\}$ with $i < j$;

(vi) lim sup $\alpha(T^n x_0, x, t) \leq 1$ for all cluster point x of $\{T^n x_0\}$.

Then conditions (i), (ii) and (iii) implies (v), and condition (iv) implies (vi).

Remark 3.4. If we replace condition (3.2) of Theorem 3.1 with above conditions (i), (ii) and (iii) and replace condition (3.4) of Theorem 3.1 with above condition (iv), then T has a fixed point.

Corollary 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$. Suppose that a mapping $T : X \to X$ satisfies*
$\alpha(x, y, t)M(Tx, Ty, t) \geq \psi(M(x, y, t))$
for all $x, y \in X$. Suppose that conditions (i)-(iii) of remark (3.3) are satisfied. Assume that either T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \leq 1$ for all $n \in N$ and $\lim_{n\to\infty} x_n = x$ then $\alpha(x_n, x, t) \leq 1$ *for all $n \in N$.*
Then T has a fixed point in X. Further if for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z, t) \leq 1$ and $\alpha(y, z, t) \leq 1$, *then T has a unique fixed point.*

Proof. By remark (3.4) T has a fixed point. Let $y_* \in X$ be another fixed point of T. Then by assumption, there exists $z \in X$ such that $\alpha(x_*, z, t) \leq 1$ and $\alpha(y_*, z, t) \leq 1$. From (ii) of remark (3.3) we obtain $\alpha(x_*, T^n z, t) \leq 1$ and $\alpha(y_*, T^n z, t) \leq 1$ for all $n \in N$. Hence we have
$M(x_*, T^n z, t) \geq \alpha(x_*, T^{n-1}z, t)M(Tx_*, TT^{n-1}z, t) \geq \psi(M(x_*, T^{n-1}z, t))$ for all $n \in N$.
This implies that $M(x_*, T^n z, t) \geq \psi^n(M(x_*, z, t))$ for all $n \in N$. Thus we have $\lim_{n\to\infty} T^n z = x_*$.
Similarly, we can show that $\lim_{n\to\infty} T^n z = y_*$. Thus we have $x_* = y_*$. □

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