On some new congruences for \(\ell\)-regular overpartitions

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Abstract. Andrews investigated the function \(C_{k,j}(n)\) which counts the number of overpartitions of \(n\) in which no part is divisible by \(k\) and only parts \(\equiv \pm j \pmod{k}\) may be overlined. Let \(\mathcal{A}_r(n)\) denote the number of \(\ell\)-regular overpartitions of \(n\). Very recently, Mahadeva Naika and Gireesh discovered some congruences for \(C_{3,1}(n)\) modulo 24/31 for some values of \(i\) and \(j\) and modulo 24 for \(\mathcal{A}_r(n)\). Furthermore, they conjectured that \(C_{3,1}(2n + 11) \equiv 0 \pmod{144}\). In this paper, we confirm this conjecture. We also establish several congruences for \(\mathcal{A}_r(n)\) and \(\mathcal{A}_r(n), r \geq 2\) modulo 24/31 for few values of \(i\) and \(j\).

1 Introduction

A partition of a positive integer \(n\) is a finite non-increasing sequence of positive integers \(\lambda_1, \lambda_2, \ldots, \lambda_t\) such that \(\sum_{i=1}^{t} \lambda_i = n\). The \(\lambda_i\) are called the parts of the partition. We shall set \(p(0) = 1\) and for \(n \geq 1\), let \(p(n)\) denote the number of partitions of \(n\). The generating function for \(p(n)\) is given by

\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{1-q}.
\]

Here and throughout this paper, we assume that \(|q| < 1\) and for any positive integer \(k\), \(f_k\) is defined by

\[
f_k := \prod_{n=1}^{\infty} (1-q^{kn}).
\]

In 1919, Ramanujan [16] found nice congruence properties for \(p(n)\) moduli 5, 7 and 11. Namely, for any nonnegative integer \(n\),

\[
p(5n + 4) \equiv 0 \pmod{5},
p(7n + 5) \equiv 0 \pmod{7},
p(11n + 6) \equiv 0 \pmod{11}.
\]

Motivated by the above congruences, many mathematicians discovered many congruence properties for different partition functions such as singular overpartitions, \(\ell\)-regular partitions, broken \(\ell\)-diamond partitions and \(\ell\)-regular overpartitions. Among these, arithmetic properties of \(\ell\)-regular overpartitions has received a great deal of attention. For a positive integer \(\ell \geq 2\), a partition is called \(\ell\)-regular if none of its parts is divisible by \(\ell\). An overpartition of \(n\) is a non-increasing sequence of natural numbers whose sum is \(n\) in which the final occurrence of a part may be overlined.

In [13], Lovejoy proved the following theorem in the theory of overpartitions.

**Theorem 1.1.** ([13]) If \(\mathcal{A}_r(n)\) denote the number of overpartitions of \(n\) of the form \(y_1 + y_2 + \cdots + y_s\), where \(y_j - y_{j+\ell-1} = 1\) if \(y_j, y_{j+\ell-1}\) is overlined and \(y_j - y_{j+\ell-1} \geq 2\) otherwise. Let \(\mathcal{A}_r(0) = 1\)
and for \( n \geq 1 \), let \( \overline{A}_\ell(n) \) denote number of overpartitions of \( n \) with no parts divisible by \( \ell \). Then \( \overline{A}_\ell(n) = B_\ell(n) \).

The generating function for \( \overline{A}_\ell(n) \) is given by [18]

\[
\sum_{n=0}^{\infty} \overline{A}_\ell(n) q^n = \frac{f_\ell f_{\ell^2}}{f_1 f_{\ell^2}}.
\] (1.1)

Setting \( \ell = 3 \) in (1.1), Shen [18] observed that \( \overline{A}_3(n) = B_{3,1}(n) \), where \( B_{k,j}(n) \) counts the number of overpartitions of \( n \) in which no part is divisible by \( k \) and only parts \( \equiv \pm j \pmod{k} \) may be overlined. This function was introduced and investigated by Andrews in [3]. As noted in [3], the generating function for \( B_{k,j}(n) \) is given by

\[
\sum_{n=0}^{\infty} B_{k,j}(n) q^n = \frac{(q^k; q^k)_\infty (-q^j; q^k)_\infty (-q^{k-j}; q^k)_\infty}{(q; q)_\infty}.
\] (1.2)

where \( k \geq 3 \) and \( 1 \leq i \leq \left\lfloor \frac{n}{k} \right\rfloor \). Using generating function dissection techniques, Shen [18] established several interesting congruences modulo 2, 6, 24 for \( \overline{A}_3(n) \) and modulo 3, 24 for \( \overline{A}_3(n) \). For example

**Theorem 1.2.** ([18]) For all non-negative integer \( n \),

\[
\overline{A}_3(9n + 3) \equiv 0 \pmod{6},
\]

\[
\overline{A}_3(9n + 6) \equiv 0 \pmod{24},
\]

\[
\overline{A}_3(12n + 8) \equiv 0 \pmod{3},
\]

\[
\overline{A}_3(12n + 7) \equiv 0 \pmod{24}.
\]

In the same paper, Shen gave a combinatorial interpretation of first two congruences in the above theorem by introducing the rank of vector partitions. Very recently, Mahadeva Naika and Gireesh [14] employed dissection formulas of certain quotients of theta functions to establish several infinite families of congruences for \( B_{k,j}(n) \) for different values of \( k \) and \( j \). They also considered the function \( \overline{A}_5(n) \) and proved some congruences modulo 16. For example, they proved the following theorems:

**Theorem 1.3.** ([14]) For all integers \( n \geq 0 \), we have

\[
\overline{A}_5(8n + 7) \equiv 0 \pmod{12},
\]

\[
\overline{A}_5(8n + 6) \equiv 0 \pmod{24},
\]

\[
\overline{A}_5(24n + 14) \equiv 0 \pmod{72}.
\]

**Theorem 1.4.** ([14]) Let \( p \geq 5 \) be prime and \( \left( \frac{-2}{p} \right) = -1 \). Then for all integers \( n \geq 0 \), \( \alpha \geq 1 \) and \( 1 \leq j \leq p - 1 \), we have

\[
\overline{A}_5(8p^{2\alpha}n + p^{2\alpha-1}(3p + 8j)) \equiv 0 \pmod{2^4}.
\]

In the same paper, they also proposed the following conjecture for \( \overline{A}_{3,1}(n) \).

**Conjecture 1.5.** [14] For all integer \( n \geq 0 \),

\[
\overline{A}_{3,1}(12n + 11) \equiv 0 \pmod{144}.
\]

Alanazi, Munagi and Sellers [2] established several Ramanujan type congruences for \( \ell \)-regular overpartitions. In particular, Alanazi et al. [2] discovered the following theorem.

**Theorem 1.6.** ([2]) For all \( n \geq 0 \), we have \( \overline{A}_5(6n + 5) \equiv 0 \pmod{3} \).
The following theorem was proved by Alanazi et al. [2] using a congruence relation due to Munagi and Sellers [15].

**Theorem 1.7.** ([2]) *For all* \( n \geq 0 \) *and all* \( j \geq 3 \), *we have* \( A_{3j}(27n + 18) \equiv 0 \pmod{3} \).

The main aim of this paper is to show that Conjecture 1.5 is true and also to prove some new congruences for \( A_5(n) \) and \( A_{35}(n) \). The paper is organized as follows: In Section 2, we recall some notations, definitions and also we collect some lemmas and theorems which are useful to prove our main results. In Section 3, we give a simple proof of Conjecture 1.5 and also establish a \( p \)-dissection formula for \( f_1^p / f_2^p \) which seems to be new. In Section 4, we derive some new congruences modulo 8 and 16 for \( A_{16}(n) \). In Section 5, we discover several infinite families of congruences modulo 6, 8 and 16 for \( A_6(n) \). We also deduce Theorem 1.6 as a special case of one of our theorems. In Section 6, we prove infinite families of congruences for \( A_{35}(n) \), \( r \geq 2 \) modulo 3, 4, 8 and 16. We also provide a short and simple proof of the Theorem 1.7.

## 2 Set of preliminary results

In this section, we present some identities which are useful to prove our main results.

Let \( p \geq 3 \) be a prime. The Legendre symbol \( \left( \frac{a}{p} \right) \) is defined by

\[
\left( \frac{a}{p} \right) := \begin{cases} 
1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\
-1 & \text{if } a \text{ is a quadratic nonresidue modulo } p \text{ and } p \nmid a, \\
0 & \text{if } p \mid a.
\end{cases}
\]

For \( |ab| < 1 \), Ramanujan’s general theta function \( f(a, b) \) is defined by [1]

\[
f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{(n-1)/2}.
\]

The following lemma is a consequence of Entry 25 of (i), (ii), (v) and (vi) in [1, pp. 35–36].

**Lemma 2.1.** *The following 2-dissection formulas are true:*

\[
\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_6^8} + 2q \frac{f_2^6 f_6^8}{f_2^5 f_8}
\]  

(2.1)

and

\[
\frac{1}{f_4^2} = \frac{f_8^{14}}{f_2^{18} f_4^{9}} + 4q^2 \frac{f_2^8 f_6^8}{f_2^8 f_8^9}.
\]

(2.2)

The following 2-dissection formula for \( \frac{f_2^2}{f_1} \) was proved by Hirschhorn, Garvan and Borwein [9] and also by Xia and Yao [19].

**Lemma 2.2.** *The following 2-dissection formulas are true:*

\[
\frac{f_3^2}{f_1} = \frac{f_6^3 f_6^6}{f_2^9 f_1^2} + q \frac{f_6^3}{f_2^2}
\]

(2.3)

and

\[
\frac{f_3^2}{f_1} = \frac{f_6^4 f_6^6}{f_2^9 f_6^2 f_4} + 2q^2 \frac{f_6^4 f_6^6}{f_2^9 f_8 f_4}.
\]

(2.4)

For a proof of (2.4), see [5] and [19].

From [8], we recall the following lemma.

**Lemma 2.3.** *The following 3-dissection formula holds:*

\[
\frac{f_2}{f_1} = \frac{f_6^3 f_6^6}{f_3^9 f_1^4} + 2q \frac{f_6^3 f_6^6}{f_3^9 f_3} + 4q^2 \frac{f_6^2 f_6^6}{f_3 f_3^2}.
\]
From [1, p.49], we recall the following $p$-dissection formula.

**Lemma 2.4.** For any prime $p$, we have
\[
\frac{f_2^5}{f_1^3 f_2^4} = \frac{f_2^5}{f_p^2 f_{4p}^2} + \sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}).
\]

**Theorem 2.5.** ([7, Theorem 2.1]) For any odd prime $p$,
\[
\frac{f_2^5}{f_1^3} = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{p^2+2m+1}{4}} f\left(q^{p(p-2m-1)/4}, q^{p(p-2m+1)/4}\right) + q^{\frac{p-1}{2}} \frac{f_{2p}^2}{f_p^2}.
\]
Furthermore, $\frac{m^2+m}{4} \not\equiv \frac{p-1}{8} \pmod p$ for $0 \leq m \leq \frac{p-3}{2}$.

For all integers $n, k \geq 0$, let $t_k(n)$ (respectively $r_k(n)$) denote the number of representations of $n$ as sum of $k$ triangular (respectively square) numbers.

**Theorem 2.6.** For $1 \leq k \leq 7$, we have
\[
\text{For all integers } n \geq 0, \text{ we have }
\]
\[
t_k(8n+k) = 2^{k-1} \left(2 + \binom{k}{4}\right) t_k(n).
\]

In [12], Hirschhorn and Sellers proved the following arithmetic identity for $a_3(n)$.

**Theorem 2.7.** Let $p \equiv 2 \pmod 3$. For all integers $n \geq 0$, we have
\[
a_3\left(p^{2\alpha} n + \frac{p^{2\alpha}-1}{3}\right) = a_3(n),
\]
where $a_3(n)$ denote the number of $3$-core partitions of $n$.

## 3 Proof of Conjecture 1.5 and a $p$-dissection formula

In this section, we give a simple proof of Conjecture 1.5 and also establish a $p$-dissection formula for $f_1^3 / f_2^5$ which will be used to prove congruence properties for $\mathcal{T}_5(n)$ and $\mathcal{T}_6(n)$.

**Theorem 3.1.** Conjecture 1.5 is true.

**Proof.** On using Lemma 2.3, Yao [20] proved that
\[
\sum_{n=0}^{\infty} c_{5,1}(6n+5)q^n = 16 \frac{f_2^5 f_3^4 f_4^4}{f_1^4}. \tag{3.1}
\]

By the binomial theorem, it is easy to check that, for all positive integers $k$ and $m$,
\[
f_k^{3m} \equiv f_k^{3m} \pmod 3, \tag{3.2}
f_k^{9m} \equiv f_k^{9m} \pmod 3. \tag{3.3}
\]

In view of congruence (3.3), we have
\[
\sum_{n=0}^{\infty} c_{5,1}(6n+5)q^n = 16 \frac{f_2^5 f_3^4 f_4^4}{f_1^4} \equiv 16 f_2^5 f_4^4 \pmod {144}. \tag{3.4}
\]

Now, comparing the odd powers of $q$ in (3.4), we obtain the required congruence. \qed

**Theorem 3.2.** Let $p \geq 5$ be a prime. Then
\[
\frac{f_2^5}{f_1^3} = \sum_{k=-\frac{p-1}{8} \leq \frac{p-1}{2}} q^{\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty} (6pn + 6k + 1)q^{p^2 n^2 + p/p_k[6k+1]} \pm pq^{\frac{p-1}{2}} \frac{f_{2p}^2}{f_2^2}. \tag{3.4}
\]

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, $k \neq \frac{p+1}{6}$, we have $\frac{3k^2+k}{2} \not\equiv \frac{p-1}{24} \pmod p$. 

Proof. From [6, Corollary 1.3.21], we recall that
\[
\frac{f_1^5}{f_2^5} = \sum_{n=-\infty}^{\infty} (6n + 1)q^{\frac{3n^2 + n}{2}}.
\]
Dissecting the right side into \( p \) terms, we find that
\[
\frac{f_1^5}{f_2^5} = \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty} (6(pn + k) + 1)q^{\frac{3(pn + k)^2 + (pn + k)}{2}}
\]
\[
= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} q^{\frac{3k^2 + k}{2}} \sum_{n=-\infty}^{\infty} (6pn + 6k + 1)q^{\frac{pn(2pn + 6k + 1)}{2}} \pm q^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty} p(6n + 1)q^{\frac{p^2(2n^2 + n)}{2}}
\]
\[
= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}, k \neq \pm \frac{p}{2}} q^{\frac{3k^2 + k}{2}} \sum_{n=-\infty}^{\infty} (6pn + 6k + 1)q^{\frac{pn(2pn + 6k + 1)}{2}} \pm pq^{\frac{p-1}{2}} \frac{f_2^5}{f_2^{2p^2}}.
\]
If \( \frac{3k^2 + k}{6} = \frac{p^2 - 1}{2^4} \) (mod \( p \)), which implies that \((6k + 1)^2 \equiv 0 \) (mod \( p \)). This implies that \( k = mp - 1 \) for some integer \( m \). Since \( -\frac{p-1}{2} \leq k \leq \frac{p-1}{2} \), we have \( m = \pm 1 \). Thus \( k = \pm \frac{p-1}{2} \) which is a contradiction. \( \square \)

4 Congruences for \( \mathcal{A}_5(n) \) modulo powers of 2

In this section, we prove infinite families of congruences modulo \( 2^3 \) and \( 2^4 \) for \( \mathcal{A}_5(n) \).

Theorem 4.1. If \( p \geq 5 \) is a prime such that \( \left( \frac{-2}{p} \right) = -1 \) and \( 1 \leq j \leq p - 1 \), then for all non-negative integers \( n \) and \( \alpha \), we have
\[
\mathcal{A}_5\left(4p^{2\alpha + 2}n + 4p^{2\alpha + 1}j + p^{2\alpha + 2}\right) \equiv 0 \pmod{2^3}, \tag{4.1}
\]
\[
\mathcal{A}_5\left(4 \cdot 5^{\alpha + 1}n + 13 \cdot 5^\alpha\right) \equiv 0 \pmod{2^3}, \tag{4.2}
\]
\[
\mathcal{A}_5\left(4 \cdot 5^{\alpha + 1}n + 17 \cdot 5^\alpha\right) \equiv 0 \pmod{2^3}. \tag{4.3}
\]

Proof. In [14], Mahadeva Naika and Gireesh showed that
\[
\sum_{n=0}^{\infty} \mathcal{A}_5(2n + 1)q^n = 8q^{\frac{f_{10}f_{14}^2}{f_2^4}} + 2q^{\frac{f_{10}f_{14}^{14}}{f_2^4 f_8^4}}. \tag{4.4}
\]
Extracting the even powers of \( q \) in (4.4), we obtain
\[
\sum_{n=0}^{\infty} \mathcal{A}_5(4n + 1)q^n = 2q^{\frac{f_{14}f_4}{f_2^{11} f_4^2}}. \tag{4.5}
\]
By the binomial theorem, for any positive integers \( m \) and \( k \), we have
\[
f_{2m}^k \equiv f_k^m \pmod{2}, \tag{4.6}
\]
\[
f_{2m}^k \equiv f_{2k}^m \pmod{2^2}. \tag{4.7}
\]
From (4.5) and (4.7), we find that
\[
\sum_{n=0}^{\infty} \mathcal{A}_5(4n + 1)q^n \equiv 2 \frac{f_5^5}{f_2^2 f_10} \pmod{2^3}. \tag{4.8}
\]
Define
\[
\sum_{n=0}^{\infty} a(n)q^n = \frac{f_5^5}{f_2^2 f_10} f_2^5 f_5 \tag{4.9}
\]
Then, in view of (4.8) and (4.9), we have
\[
\mathcal{A}_5(4n + 1) \equiv 2a(n) \pmod{2^3}. \tag{4.10}
\]
Using Lemma 3.2, we can rewrite (4.9) as
\[
\sum_{n=0}^{\infty} a(n)q^n = \left\{ \begin{array}{ll}
\sum_{j=0}^{m-1} q^{\frac{j^2}{2}} \sum_{n=-\infty}^{\infty} (6pn + 6j + 1)q^{\frac{pn^2(pn+1)}{4} \pm p\frac{j^{p-1}}{2^p}} \\
\sum_{m=0}^{m-1} q^{3m^2+3} \sum_{n=-\infty}^{\infty} (6pn + 6m + 1)q^{\frac{pn^2(pn+1)}{4} \pm p\frac{j^{p-1}}{2^p}} \end{array} \right. \tag{4.11}
\]
Let \( p \geq 5 \) be a prime with \( \left(\frac{-5}{p}\right) = -1 \). For \( -\frac{p-1}{2} \leq j, m \leq \frac{p+1}{2} \), consider the following congruence equation
\[
\frac{3j^2 + j}{2} + 5\frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{4} \pmod{p}, \tag{4.12}
\]
which is equivalent to
\[
(6j + 1)^2 + 5(6m + 1)^2 \equiv 0 \pmod{p}.
\]
Since \( \left(\frac{-5}{p}\right) = -1 \), the above congruence holds if and only if \( j = m = \frac{\pm p-1}{6} \). So, in (4.11), extracting the terms involving \( q^{\frac{j^{p-1}}{2^p}} \) and then replacing \( q^p \) by \( q \) in the resulting congruence, we obtain
\[
\sum_{n=0}^{\infty} a\left(pn + \frac{p^2 - 1}{4}\right)q^n = (-1)^{\frac{p+1}{6}} p^2 \left(\frac{f_5^5}{f_2^2 f_10} f_2^5 f_5 \right).
\]
This implies that, for \( 1 \leq t \leq p - 1 \),
\[
a\left(p(pn + t)n + \frac{p^2 - 1}{4}\right) = 0 \tag{4.13}
\]
and
\[
\sum_{n=0}^{\infty} a\left(p^2 n + \frac{p^2 - 1}{4}\right)q^n = (-1)^{\frac{p-1}{6}} p^2 \left(\frac{f_5^5}{f_2^2 f_10} f_2^5 f_5 \right).
\]
From the above identity and (4.9), we find that
\[
a\left(p^2 n + \frac{p^2 - 1}{4}\right) = (-1)^{\frac{p+1}{6}} p^2 a(n),
\]
and by induction on $\alpha \geq 0$, we deduce
\[
a \left( \frac{p^{2\alpha + 2} n + p^{2\alpha + 1} t}{4} \right) = (\frac{p}{2})^{\alpha + 1} a(n).
\]
Replacing $n$ by $p^2 n + pt + \frac{p^2 \alpha + 1}{4} (1 \leq t \leq p - 1)$ in the above identity and then invoking (4.13), we deduce that for $\alpha \geq 0$ and $n \geq 0$,
\[
a \left( \frac{p^{2\alpha + 2} n + p^{2\alpha + 1} t + \frac{p^{2\alpha + 1} - 1}{4}}{4} \right) = 0. \tag{4.14}
\]
Replacing $n$ by $p^{2\alpha + 2} n + p^{2\alpha + 1} t + \frac{p^{2\alpha + 1} - 1}{4}$ in (4.10) and then using (4.14), we obtain (4.1).

From [1, pp.82], we recall that
\[
f_1 = f_{2S} f(-q^{15}, -q^{10}) - q^2 f(-q^{20}, -q^5) f_{2S} - q f_{2S}.
\tag{4.15}
\]
In view of (4.8), (4.15) and by induction, we find that for all non-negative integers $n$ and $\alpha$
\[
\sum_{n=0}^{\infty} \mathcal{A}_5(4 \cdot 5^\alpha n + 5^\alpha)q^n = 2(-1)^{\alpha} f_1 f_5 \quad \text{(mod $2^3$)}.
\]
Substituting (4.15) into the above congruence and then equating the coefficients of $q^{5n+3}$ and $q^{5n+4}$ in the resulting congruence, we obtain the remaining two congruences of the above theorem. \hfill \Box

**Theorem 4.2.** Let $p$ be an odd prime and $N$ be a positive integer with $p \nmid N$ such that $pN \equiv 3 \pmod{2^3}$. Let $\alpha \geq 0$ be an integer.

(1) If $p \equiv -1 \pmod{2^2}$, then $\mathcal{A}_5(p^{4\alpha+3}N) \equiv 0 \pmod{2^3}$,

(2) If $p \equiv 3, 11 \pmod{2^2}$, then $\mathcal{A}_5(p^{16\alpha+15}N) \equiv 0 \pmod{2^3}$,

(3) If $p \equiv 1, 5, 9 \pmod{2^2}$, then $\mathcal{A}_5(p^{2\alpha+31}N) \equiv 0 \pmod{2^3}$,

(4) If $p \equiv 7 \pmod{2^2}$, then $\mathcal{A}_5(p^{6\alpha+7}N) \equiv 0 \pmod{2^3}$,

(5) If $p \equiv 13 \pmod{2^2}$, then $\mathcal{A}_5(p^{6\alpha+63}N) \equiv 0 \pmod{2^3}$.

**Proof.** Hirschhorn and Sellers [11] obtained the following $2$-dissection formula:
\[
\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2 f_4} + \frac{f_2^2 f_{10} f_{40}}{f_2^2 f_8 f_{20}}. \tag{4.16}
\]
From (2.1), (4.4) and (4.16), we find that
\[
\sum_{n=0}^{\infty} \mathcal{A}_5(4n + 3)q^n = 8 f_8 f_{20}^2 + 8 q f_2^2 f_{10} f_{40} f_2^2 f_8 f_{20} \quad \text{(mod $2^4$)}.
\tag{4.17}
\]
Extracting the even powers of $q$ in (4.17) and then using (4.6), we find that
\[
\sum_{n=0}^{\infty} \mathcal{A}_5(8n + 3)q^n = 8 \frac{f_8}{f_1} = 8 \sum_{n=0}^{\infty} t_3(n)q^n \quad \text{(mod $2^4$)}.
\]
Equating the coefficients of $q^n$ on both sides of the above congruence, we obtain
\[
\mathcal{A}_5(8n + 3) \equiv 8t_3(n) \quad \text{(mod $2^4$)}.
\]
Setting $k = 3$ in Theorem 2.6, we obtain $r_3(8n + 3) = \mathcal{A}_5(8n + 3)$. Hence
\[
\mathcal{A}_5(8n + 3) \equiv r_3(8n + 3) \quad \text{(mod $2^4$)}.
\tag{4.18}
\]
Hirschhorn and Sellers [10] proved that if $p \geq 3$ is a prime and $n$ is a positive integer, then
\[
r_3(p^{2\alpha}n) = \left( \frac{p^{\alpha + 1} - 1}{p - 1} \right) r_3(n) - p \frac{p^{\alpha} - 1}{p - 1} r_3(n/p^2), \quad \alpha \geq 0. \quad \tag{4.19}
\]
Here \( \left( \frac{p}{n} \right) \) is the Legendre symbol and we take \( r_3(n/p^2) = 0 \) if \( p^2 \nmid n \).

Replacing \( n \) by \( pN \) (\( p \nmid N \)) in (4.19), we get

\[
r_3(p^{2\alpha+1}N) = \left( \frac{p^{\alpha+1} - 1}{p - 1} \right) r_3(pN). \tag{4.20}
\]

By (4.20), if \( p \equiv -1 \pmod{2^4} \), then

\[
r_3(p^{2\alpha+1}N) \equiv \begin{cases} 0 \pmod{16} & \text{if } \alpha \text{ is odd}, \\ r_3(pN) \pmod{16} & \text{if } \alpha \text{ is even}. \end{cases}
\]

The above congruence implies that \( r_3(p^{\alpha+3}N) \equiv 0 \pmod{2^4} \). Setting \( n = \frac{p^{\alpha+3}N - 3}{8} \) in (4.18), we obtain

\[
\overline{A}_5(p^{\alpha+3}N) \equiv r_3(p^{\alpha+3}N) \equiv 0 \pmod{2^4}.
\]

This completes the proof of (1).

Let \( p \equiv 3, 11 \pmod{16} \). Replacing \( \alpha \) by \( 8\alpha + 7 \) in (4.20) and using the fact that

\[
\frac{p^{8\alpha+8} - 1}{p - 1} = 1 + p + \cdots + p^{8\alpha+7} \equiv 0 \pmod{2^4},
\]

we obtain

\[
r_3(p^{16\alpha+15}N) \equiv 0 \pmod{2^4}. \tag{4.21}
\]

Putting \( n = \frac{p^{\alpha+3}N - 3}{8} \) in (4.18) and then using the above congruence, we get (2). The other statements follow in a similar way.

**Theorem 4.3.** Let \( p \geq 3 \) be a prime and \( N, \alpha \geq 1 \) are integers.

1. If \( p \equiv 7 \pmod{2^4} \), then \( \overline{A}_5(p^{8\alpha}(8N + 3)) \equiv \overline{A}_5(8N + 3) \pmod{2^4} \).
2. If \( p \equiv 1, 5, 9 \pmod{2^4} \), then \( \overline{A}_5(p^{2\alpha}(8N + 3)) \equiv \overline{A}_5(8N + 3) \pmod{2^4} \).
3. If \( p \equiv -1 \pmod{2^4} \), then \( \overline{A}_5(p^{8\alpha}(8N + 3)) \equiv \overline{A}_5(8N + 3) \pmod{2^4} \).
4. If \( p \equiv 3, 11 \pmod{2^4} \), then \( \overline{A}_5(p^{16\alpha}(8N + 3)) \equiv \overline{A}_5(8N + 3) \pmod{2^4} \).
5. If \( p \equiv 13 \pmod{2^4} \), then \( \overline{A}_5(p^{16\alpha}(8N + 3)) \equiv \overline{A}_5(8N + 3) \pmod{2^4} \).

**Proof.** We give a proof of (1). The proof of other congruences follows similarly. Replacing \( n \) by \( p^2(8N + 3) \) and \( \alpha \) by \( 4\alpha + 3 \) in (4.19), we obtain

\[
r_3(p^{8\alpha+8}(8N + 3)) = r_3(p^2(8N + 3)) \frac{p^{8\alpha+8} - 1}{p - 1} - r_3(8N + 3) p \frac{p^{8\alpha+7} - 1}{p - 1} \quad (\alpha \geq 0). \tag{4.22}
\]

If \( p \equiv 7 \pmod{16} \), then we have

\[
\frac{p^{8\alpha+8} - 1}{p - 1} = 1 + p + \cdots + p^{8\alpha+7} \equiv 0 \pmod{2^4}
\]

and

\[
p \frac{p^{8\alpha+7} - 1}{p - 1} = p + p^2 + \cdots + p^{8\alpha+6} \equiv -1 \pmod{2^4}.
\]

Using above two congruences in (4.22), we get

\[
r_3(p^{8\alpha+8}(8N + 3)) \equiv r_3(8N + 3) \pmod{2^4}. \tag{4.23}
\]

Putting \( n = \frac{p^{2\alpha+3}(8N + 3) - 3}{8} \) in (4.18) and then using (4.23) and (4.18), we get the required result.
Theorem 4.4. If \( p \geq 3 \) is a prime with \( \left( \frac{-10}{p} \right) = -1 \), then for all non-negative integers \( n \) and \( \alpha \),

\[
\overline{A}_5\left(p^{2\alpha}8n + 7p^{2\alpha}\right) \equiv 8f_2^3f_3^3 \pmod{2^4}.
\] (4.24)

Moreover, for \( 1 \leq r \leq p - 1 \),

\[
\overline{A}_5\left(p^{2\alpha+2}(8n + 7) + 8p^{2\alpha+1}r\right) \equiv 0 \pmod{2^4}.
\]

Proof. Extracting the terms involving \( q^{2n+1} \) in (4.17) and then using (4.6), we deduce that

\[
\sum_{n=0}^{\infty} \overline{A}_5(8n + 7)q^n = 8f_2^3f_3^3 \pmod{2^4}.
\] (4.25)

Thus (4.24) is true for \( \alpha = 0 \). In view of Theorem 2.5 and (4.7), we have

\[
f_3^1 \equiv \sum_{m=0}^{p-1} q^{m^2 + m} f\left(\frac{q^2 - (2m + 1)p}{2}, \frac{q^{2 - (2m + 1)p}}{2}\right) + q^{\frac{p^2}{4} - 1}f_2^3 \pmod{2^4}.
\] (4.26)

Assume that (4.24) holds for \( \alpha = j \). With the aid of (4.26), we can rewrite (4.24) with \( \alpha = j \) as

\[
\sum_{n=0}^{\infty} \overline{A}_5(p^{2j}8n + 7p^{2j})q^n = 8\sum_{m=0}^{p-1} q^{m^2 + m} f\left(\frac{q^2 - (2m + 1)p}{2}, \frac{q^{2 - (2m + 1)p}}{2}\right) + q^{\frac{p^2}{4} - 1}f_2^3
\]

\[
\times \left[ \sum_{k=0}^{\frac{p-1}{2}} q^{\frac{k^2}{2}} f\left(\frac{q^{2 - (2k+1)p}}{2}, \frac{q^{2 - (2k+1)p}}{2}\right) + q^{\frac{p^2}{4} - 1}f_2^3 \right] \pmod{2^4}.
\] (4.27)

Now consider the congruence equation,

\[
m^2 + m + 5 \cdot \frac{k^2 + k}{2} \equiv 7 \cdot \frac{p^2 - 1}{8} \pmod{p}.
\]

where \( 0 \leq m, k \leq \frac{p-1}{2} \) and \( p \) is a prime such that \( \left( \frac{-10}{p} \right) = -1 \). We can rewrite the above congruence as follows:

\[
(4m + 2)^2 + 10(2k + 1)^2 \equiv 0 \pmod{p}.
\]

Since \( \left( \frac{-10}{p} \right) = -1 \), it implies that

\[
4m + 2 = 2k + 1 \equiv 0 \pmod{p}.
\]

Thus \( m = k = \frac{p-1}{2} \). Using the above fact in (4.27), extracting the terms involving \( q^{pn+2} \) and then replacing \( q^p \) by \( q \), we obtain

\[
\sum_{n=0}^{\infty} \overline{A}_5\left(8p^{2j+1}n + 7p^{2j+2}\right)q^n \equiv 8f_2^3f_3^3 \pmod{2^4}.
\] (4.28)

Again Extracting the terms involving \( q^n \) in the above congruence, we see that (4.24) is true for

\[
\alpha = j + 1. \quad \text{Hence the proof of (4.24).}
\]

Next, comparing the coefficients of \( q^{pn+r} \) for \( 1 \leq r \leq p - 1 \) in (4.28), we obtain

\[
\overline{A}_5\left(8p^{2j+1}(pn + r) + 7p^{2j+2}\right) \equiv 0 \pmod{2^4}.
\]
Theorem 4.5. For all integers \( n, \alpha \geq 0, j \in \{642, 842\} \) and \( k \in \{242, 3242\} \), we have

\[
\overline{A}_5\left(5^{2\alpha}\left(10^n + j\right) - 35\right) \equiv 0 \pmod{2^4}
\]  
(4.29)

and

\[
\overline{A}_5\left(5^{2\alpha}\left(5 \cdot 10^n + k\right) - 35\right) \equiv 0 \pmod{2^4}.
\]  
(4.30)

Proof. Setting \( p = 5 \) in (4.26), we obtain

\[f_1^3 \equiv f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3 f_5^3 \pmod{2^2}.
\]
(4.31)

Let \( b(n) \) be defined by

\[
\sum_{n=0}^{\infty} b(n)q^n = f_2 f_3.
\]  
(4.32)

Then from (4.25), we have

\[
\overline{A}_5(8n + 7) \equiv 8b(n) \pmod{2^4}.
\]  
(4.33)

In view of (4.31) and (4.32), we see that

\[
\sum_{n=0}^{\infty} b(n)q^n \equiv f(q^{10}, q^{15})f_2^3 + q^2f(q^{10}, q^{40})f_3^3 + q^6f_{30}f_2^3 \pmod{2^2}.
\]

Equating the coefficients of \( q^{5n+3}, q^{5n+4} \) and \( q^{5n+1} \) in the above congruence, we find that

\[
b(5n + 3) \equiv b(5n + 4) \equiv 0 \pmod{2^2},
\]  
(4.34)

\[
\sum_{n=0}^{\infty} b(5n + 1)q^n \equiv qf_1^3 f_{10}^3 \pmod{2^2}.
\]
(4.35)

Employing (4.31) in the above congruence and then equating the coefficients of \( q^{5n}, q^{5n+3} \) and \( q^{5n+4} \), we obtain

\[
b(25n + 1) \equiv b(25n + 16) \equiv 0 \pmod{2^2},
\]  
(4.36)

\[
\sum_{n=0}^{\infty} b(25n + 21)q^n \equiv f_2^3 f_3^3 \pmod{2^2}.
\]
(4.37)

In view of (4.32), (4.36) and by mathematical induction, we find that for \( \alpha, n \geq 0 \)

\[
b\left(5^{2\alpha+2}n + 21 \cdot \frac{5^{2\alpha} - 1}{4}\right) \equiv b(n) \pmod{2^2}.
\]  
(4.38)

Replacing \( n \) by \( 5n + 3 \) and \( 5n + 4 \) in (4.37) and then using (4.34), we obtain

\[
b\left(5^{2\alpha+2}(5n + 3) + 21 \cdot \frac{5^{2\alpha} - 1}{4}\right) \equiv b\left(5^{2\alpha+2}(5n + 4) + 21 \cdot \frac{5^{2\alpha} - 1}{4}\right) \equiv 0 \pmod{2^2}.
\]
(4.39)

From (4.33) and (4.38), we deduce that

\[
\overline{A}_5\left(5^{2\alpha}(10^n + 642) - 35\right) \equiv \overline{A}_5\left(5^{2\alpha}(10^n + 842) - 35\right) \equiv 0 \pmod{2^4}.
\]

This completes the proof of (4.29). In a similar way, remaining one follows from (4.33), (4.35) and (4.37). \( \square \)
5 Congruences modulo powers of 2 and 6 for $\overline{A}_9(n)$

In this section, we prove several infinite families of congruences for $\overline{A}_9(n)$ modulo $2^3, 6, 2^3$ and $2^4$. The following lemma gives the generating functions for $\overline{A}_9(4n+1)$ and $\overline{A}_9(4n+3)$.

**Lemma 5.1.** We have

$$\sum_{n=0}^{\infty} \overline{A}_9(4n+1)q^n = 2 \frac{f_3^2 f_2^{14}}{f_1^3 f_4}. \quad (5.1)$$

and

$$\sum_{n=0}^{\infty} \overline{A}_9(4n+3)q^n = 8 \frac{f_3^2 f_2^2 f_4^4}{f_1^8}. \quad (5.2)$$

**Proof.** Setting $l = 9$ in (1.1), we have

$$\sum_{n=0}^{\infty} \overline{A}_9(n)q^n = \frac{f_9^2 f_2}{f_1^2 f_8}. \quad (5.3)$$

Xia and Yao [19] found the following 2-dissection formula for $\frac{f_9}{f_1}$:

$$\frac{f_9}{f_1} = \frac{f_3^2 f_2 f_6^2 f_3^4 + q f_2 f_6^2 f_3^3}{f_2^2 f_6^4}. \quad (5.4)$$

In view of (5.4), we have

$$\frac{f_2 f_9}{f_1 f_8} = \frac{f_2}{f_8} \left( \frac{f_3^2 f_2 f_6^2 f_3^4 + q f_2 f_6^2 f_3^3}{f_2^2 f_6^4} \right)^2 = \frac{f_1 f_2^2 f_6^4}{f_2^3 f_6^2 f_3^2} + \frac{2 q f_3^2 f_6^4}{f_2} + q^2 f_3^2 f_6^4 f_2^2 f_1 f_6. \quad (5.5)$$

Combining (5.5) and (5.3) and then extracting the terms involving $q^{2n+1}$ in the resulting identity, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_9(2n+1)q^n = 2 \frac{f_6^2 f_2^2}{f_1}. \quad (5.6)$$

With the help of (2.2), we can rewrite the above identity as follows:

$$\sum_{n=0}^{\infty} \overline{A}_9(2n+1)q^n = 8 q \frac{f_2^2 f_6^2 f_2^{14}}{f_2^8} + 2 \frac{f_2^2 f_6^2 f_2^{14}}{f_2^{12} f_8}. \quad (5.7)$$

Extracting the even powers of $q$ and the odd powers of $q$ in (5.7), we arrive at (5.1) and (5.2) respectively.

**Theorem 5.2.** If $p \geq 5$ is a prime with $\left(\frac{-3}{p}\right) = -1$ and $1 \leq j \leq p-1$, then for all non-negative integers $n$ and $\alpha$, we have

$$\overline{A}_9 \left( p^{2\alpha+2} (8n+3) + 8 p^{2\alpha+1} \right) \equiv 0 \pmod {2^4}. \quad (5.8)$$

**Proof.** Substituting (2.1) and (2.4) into (5.2), we get

$$\sum_{n=0}^{\infty} \overline{A}_9(4n+3)q^n = 8 \frac{f_2^2 f_6^2 f_2^{14}}{f_2^8 f_2^4 f_6^4} \equiv \frac{f_3^2 f_6^2 f_2^{14}}{f_1 f_4^3 f_6^4} \pmod {2^4}. \quad (5.8)$$

Employing (4.6), we deduce that

$$\frac{f_3^2 f_6^2 f_2^{14}}{f_1 f_4^3 f_6^4} \equiv f_2 \frac{f_3}{f_6} \pmod 2.$$

On some new congruences for $\ell$-regular overpartitions 355
Extracting the even powers of \( q \) in (5.8) and then using the above congruence, we find that

\[
\sum_{n=0}^{\infty} \overline{A}_0(8n+3)q^n \equiv 8f_2^3f_5^3 \quad (\text{mod } 2^4). \tag{5.9}
\]

Using Lemma 3.2 and (4.26), we can rewrite the above congruence as

\[
\overline{A}_0(8n+3)q^n \equiv 8 \left[ \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} q^{\frac{1}{2}k^2} \sum_{n=-\infty}^{\infty} (6pn+6k+1)q^{\frac{3}{2}(3pn+6k+1)\frac{p-1}{2}} \pm pq^{\frac{p+1}{2}f_6^p} \right] \\
\times \left[ \sum_{m=0}^{\infty} q^{m^2+m} f \left( q^{\frac{p^2-1}{2}}, q^{\frac{p^2-2}{2}} \right) + q^{\frac{p^2-1}{4}f_2^p} \right] \quad (\text{mod } 2^4). \tag{5.10}
\]

Let \( p \geq 5 \) be prime with \( \left( \frac{-2}{p} \right) = -1 \). For \( 0 \leq m \leq \frac{p^2-2}{2} \) and \( -\frac{p-1}{2} \leq k \leq \frac{p-1}{2} \), we consider the congruence equation

\[
m^2 + m + 3 \cdot \frac{3k^2 + k}{2} \equiv 3 \cdot \frac{p^2 - 1}{8} \quad (\text{mod } p). \tag{5.11}
\]

We can rewrite the above congruence as follows:

\[
2(2m+1)^2 + (6k+1)^2 \equiv 0 \quad (\text{mod } p).
\]

Since \( \left( \frac{-2}{p} \right) = -1 \), it implies that

\[
2m+1 = 6k+1 \equiv 0 \quad (\text{mod } p).
\]

Thus, the congruence (5.11) holds if and only if \( m = \frac{p-1}{2} \) and \( k = \frac{p-1}{6} \). Using the above fact in (5.10), extracting the terms involving \( q^{\frac{-1}{2}(2m+1)p} \) and then replacing \( q^p \) by \( q \), we obtain

\[
\sum_{n=0}^{\infty} \overline{A}_0 \left( 8q^2n + 3q^2 \right) q^n \equiv 8f_2^3f_5^3 \quad (\text{mod } 2^4). \tag{5.12}
\]

From (5.9), (5.12) and by mathematical induction, we find that for \( \alpha \geq 0 \) and \( n \geq 0 \)

\[
\sum_{n=0}^{\infty} \overline{A}_0 \left( 8q^{2\alpha n} + 3q^{2\alpha} \right) q^n \equiv 8f_2^3f_5^3 \quad (\text{mod } 2^4). \tag{5.13}
\]

Again employing Lemma 3.2 and (4.26) into (5.13), extracting the terms involving \( q^{pn+3\frac{p^2-1}{8}} \) in the resulting congruence and then replacing \( q^p \) by \( q \), we obtain

\[
\sum_{n=0}^{\infty} \overline{A}_0 \left( 8q^{2\alpha} \left( pn + 3 \frac{p^2 - 1}{8} \right) + 3q^{2\alpha} \right) q^n \equiv 8f_2^3f_5^3 \quad (\text{mod } 2^4).
\]

Equating the coefficients of \( q^{pn+j} \) for \( 1 \leq j \leq p-1 \), we obtain the required congruence. \( \square \)

**Remark 5.3.** Equating the coefficients of odd powers of \( q \) in (5.8), we see that for \( n \geq 0 \)

\[
\overline{A}_0(8n + 7) \equiv 0 \quad (\text{mod } 2^4).
\]

**Theorem 5.4.** If \( p \geq 5 \) is a prime with \( \left( \frac{-1}{p} \right) = -1 \) and \( 1 \leq j \leq p-1 \), then for all non-negative integers \( n \) and \( \alpha \),

\[
\overline{A}_0 \left( p^{2\alpha + 2}(8n + 5) + 8p^{2\alpha + 1}j \right) \equiv 0 \quad (\text{mod } 2^3).
\]
Proof. In view of (2.1), (2.4) and (5.1), modulo 4, we find that

\[
\frac{f_4 f_{14}}{f_2 f_4} = 2q \left( \frac{f_4 f_{16} f_{12} f_{18}}{f_2^3 f_6 f_{24} f_{10}} + \frac{f_4 f_{26} f_{24}}{f_2^3 f_6^3 f_{24} f_{10}} \right) + \frac{f_6 f_{12} f_{18}}{f_2^6 f_{24} f_{10}}.
\] (5.14)

Combining (5.1) and (5.14), extracting the odd powers of \( q \) and then using (4.6), we deduce

\[
\sum_{n=0}^{\infty} \mathcal{A}_9(8n + 5) q^n \equiv 4f_4^3 \frac{f_6^5}{f_2^6} + 4f_4^3 \frac{f_6^5}{f_2^6} \quad (\text{mod } 2^3).
\] (5.15)

Now, we consider the following two congruences:

\[
3 \frac{j^2 + j}{2} + 2m^2 + 2m \equiv \frac{5j^2 - 1}{8} \quad (\bmod \ p), \quad (5.16)
\]

\[
18j^2 + 6j + \frac{m^2 + m}{2} \equiv \frac{5j^2 - 1}{8} \quad (\bmod \ p), \quad (5.17)
\]

where \( 0 \leq m \leq \frac{p-3}{2}, -\frac{p-1}{2} \leq j \leq \frac{p-1}{2} \) and \( p \geq 5 \) is a prime such that \( \left( \frac{-1}{p} \right) = -1 \). We can rewrite above congruences as follows:

\[
(6j + 1)^2 + (4m + 2)^2 \equiv 0 \quad (\bmod \ p), \quad (5.16)
\]

\[
(12j + 2)^2 + (2m + 1)^2 \equiv 0 \quad (\bmod \ p). \quad (5.17)
\]

Since \( \left( \frac{-1}{p} \right) = -1 \), above two congruence implies that

\[
6j + 1 = 2m + 1 \equiv 0 \quad (\bmod \ p).
\]

Thus, the congruences (5.16) and (5.17) holds if and only if \( m = \frac{p-1}{6} \) and \( j = \frac{p-1}{2} \). Substituting Lemma 3.2 and (4.26) into (5.15), using the above fact in the resulting congruence and then extracting the terms involving \( q^{pq^2n+5\frac{p^2-1}{8}} \), we obtain

\[
\sum_{n=0}^{\infty} \mathcal{A}_9 \left( 8p^2 n + 5p^2 \right) q^n \equiv 4f_4^3 \frac{f_6^5}{f_2^6} + 4f_4^3 \frac{f_6^5}{f_2^6} \quad (\text{mod } 2^3). \quad (5.18)
\]

From (5.15), (5.18) and by mathematical induction, we see that for \( \alpha \geq 0 \) and \( n \geq 0 \)

\[
\sum_{n=0}^{\infty} \mathcal{A}_9 \left( 8p^{2\alpha} n + 5p^{2\alpha} \right) q^n \equiv 4f_4^3 \frac{f_6^5}{f_2^6} + 4f_4^3 \frac{f_6^5}{f_2^6} \quad (\text{mod } 2^3). \quad (5.19)
\]

Again employing Lemma 3.2 and (4.26) into (5.19), extracting the terms involving \( q^{pn+5\frac{p^2-1}{8}} \) in the resulting congruence and then replacing \( q^p \) by \( q \), we obtain

\[
\sum_{n=0}^{\infty} \mathcal{A}_9 \left( 8p^{2\alpha}(pn + 5\frac{j^2 - 1}{8}) + 5p^{2\alpha} \right) q^n \equiv 4f_4^3 \frac{f_6^5}{f_2^6} + 4f_4^3 \frac{f_6^5}{f_2^6} \quad (\text{mod } 2^3).
\]

Equating the coefficients of \( q^{pn+j} \) for \( 1 \leq j \leq p - 1 \) in the above congruence, we obtain the required result. \( \square \)

**Theorem 5.5.** If \( p \geq 5 \) is a prime with \( \left( \frac{-1}{p} \right) = -1 \) and \( 1 \leq j \leq p - 1 \), then for all non-negative integers \( n \) and \( \alpha \),

\[
\mathcal{A}_9 \left( p^{2\alpha+2}(8n + 1) + 8p^{2\alpha+1}j \right) \equiv 0 \quad (\text{mod } 2^3).
\]
Proof. Combining (5.1) and (5.14), extracting the even powers of \( q \) and then using (4.7), we see that
\[
\sum_{n=0}^{\infty} \mathcal{A}_0(8n+1) q^n \equiv 2 \frac{f_3^5}{f_6^2} \frac{f_5^2}{f_{12}^3} \pmod 2^3. \tag{5.20}
\]
Using Lemma 2.4 with \( q \) replaced by \(-q\) and Lemma 3.2 in (5.20), we have
\[
\sum_{n=0}^{\infty} \mathcal{A}_0 \left( p^2 n + p^2 \right) q^n \equiv 2 \left[ \sum_{k=\frac{p-1}{2}}^{\infty} q^{3k^2 + k} \sum_{n=-\infty}^{n=\infty} \frac{(6pn + 6k + 1) q^{pn + 2n + 1}}{2^{n+1}} \right] \pm p q^{3p+1} \frac{f_p^5}{f_{12}^3} \pmod 2^3.
\]
\[
\times \left[ \frac{f_6^2}{f_{12}^3} + \sum_{r=1}^{p-1} \left( -1 \right)^r q^r f \left( -q^{6p(p-2r)} \right) \right] \pmod 2^3. \tag{5.21}
\]
Let \( p \geq 5 \) be a prime with \( \left( \frac{-p}{p} \right) = -1 \). For \(-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}\) and \( 1 \leq r \leq p-1 \), consider the congruence equation
\[
3 \frac{3k^2 + k}{2} + r^2 \equiv \frac{p^2 - 1}{8} \pmod p,
\]
which is equivalent to
\[
(6k + 1)^2 + 2(2r)^2 \equiv 0 \pmod 2^3.
\]
Since \( \left( \frac{-p}{p} \right) = -1 \), the only solution of the congruence (5.22) is \( k = \frac{3p-1}{6} \) and \( r = 0 \). Using the above fact in (5.21), extracting the terms involving \( q^{p^2 n + \frac{p-1}{2}} \) and then replacing \( q^r \) by \( q \), we obtain
\[
\sum_{n=0}^{\infty} \mathcal{A}_0 \left( 8p^2 n + p^2 \right) q^n \equiv \pm 2 p \frac{f_3^5}{f_6^2} \frac{f_5^2}{f_{12}^3} \pmod 2^3. \tag{5.23}
\]
From (5.20), (5.23) and by induction, we find that for \( n \geq 0 \) and \( \alpha \geq 0 \),
\[
\sum_{n=0}^{\infty} \mathcal{A}_0 \left( 8p^{2\alpha} n + p^{2\alpha} \right) q^n \equiv 2 \left( \pm p \right)^{\alpha} \frac{f_3^5}{f_6^2} \frac{f_5^2}{f_{12}^3} \pmod 2^3. \tag{5.24}
\]
Substituting Lemma 2.4 with \( q \) replaced by \(-q\) and Lemma 3.2 into (5.24), extracting the terms involving \( q^{p^2 n + \frac{p-1}{2}} \) in the resulting congruence, we deduce that
\[
\sum_{n=0}^{\infty} \mathcal{A}_0 \left( 8p^{2\alpha} \left( pn + \frac{p^2 - 1}{8} + p^{2\alpha} \right) \right) q^n \equiv 2 \left( \pm p \right)^{\alpha+1} \frac{f_3^5}{f_6^2} \frac{f_5^2}{f_{12}^3} \pmod 2^3. \tag{5.25}
\]
Equating the coefficients of \( q^{pk+j} \) for \( 1 \leq j \leq p-1 \) in (5.25), we obtain
\[
\mathcal{A}_0 \left( p^{2\alpha+1} (pn + j) + p^{2\alpha+2} \right) \equiv 0 \pmod 2^3.
\]
Hence the proof. \( \square \)

Theorem 5.6. If \( p \) is an odd prime such that \( \left( \frac{-p}{p} \right) = -1 \) and \( 1 \leq k \leq p-1 \), then for all integers \( n \geq 0 \) and \( \alpha \geq 0 \)
\[
\mathcal{A}_0 \left( 2p^{2\alpha+2} n + 2p^{2\alpha+1} k + p^{2\alpha+2} \right) \equiv 0 \pmod 2^3, \tag{5.26}
\]
\[
\mathcal{A}_0 \left( 3\alpha (2n + 1) \right) \equiv \mathcal{A}_0 (2n + 1) \pmod 2^3, \tag{5.27}
\]
\[
\mathcal{A}_0 \left( 3\alpha (6n + 5) \right) \equiv 0 \pmod 2^3. \tag{5.28}
\]
Proof. It follows from (3.2) and (5.6) that
\[
\sum_{n=0}^{\infty} \mathcal{A}_0(2n + 1)q^n \equiv 2 \frac{f_2^2 f_3^2}{f_5 f_1} \pmod{6}.
\] (5.29)

Let \( p \) be odd prime such that \( \left( \frac{-3}{p} \right) = -1 \) and for \( 0 \leq m, j \leq \frac{p^2 - 1}{2} \), the following relation
\[
3 \cdot \frac{m^2 + m}{2} + \frac{j^2 + j}{2} \equiv \frac{p^2 - 1}{2} \pmod{p}
\]
holds if and only if \( m = j = \frac{p-1}{2} \). From Theorem 2.5, (5.29) and by induction \( \alpha \), we find that for all integer \( n \geq 0 \)
\[
\sum_{n=0}^{\infty} \mathcal{A}_0 \left( 2p^{2\alpha}n + p^{2\alpha} \right) q^n \equiv 2 \frac{f_2^2 f_3^2}{f_5 f_1} \pmod{6}.
\]

Now, substituting Theorem 2.5 into the above congruence and then extracting the terms involving \( q^{pn+k} \), we deduce
\[
\sum_{n=0}^{\infty} \mathcal{A}_0 \left( 2p^{2\alpha} \left( pn + \frac{p^2 - 1}{2} \right) + p^{2\alpha} \right) q^n \equiv 2 \frac{f_2^2 f_3^2}{f_3 p f_1} \pmod{6}.
\]

Equating the coefficients of \( q^{pn+k} \) for \( 1 \leq k \leq p-1 \) in the above congruence, we arrive at (5.26).

Form [1, pp.49], we recall that
\[
\frac{f_2^2}{f_1} = f(q^3, q^9) + q \frac{f_{18}}{f_9}.
\] (5.30)

In view of (5.30), (5.29) and by induction, we arrive at (5.27) and (5.28).

\( \square \)

Remark 5.7. Setting \( \alpha = 0 \) in (5.28), we obtain Theorem 1.6.

6 Congruences modulo powers of 2 and 3 for \( \mathcal{A}_{3^r}(n) \)

In this section, by employing (2.3) and Lemma 2.3, we find several congruences modulo \( 2^2, 2^3, 2^4 \) and 3 for \( \mathcal{A}_{3^r}(n), r \geq 2 \).

Lemma 6.1. We have
\[
\mathcal{A}_{3^r}(9n + 3) \equiv 8a_3(n) \pmod{2^4},
\] (6.1)
\[
\mathcal{A}_{3^r}(6n + 2) \equiv 4a_3(n) \pmod{2^3},
\] (6.2)
\[
\mathcal{A}_{3^r}(3n + 1) \equiv 2a_3(n) \pmod{2^2},
\] (6.3)

where \( a_3(n) \) denote the number of 3–cores of \( n \).

Proof. Setting \( l = 3^r (r \geq 2) \) in (1.1) and then employing Lemma 2.3, we find that
\[
\sum_{n=0}^{\infty} \mathcal{A}_{3^r}(n)q^n = \frac{f_{3^{r-1}}^2 f_6 f_9}{f_{2^{3^r-1}} f_9 f_6} + 2 q \frac{f_{3^{r-1}}^2 f_6 f_9}{f_{2^{3^r-1}} f_9 f_6} + 4 q^2 \frac{f_{3^{r-1}}^2 f_6 f_9}{f_{2^{3^r-1}} f_9 f_6}.
\] (6.4)

Extracting the terms involving \( q^{3n}, q^{3n+1} \) and \( q^{3n+2} \) in (6.4), we obtain
\[
\sum_{n=0}^{\infty} \mathcal{A}_{3^r}(3n)q^n = \frac{f_{3^{r-1}}^2 f_6 f_9}{f_{2^{3^r-1}} f_9 f_6},
\] (6.5)
\[
\sum_{n=0}^{\infty} \mathcal{A}_{3^r}(3n + 1)q^n = 2 \frac{f_{3^{r-1}}^2 f_6 f_9}{f_{2^{3^r-1}} f_9 f_6},
\] (6.6)
and

\[ \sum_{n=0}^{\infty} \overline{A}_{3^r}(3n + 2)q^n = 4 \frac{f_2^{3r-1}f_3^{2r+1}f_6^{r+1}}{f_2^{2r-1}f_6^{r+1}f_1^2}. \]  

(6.7)

In view of Lemma 2.3, modulo 16, we find that

\[
\frac{f_2^{3r-1}f_3^{2r+1}f_6^{r+1}}{f_2^{2r-1}f_6^{r+1}f_1^2} = \frac{f_2^{3r-1}f_3^r}{f_2^{2r-1}f_6^{r+1}f_1^2} \left( \frac{f_6^2 f_6^2}{f_3^{2r+1}f_6^{r+1}f_1^2} + 2q \frac{f_6^2 f_6^2}{f_3^{2r+1}f_6^{r+1}f_1^2} + 4q^2 \frac{f_6^2 f_6^2}{f_3^{2r+1}f_6^{r+1}f_1^2} \right)^4
\]

\[
= \frac{f_2^{3r-1}f_3^{2r+1}f_6^{r+1}}{f_2^{2r-1}f_6^{r+1}f_1^2} + 8q \frac{f_2^{3r-1}f_3^{2r+1}f_6^{r+1}}{f_2^{2r-1}f_6^{r+1}f_1^2} + 8q^2 \frac{f_2^{3r-1}f_3^{2r+1}f_6^{r+1}}{f_2^{2r-1}f_6^{r+1}f_1^2}. \]  

(6.8)

Combining (6.5) and (6.8), extracting the terms of the form \( q^{3n+1} \) and then using (4.6), we obtain

\[
\sum_{n=0}^{\infty} \overline{A}_{3^r}(9n + 3)q^n \equiv 8 \frac{f_3^3}{f_1^3} = \sum_{n=0}^{\infty} a_3(n)q^n \pmod{2^4}.
\]

Equating the coefficients of \( q^n \) on both sides of the above congruence, we arrive at (6.1).

Employing (4.6) in (6.7), we see that

\[
\sum_{n=0}^{\infty} \overline{A}_{3^r}(3n + 2)q^n \equiv \frac{4 f_3^3}{f_2} = \sum_{n=0}^{\infty} a_3(n)q^{2n} \pmod{2^3}.
\]  

(6.9)

Extracting even powers of \( q \) in (6.9), we obtain (6.2).

In view of (6.6) and (4.6), we deduce (6.3).

\[ \square \]

**Remark 6.2.** Equating the odd powers of \( q \) in (6.9), we find that

\[ \overline{A}_{3^r}(6n + 5) \equiv 0 \pmod{2^3}, \quad n \geq 0. \]

Utilizing (2.3), we can easily derive the following corollary.

**Corollary 6.3.** For all non-negative integers \( n, \alpha \) and \( 1 \leq j \leq 3 \), we have

\[ a_3(4n + 1) = 0, \]  

(6.10)

\[ a_3(8n + 2j) \equiv 0 \pmod{2} \]  

(6.11)

and

\[ a_3(8n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = k(3k - 1)/2 \text{ for some integer } k, \\ 0 \pmod{2}, & \text{otherwise} \end{cases}. \]

**Theorem 6.4.** If \( p \equiv 2 \pmod{3} \) and \( j \in \{1, 2, 3\}, \) then for all non-negative integers \( n \) and \( \alpha, \) we have

\[ \overline{A}_{3^r}\left(p^{2\alpha}(9n + 3)\right) \equiv \overline{A}_{3^r}(9n + 3) \pmod{2^4}, \]  

(6.12)

\[ \overline{A}_{3^r}\left(p^{2\alpha}(36n + 30)\right) \equiv 0 \pmod{2^4}, \]  

(6.13)

\[ \overline{A}_{3^r}\left(p^{2\alpha}(72n + 18j + 3)\right) \equiv 0 \pmod{2^4} \]  

(6.14)

and

\[ \overline{A}_{3^r}\left(p^{2\alpha}(72n + 3)\right) \equiv \begin{cases} 2^3 \pmod{2^4}, & \text{if } n = k(3k - 1)/2 \text{ for some integer } k, \\ 0 \pmod{2^4}, & \text{otherwise}. \end{cases} \]

**Proof.** Proof follows from Corollary 2.7, Corollary 6.3 and (6.1).

\[ \square \]
Remark 6.5. Employing Corollary 2.7 and Corollary 6.3 in (6.2) and (6.2), we can also find infinite families of congruences modulo 8 and 4 for $A_{3\ell'}(n)$ which are similar to congruences in Theorem 6.4.

Next, we present a short and simple proof of the Theorem 1.7.

Theorem 6.6. For all non-negative integers $r \geq 3$ and $n$, we have
\[
A_{3\ell'}(27n + 18) \equiv 0 \pmod{3}.
\]

Proof. From (3.2) and (6.5), it follows that
\[
\sum_{n=0}^{\infty} A_{3\ell'}(3n)q^n \equiv \frac{f_3^{2x-1}f_3^3 f_2}{f_2 f_3^{x-1} f_3 f_1} \pmod{3}.
\]

In view of above congruence, Lemma 2.3 and (3.2), we find that
\[
\sum_{n=0}^{\infty} A_{3\ell'}(9n)q^n \equiv \frac{f_3^{2x-1}f_3^3 f_2}{f_2 f_3^{x-1} f_3 f_1} \pmod{3}. \quad (6.15)
\]

Substituting (5.30) into (6.15) and then equating the coefficients of $q^{3n+2}$, we obtain the required congruence. Hence the proof. $lacksquare$

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