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COUPLED FIXED POINT THEOREM FOR RATIONAL CONTRACTION CONDITIONS IN DISLOCATED QUASI-METRIC SPACE

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Abstract In this article, we have establish a coupled fixed point theorem satisfying rational contraction conditions in dislocated quasi-metric space. In order to validate our establish theorem and corollaries we have provide an example.

1 Introduction and Preliminaries

The concept of dislocated metric space was introduced by Hitzler [1]. In such a space self-distance between points need not to be zero necessarily. They also generalized famous Banach contraction principle in dislocated metric space. Dislocated metric space play a vital role in topology, logical programming, computer science and electronic engineering etc. In 2005, Zeyada, Hassan and Ahmad [2] initiated the notion of complete dislocated quasi-metric space and generalized the result of Hitzler [1] in dislocated quasi-metric space. With the passage of time many papers have been published containing fixed point results for a single and a pair of mappings for different type of contractive conditions in dislocated quasi-metric space (see [3, 4, 5, 6]).

In 2006, Bhaskar and Lakshmikantham [7] initiated the concept of coupled fixed point for non-linear contractions in partially ordered metric spaces. Furthermore, after the work of Bhaskar and Lakshmikantham [7] coupled fixed point theorems have been established by many authors in different type of spaces (see [8, 9, 10]).

In this paper, we have establish a coupled fixed point theorem satisfying rational contraction conditions in the context of dislocated quasi-metric space. An example is given in the support of our main results. Throughout the paper $\mathbb{R}^+$ represent the set of non-negative real numbers.

Definition[2]. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a function satisfying the conditions

d_1) $d(x, x) = 0$;

d_2) $d(x, y) = d(y, x) = 0$ implies $x = y$;

d_3) $d(x, y) = d(y, x)$;

d_4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfy the conditions from d_1 to d_4 then it is called metric on X. If d satisfy conditions d_2 to d_4 then it is called dislocated metric (d-metric) on X and if d satisfy conditions d_2 and d_4 only then it is called dislocated quasi-metric (dq-metric) on X. The pair (X, d) is called dislocated quasi-metric space.

Clearly every metric space is a dislocated metric space but the converse is not necessarily true as clear form the following example:

Example. Let X=$\mathbb{R}^+$ define the distance function $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = max\{x, y\}$$

Clearly X is dislocated metric space but not a metric space.

Also every metric and dislocated metric spaces are dislocated quasi-metric spaces but the converse is not true.
Example. Let $X = \mathbb{R}$ define the distance function $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |x| \quad \text{for all } x, y \in X.$$

Evidently X is dq-metric space but not a metric space nor dislocated metric space.
In our main work we will use the following definitions which can be found in [2].
Definition. A sequence $\{x_n\}$ is called dislocated quasi convergent (dq-convergent) in X if for $n \in N$ we have

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In this case x is called dislocated quasi limit (dq-limit) of the sequence $\{x_n\}$.
Definition. A sequence $\{x_n\}$ in dq-metric space is called Cauchy sequence if for $\epsilon > 0$ there exists a positive integer n_0 such that for $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$ i.e

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Definition. A dq-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.
Definition[7]. An element $(x, y) \in X^2$ is called coupled fixed point of the mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$ for $x, y \in X$.
Example. Let $X = \mathbb{R}$ and $T : X \times X \to X$ defined by

$$T(x_1, x_2) = \frac{x_1 x_2}{2}.$$

Here $(0, 0)$ is the coupled fixed point of T.
The following well-known results can be seen in [2].

Lemma 1.1. *Limit of a convergent sequence in dq- metric space is unique.*

Theorem 1.2. *Let (X, d) be a complete dq-metric space $T : X \to X$ be a contraction. Then T has a unique fixed point.*

Remark.

- It is obvious that the the following statement hold.
  for real numbers a, b and c,
  if $a < b$ and $c > 0$. Then $ac < bc$.

2 Main Results

Theorem 2.1. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \times X \to X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha \cdot [d(x, u) + d(y, v)] + \beta \cdot \frac{d(x, T(x, y)).d(x, T(u, v))}{1 + d(x, u) + d(y, v)} + \gamma \cdot \frac{d(x, T(x, y)).d(u, T(u, v))}{1 + d(x, u)} \tag{2.1}$$

for all $x, y, u, v \in X$ and α, β and γ are non-negative constants with $2(\alpha + \beta) + \gamma < 1$. Then T has a unique coupled fixed point in $X \times X$.

proof. Let x_0 and y_0 are arbitrary in X, we define the sequences $\{x_n\}$ and $\{y_n\}$ as following,

$$x_{n+1} = T(x_n, y_n) \quad \text{and} \quad y_{n+1} = T(y_n, x_n) \quad \text{for} \quad n \in N.$$

Consider

$$d(x_n, x_{n+1}) = d(T(x_{n-1}, y_{n-1}), T(x_n, y_n))$$

Now by (2.1) we have

$$d(x_n,x_{n+1}) \le \alpha\cdot[d(x_{n-1},x_n)+d(y_{n-1},y_n)]+\beta\cdot\frac{d(x_{n-1},T(x_{n-1},y_{n-1})).d(x_{n-1},T(x_n,y_n))}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)}$$

$$+\gamma\cdot\frac{d(x_{n-1},T(x_{n-1},y_{n-1})).d(x_n,T(x_n,y_n))}{1+d(x_{n-1},x_n)}.$$

Using the definition of the sequences $\{x_n\}$ and $\{y_n\}$ we have

$$\le \alpha\cdot[d(x_{n-1},x_n)+d(y_{n-1},y_n)]+\beta\cdot\frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)}$$

$$+\gamma\cdot\frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}.$$

Simplifying and using Remark 1 we have

$$< \alpha\cdot[d(x_{n-1},x_n)+d(y_{n-1},y_n)]+\beta\cdot d(x_{n-1},x_{n+1})+\gamma\cdot d(x_n,x_{n+1})$$

$$\le \alpha\cdot[d(x_{n-1},x_n)+d(y_{n-1},y_n)]+\beta\cdot[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+\gamma\cdot d(x_n,x_{n+1}).$$

Simplification yeilds

$$d(x_n,x_{n+1}) \le \frac{\alpha+\beta}{1-(\beta+\gamma)}d(x_{n-1},x_n)+\frac{\alpha}{1-(\beta+\gamma)}d(y_{n-1},y_n). \tag{2.2}$$

Similarly we can show that

$$d(y_n,y_{n+1}) \le \frac{\alpha+\beta}{1-(\beta+\gamma)}d(y_{n-1},y_n)+\frac{\alpha}{1-(\beta+\gamma)}d(x_{n-1},x_n). \tag{2.3}$$

Adding (2.2) and (2.3) we have

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \le \frac{2\alpha+\beta}{1-(\beta+\gamma)}[d(x_{n-1},x_n)+d(y_{n-1},y_n).$$

Since $2\alpha+2\beta+\gamma<1$, so $h=\frac{2\alpha+\beta}{1-(\beta+\gamma)}<1$. Therefore the above inequality becomes,

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \le h\cdot[d(x_{n-1},x_n)+d(y_{n-1},y_n).$$

Also

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \le h^2\cdot[d(x_{n-2},x_{n-1})+d(y_{n-2},y_{n-1}).$$

Similarly proceeding we have

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \le h^n\cdot[d(x_0,x_1)+d(y_0,y_1).$$

Since $h<1$ taking limit $n\to\infty$ we have

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})]\to 0.$$

Implies

$$d(x_n,x_{n+1})\to 0 \text{ and } d(y_n,y_{n+1})\to 0.$$

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in complete dislocated quasi-metric space X. So there must exist $w,z\in X$ such that

$$\lim_{n\to\infty} x_n = w \text{ and } \lim_{n\to\infty} y_n = z.$$

Also since T is continuous and $T(x_n,y_n)=x_{n+1}$ so taking limit $n\to\infty$. We have

$$\lim_{n\to\infty} T(x_n,y_n) = \lim_{n\to\infty} x_{n+1}$$

$$T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} x_{n+1}$$
$$T(w, z) = w.$$

Also from $T(y_n, x_n) = y_{n+1}$. We can show that

$$d(z, w) = z.$$

Thus $(w, z) \in X \times X$ is the coupled fixed point of T in X.

Uniqueness. Let (w, z) and $(w^{'}, z^{'})$ are two distinct coupled fixed points of T in $X \times X$. Then by use of (2.1) we have

$$d(w, w) = d(T(w, z), T(w, z)) \le \alpha \cdot [d(w, w) + d(z, z)]+$$
$$\beta \cdot \frac{d(w, T(w, z)).d(w, T(w, z))}{1 + d(w, w) + d(z, z)} + \gamma \cdot \frac{d(w, T(w, z)).d(w, T(w, z))}{1 + d(w, w)}$$
$$\le \alpha \cdot [d(w, w) + d(z, z)] + \beta \cdot \frac{d(w, w).d(w, w)}{1 + d(w, w) + d(z, z)} + \gamma \cdot \frac{d(w, w)).d(w, w))}{1 + d(w, w)}.$$

Using Remark 2.1 and then simplifying we have

$$d(w, w) \le (\alpha + 2\beta + \gamma)d(w, w) + \alpha d(z, z). \tag{2.4}$$

Similarly we can show that

$$d(z, z) \le (\alpha + 2\beta + \gamma)d(z, z) + \alpha d(w, w). \tag{2.5}$$

Adding (2.4) and (2.5) we have

$$[d(w, w) + d(z, z)] \le (2\alpha + 2\beta + \gamma)[d(w, w) + d(z, z)].$$

Since $2\alpha + 2\beta + \gamma < 1$ so the above inequality is possible only if

$$[d(w, w) + d(z, z)] = 0.$$

Implies

$$d(w, w) = d(z, z) = 0. \tag{2.6}$$

Now consider

$$d(w, w^{'}) = d(T(w, z), T(w^{'}, z^{'}) \le \alpha \cdot [d(w, w^{'}) + d(z, z^{'})]$$
$$+\beta \cdot \frac{d(w, T(w, z)).d(w, T(w^{'}, z^{'}))}{1 + d(w, w') + d(z, z')} + \gamma \cdot \frac{d(w, T(w, z)).d(w^{'}, T(w^{'}, z^{'}))}{1 + d(w, w')}$$
$$\le \alpha \cdot [d(w, w^{'}) + d(z, z^{'})] + \beta \cdot \frac{d(w, w).d(w, w^{'})}{1 + d(w, w') + d(z, z')} + \gamma \cdot \frac{d(w, w).d(w^{'}, w^{'})}{1 + d(w, w')}.$$

Now using (2.6) we have the following

$$d(w, w^{'}) \le \alpha \cdot [d(w, w^{'}) + d(z, z^{'})]. \tag{2.7}$$

By following similar procedure we can get

$$d(z, z^{'}) \le \alpha \cdot [d(z, z^{'}) + d(w, w^{'})]. \tag{2.8}$$

Adding (2.7) and (2.8) we have

$$d(w, w^{'}) + d(z, z^{'}) \le 2\alpha \cdot [d(w, w^{'}) + d(z, z^{'})].$$

Since $2\alpha < 1$ so the above inequality is possible only if

$$d(w, w^{'}) + d(z, z^{'}) = 0.$$

Which implies that

$$d(w, w^{'}) = d(z, z^{'}) = 0.$$

Implies

$$w = w^{'} \text{ and } z = z^{'}.$$

Hence

$$(w, z) = (w^{'}, z^{'}).$$

Thus coupled fixed point of T in $X \times X$ is unique.
We deduce the following corollaries from Theorem 2.1.

Corollary 2.2. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \times X \to X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha \cdot [d(x, u) + d(y, v)] + \beta \cdot \frac{d(x, T(x, y)).d(x, T(u, v))}{1 + d(x, u) + d(y, v)}$$

for all $x, y, u, v \in X$ and α, β are non-negative constants with $2(\alpha + \beta) < 1$. Then T has a unique coupled fixed point in $X \times X$.

Corollary 2.3. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \times X \to X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha \cdot [d(x, u) + d(y, v)] + \beta \cdot \frac{d(x, T(x, y)).d(u, T(u, v))}{1 + d(x, u)}$$

for all $x, y, u, v \in X$ and α, β are non-negative constants with $2\alpha + \beta < 1$. Then T has a unique coupled fixed point in $X \times X$.

Corollary 2.4. *Let (X, d) be a complete dislocated quasi=metric space. $T : X \times X \to X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha[d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ and $\alpha > 0$ with $2\alpha < 1$. Then T has a unique coupled fixed point in $X \times X$.

Example. Let $X = [0, 1]$. Define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |x - y| + |x|$$

for all $x, y \in X$. Then (X, d) is a complete dislocated quasi-metric space. Define a continuous self-map $T : X \times X \to X$ by $T(x, y) = \frac{1}{6}xy$ for all $x, y \in X$. Since

$$|xy - uv| \leq |x - u| + |y - v| \text{ and } |xy| \leq |x| + |y|.$$

Hold for all $x, y, u, v \in X$. Then

$$d(T(x, y), T(u, v)) = |\frac{1}{6}xy - \frac{1}{6}uv| + |\frac{1}{6}xy|$$

$$\leq \frac{1}{6}(|x - u| + |y - v|) + \frac{1}{6}(|x| + |y|)$$

$$\leq \frac{1}{6}(|x - u| + |y - v| + |x| + |y|)$$

$$\leq \frac{1}{3}[(|x - u| + |x|) + (|y - v| + |y|)]$$

$$d(T(x, y), T(u, v)) \leq \alpha \cdot [d(x, u) + d(y, v)].$$

So for $\alpha = \frac{1}{3}$ and $\beta = \gamma = 0$ all the conditions of Theorem 2.1 are satisfied having $(0, 0) \in X \times X$ is the unique coupled fixed point of T in $X \times X$.

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