Palestine Journal of Mathematics

Vol. 5(2) (2016) , 164–174

Slant Helices Generated by Plane Curves in Euclidean 3-space

Mesut Altınok and Levent Kula

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 53A04 ; Secondary 14H50.

Keywords and phrases: Slant helix, plane curve and principal normal indicatrix.

Abstract In this paper, we investigate the relationship between the plane curves and slant helices in $\mathbb{R}^3$. Moreover, we show how could be obtained to a slant helix from a plane curve. Finally, we give some slant helix examples generated by plane curves in Euclidean 3-space.

1 Introduction

In [3], A slant helix in Euclidean space $\mathbb{R}^3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that γ is a slant helix in $\mathbb{R}^3$ if and only if the geodesic curvature of the principal normal of a space curve γ is a constant function.

In [5], Kula and Yayli have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix.

In [4], Kula, Ekmekci, Yayli and Ilarslan have studied the relationship between the plane curves and slant helices in $\mathbb{R}^3$. They obtained that the differential equations which are characterizations of a slant helix.

In this paper we consider the relationship between the plane curves and slant helices in $\mathbb{R}^3$. Moreover, we get slant helix from plane curve. Also, we give some slant helix examples in Euclidean 3-space.

2 Preliminaries

We now recall some basic concept on classical geometry of space curves and the definition of slant helix in $\mathbb{R}^3$. A curve $\tilde{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^3$, with unit speed, is a space curve. $T(s) = \tilde{\gamma}'(s)$ is a unit tangent vector of $\tilde{\gamma}$ at s. We define the curvature of $\tilde{\gamma}$ by $\kappa = \|\tilde{\gamma}''\|$. If $\kappa(s) \neq 0$, then the unit principal normal vector $N(s)$ of the curve $\tilde{\gamma}$ at s is given by $\tilde{\gamma}''(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \wedge N(s)$ is called the unit binormal vector of $\tilde{\gamma}$ at s. For the derivatives of the Frenet frame the Serret-Frenet formula hold:

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s), \\ B'(s) &= -\tau(s)N(s), \end{aligned} \tag{2.1}$$

where $\tau(s)$ is the torsion of the curve $\tilde{\gamma}$ at s.

Definition 2.1. A curve $\tilde{\gamma}$ with $\kappa(s) \neq 0$ is called a slant helix if the principal normal vector line of $\tilde{\gamma}$ make a constant angle with a fixed direction [3].

Theorem 2.2. *$\tilde{\gamma}$ is a slant helix if and only if the geodesic curvature of the spherical image of the principal normal indicatrix (N) of $\tilde{\gamma}$*

$$\kappa_g(s) = \left(\frac{\kappa^2}{\nu(\kappa^2+\tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)' \right)(s) \tag{2.2}$$

is a constant function and $\nu = \|\tilde{\gamma}'\|$. Also, axis of the slant helix is

$$\vec{a} = \pm \frac{\tau \sin\theta}{(\kappa^2+\tau^2)^{\frac{1}{2}}} T + \cos\theta N \pm \frac{\kappa \sin\theta}{(\kappa^2+\tau^2)^{\frac{1}{2}}} B \tag{2.3}$$

[3].

Theorem 2.3. *Let $\tilde{\gamma} : I \to \mathbb{R}^3$ be a space curve and $\beta : I \to \mathbb{R}^3$ be a space curve such that $\beta' = \frac{\gamma'}{\|\gamma'\|}$.*
$\tilde{\gamma}$ is a slant helix if and only if β is a slant helix.

Proof. If we use that

$$\begin{aligned} \|\gamma'\|\kappa_\gamma &= \kappa_\beta, \\ \|\gamma'\|\tau_\gamma &= \tau_\beta. \end{aligned}$$

and by simple calculation, we show that β is a slant helix and thus the proof of theorem 2.3 is completed. □

Now we can give the following theorem.

Theorem 2.4. *Curves $\tilde{\gamma}$ and β have the same axis.*

Proof. Let $\tilde{\gamma}$ be a slant helix with Frenet frame $\{T_{\tilde{\gamma}}, N_{\tilde{\gamma}}, B_{\tilde{\gamma}}\}$, curvature $\kappa_{\tilde{\gamma}}$ and torsion $\tau_{\tilde{\gamma}}$. β is a slant helix with Frenet frame $\{T_\beta, N_\beta, B_\beta\}$, curvature κ_β and torsion τ_β. By simple calculation, we get

$$\begin{aligned} \vec{a}_\beta &= \pm \frac{\tau_\beta \sin\theta}{(\kappa_\beta^2 + \tau_\beta^2)^{\frac{1}{2}}} T_\beta + \cos\theta N_\beta \pm \frac{\kappa_\beta \sin\theta}{(\kappa_\beta^2 + \tau_\beta^2)^{\frac{1}{2}}} B_\beta \\ &= \vec{a}_{\tilde{\gamma}}. \end{aligned}$$

Thus the proof of theorem 2.4 is completed. □

Definition 2.5. The epitrochoid is traced by a point P attached at a distance h to the center of a circle of radius r_2 rolling along another circle of radius r_1 and its equation is

$$\gamma(t) = ((r_1 + r_2)\sin t - h\sin((\frac{r_1 + r_2}{r_2})t), -(r_1 + r_2)\cos t + h\cos((\frac{r_1 + r_2}{r_2})t)) \tag{2.4}$$

[6].

Definition 2.6. The epicycloid is traced by a point P attached at a distance r_2 to the center of a circle of radius r_2 rolling along another circle of radius r_1 and its equation is

$$\gamma(t) = ((r_1 + r_2)\sin t - r_2\sin((\frac{r_1 + r_2}{r_2})t), -(r_1 + r_2)\cos t + r_2\cos((\frac{r_1 + r_2}{r_2})t)) \tag{2.5}$$

[6].

Definition 2.7. In definition 2.5 if we choose $r_1 = r_2$, then we get special case of epitrochoid curve. So we call this curve as limacon curve.

$$\gamma(t) = ((2r_1 + 2h\cos t)\cos t, (2r_1 + 2h\cos t)\sin t) \tag{2.6}$$

[6].

Definition 2.8. If we choose $r_1 = r_2$ in equation (2.5), then the epicycloid curve is called a cardioid curve with equation

$$\gamma(t) = (2r_1\cos t - r_1\cos(2t), 2r_1\sin t - r_1\sin(2t)). \tag{2.7}$$

Definition 2.9. If we choose $r_1 = 2r_2$ in equation (2.5), then the epicycloid curve is called a nephroid curve with equation

$$\gamma(t) = (3r_2\cos t - r_2\cos(3t), 3r_2\sin t - r_2\sin(3t)). \tag{2.8}$$

3 Plane curves and slant helix

For a space curve, if the torsion always vanishes, then the curve is contained in a plane. In this case we denote the curve γ instead of $\tilde{\gamma}$ and the curvature κ_p instead of κ. Let $\tilde{\gamma}$ be a space curve with Frenet frame $\{T, N, B\}$ and with curvatures κ, τ in $\mathbb{R}^3$. The curve γ is given by

$$\gamma = \widetilde{\gamma} - \langle \widetilde{\gamma}, \vec{a} \rangle \vec{a}, \tag{3.1}$$

where $\vec{a}$ is a constant vector.

Theorem 3.1. *Under the above notation, γ is a plane curve. Moreover, Frenet frame of the curve γ are*

$$\begin{aligned} \vec{t} &= \frac{1}{\sqrt{1 - \langle T, \vec{a} \rangle^2}} (T - \langle T, \vec{a} \rangle \vec{a}) \\ \vec{n} &= \frac{1}{\sqrt{1 - \langle T, \vec{a} \rangle^2}} (-\langle N, \vec{a} \rangle B + \langle B, \vec{a} \rangle N) \end{aligned} \tag{3.2}$$

and curvature of the curve γ is

$$\kappa_p = \frac{\kappa \sqrt{1 - \langle T, \vec{a} \rangle^2 - \langle N, \vec{a} \rangle^2}}{\sqrt{1 - \langle T, \vec{a} \rangle^2}}. \tag{3.3}$$

Proof. Differentiating the equation (3.1), we get

$$\begin{aligned} \gamma' &= \nu T - \nu \langle T, \vec{a} \rangle \vec{a} \\ \gamma'' &= \nu' T + \nu^2 \kappa N - \nu' \langle T, \vec{a} \rangle \vec{a} - \nu^2 \kappa \langle N, \vec{a} \rangle \vec{a}. \end{aligned} \tag{3.4}$$

and

$$\|\gamma'\|^2 = \nu^2 (1 - \langle T, \vec{a} \rangle^2). \tag{3.5}$$

Where $\nu = \|\tilde{\gamma}'\|$. Thus tangent vector of the curve γ is found as

$$\vec{t} = \frac{1}{\sqrt{1 - \langle T, \vec{a} \rangle^2}} (T - \langle T, \vec{a} \rangle \vec{a}). \tag{3.6}$$

We can calculate that

$$\gamma' \wedge \gamma'' = \nu^3 \kappa (B - \langle N, \vec{a} \rangle (T \wedge \vec{a}) + \langle T, \vec{a} \rangle (N \wedge \vec{a})) \tag{3.7}$$

and

$$\|\gamma' \wedge \gamma''\| = \nu^3 \kappa \sqrt{1 - \langle T, \vec{a} \rangle^2 - \langle N, \vec{a} \rangle^2}. \tag{3.8}$$

By using (3.7) and (3.8), we obtain that principal binormal vector of the curve γ is

$$\vec{b} = \frac{1}{1 - \langle N, \vec{a} \rangle^2 - \langle T, \vec{a} \rangle^2} (B - \langle N, \vec{a} \rangle (T \wedge \vec{a}) + \langle T, \vec{a} \rangle (N \wedge \vec{a})).$$

Differentiating the equation (3.4), we get

$$\begin{aligned} \gamma''' &= (\nu'' - \nu^3 \kappa^2) T + (3\nu' \nu \kappa + \nu^2 \kappa') N + \nu^2 \kappa \tau B \\ & \quad (\nu'' \langle T, \vec{a} \rangle + 3\nu' \nu \kappa \langle N, \vec{a} \rangle + \nu^2 \kappa' \langle N, \vec{a} \rangle - \nu^3 \kappa^2 \langle T, \vec{a} \rangle + \nu^3 \kappa \tau \langle B, \vec{a} \rangle) \vec{a}. \end{aligned}$$

Also

$$\begin{aligned} \det(\gamma', \gamma'', \gamma''') &= \nu^6 \kappa^2 \tau [1 - \langle T, \vec{a} \rangle^2 - \langle N, \vec{a} \rangle^2 - \langle B, \vec{a} \rangle^2] \\ &= 0. \end{aligned}$$

Then we know that

$$\vec{a} = \langle T, \vec{a} \rangle T + \langle N, \vec{a} \rangle N + \langle B, \vec{a} \rangle B. \tag{3.9}$$

By using (3.9), we get

$$\begin{aligned} T \wedge \vec{a} &= \langle N, \vec{a} \rangle B - \langle B, \vec{a} \rangle N \\ N \wedge \vec{a} &= -\langle T, \vec{a} \rangle B + \langle B, \vec{a} \rangle T \\ B \wedge \vec{a} &= \langle T, \vec{a} \rangle N - \langle N, \vec{a} \rangle T. \end{aligned} \tag{3.10}$$

Also by using (3.9) and (3.11), we obtain that

$$\vec{b} = \vec{a}.$$

If we calculate $\vec{n} = \vec{b} \wedge \vec{t}$, The principal normal vector is found as

$$\vec{n} = \vec{b} \wedge \vec{t} = \frac{1}{\sqrt{1 - \langle T, \vec{a} \rangle^2}} (-\langle N, \vec{a} \rangle B + \langle B, \vec{a} \rangle N).$$

Finally

$$\begin{aligned} \kappa_p &= \frac{\|\gamma' \wedge \gamma''\|}{\|\gamma'\|^3} \\ &= \frac{\kappa \sqrt{1 - \langle T, \vec{a} \rangle^2 - \langle N, \vec{a} \rangle^2}}{\sqrt{1 - \langle T, \vec{a} \rangle^2}} \end{aligned}$$

and

$$\tau_p = \frac{\det(\gamma', \gamma'', \gamma''')}{\|\gamma' \wedge \gamma''\|^2} = 0,$$

which means that γ is a plane curve. □

Corollary 3.2. *If space curve is a unit speed curves in theorem 3.1,*

$$\|\gamma'\| \leq 1.$$

Corollary 3.3. *Principal normal vector of the plane curve γ, defined by (3.1), is perpendicular to tangent vector of the space curve $\tilde{\gamma}$, i.e.*

$$\langle \vec{n}, T \rangle = 0.$$

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a plane curve.

$$\tilde{\gamma} = \gamma - (\tan\theta \int_{t_0}^{t} \frac{\langle \gamma', \gamma'' \rangle}{\|\gamma''\|} du)\vec{a} + \vec{c} \tag{3.11}$$

is a space curve. Where θ is a constant and $\vec{a}$, $\vec{c}$ are constant vectors with $\langle \gamma'(u), \vec{a} \rangle = 0$ and $\|\vec{a}\| = 1$.

Theorem 3.4. *Under the above notation, if $-\frac{\|\gamma''\|\sqrt{1-\|\gamma'\|^2}}{\langle \gamma', \gamma'' \rangle} = \tan\theta$ and $\|\gamma'\| < 1$, then $\tilde{\gamma}$ is a unit speed slant helix.*

Proof. Suppose that γ is a plane curve with Frenet frame $\{\vec{t}, \vec{n}\}$ and with curvature κ_p. In this case, we will show that $\langle N, \vec{a} \rangle = \cos\theta$.

Differentiating the equation (3.11), we get

$$\begin{aligned} \tilde{\gamma}' &= \|\gamma'\|\vec{t} - \tan\theta (\frac{\langle \gamma', \gamma'' \rangle}{\|\gamma''\|})\vec{a} \\ \tilde{\gamma}'' &= \|\gamma'\|'\vec{t} + \|\gamma'\|^2 \kappa_p \vec{n} - \frac{\|\gamma'\| \|\gamma'\|'}{\sqrt{1 - \|\gamma'\|^2}} \vec{a}. \end{aligned} \tag{3.12}$$

Since

$$\tan\theta = -\frac{\|\gamma''\|\sqrt{1 - \|\gamma'\|^2}}{\langle \gamma', \gamma'' \rangle}, \tag{3.13}$$

we can write

$$\sqrt{1-\|\gamma'\|^2} = -\frac{\tan\theta\langle\gamma',\gamma''\rangle}{\|\gamma''\|}. \tag{3.14}$$

By using equation (3.14) in equation (3.12), we obtain

$$\tilde{\gamma}'' = \|\gamma'\|'\vec{t} + \|\gamma'\|^2\kappa_p\vec{n} + \cot\theta\|\gamma''\|\vec{a}.$$

Then

$$\begin{aligned}\|\tilde{\gamma}'\| &= \sqrt{\|\gamma'\|^2 + \tan^2\theta\frac{\langle\gamma',\gamma''\rangle^2}{\|\gamma''\|^2}} \\ &= \sqrt{\|\gamma'\|^2 + \frac{\|\gamma''\|^2(1-\|\gamma'\|^2)}{\langle\gamma',\gamma''\rangle^2}\frac{\langle\gamma',\gamma''\rangle^2}{\|\gamma''\|^2}} \\ &= 1\end{aligned}$$

and

$$T = \|\gamma'\|\vec{t} + \sqrt{1-\|\gamma'\|^2}\vec{a}. \tag{3.15}$$

Also

$$\begin{aligned}\|\tilde{\gamma}''\| &= \sqrt{(\|\gamma'\|')^2 + \|\gamma'\|^4\kappa_p^2 + \cot^2\theta\|\gamma''\|^2} \\ &= \frac{\|\gamma''\|}{\sin\theta}.\end{aligned}$$

So

$$N = \frac{\sin\theta}{\|\gamma''\|}(\|\gamma'\|'\vec{t} + \|\gamma'\|^2\kappa_p\vec{n} + \cot\theta\|\gamma''\|\vec{a}) \tag{3.16}$$

and

$$B = \frac{\sin\theta}{\|\gamma''\|}(-\|\gamma'\|^2\sqrt{1-\|\gamma'\|^2}\kappa_p\vec{t} + (\|\gamma'\|'\sqrt{1-\|\gamma'\|^2} - \cot\theta\|\gamma'\|\|\gamma''\|)\vec{n} + \|\gamma'\|^3\kappa_p\vec{a}). \tag{3.17}$$

Moreover

$$\begin{aligned}\kappa &= \frac{\|\gamma''\|}{\sin\theta} \\ \tau &= \frac{\sqrt{1-\|\gamma'\|^2}}{\cos\theta}\end{aligned}$$

and

$$\kappa_g = \cot\theta. \tag{3.18}$$

Consequently

$$\langle N,\vec{a}\rangle = \frac{\sin\theta}{\|\gamma''\|}\cot\theta\|\gamma''\| = \cos\theta, \tag{3.19}$$

which means that $\tilde{\gamma}$ is a slant helix. □

Theorem 3.5. *Let $\tilde{\gamma}$ be a slant helix. The spherical image of the tangent indicatrix (T) of $\tilde{\gamma}$ is a spherical helix [5].*

Corollary 3.6. *We denote the curvatures of (T) of $\tilde{\gamma}$ generated by plane curve γ by κ_1, τ_1.*

$$\kappa_1 = \frac{\sin\theta}{\sqrt{\|\gamma'\|^2 - \cos^2\theta}} \tag{3.20}$$

and

$$\tau_1 = \frac{\cos\theta}{\sqrt{\|\gamma'\|^2 - \cos^2\theta}}. \tag{3.21}$$

We can calculate that

$$\frac{\tau_1}{\kappa_1} = \cot\theta.$$

$\cot\theta$ *is a constant.*

Theorem 3.7. *Let $\tilde{\gamma}$ be a slant helix. The spherical image of the binormal indicatrix (B) of $\tilde{\gamma}$ is a spherical helix [5].*

Corollary 3.8. *We denote the curvatures of (B) of $\tilde{\gamma}$ generated by plane curve γ by κ_2, τ_2.*

$$\kappa_2 = \frac{\sin\theta}{\sqrt{1 - \|\gamma'\|^2}} \tag{3.22}$$

and

$$\tau_2 = \frac{\cos\theta}{\sqrt{1 - \|\gamma'\|^2}}. \tag{3.23}$$

We can calculate that

$$\frac{\tau_2}{\kappa_2} = \cot\theta.$$

$\cot\theta$ *is a constant.*

Lemma 3.9. *Let $\gamma : I \to \mathbb{R}^3$ be an epitrochoid curve with*

$$\gamma(t) = ((r_1 + r_2)\sin t - h\sin((\frac{r_1 + r_2}{r_2})t), -(r_1 + r_2)\cos t + h\cos((\frac{r_1 + r_2}{r_2})t), 0). \tag{3.24}$$

For the curve γ, equation (3.13) is constant if and only if $r_1 + 2r_2 = 1$ and $h = \frac{r_2^2}{r_1+r_2}$.

Proof. From (3.11), if we calculate

$$\tan\theta = -\frac{\|\gamma''\|\sqrt{1 - \|\gamma'\|^2}}{\langle\gamma', \gamma''\rangle},$$

since $\tan\theta$ is a constant, we find that $r_1 + 2r_2 = 1$ and $h = \frac{r_2^2}{r_1+r_2}$. Moreover, we obtain that $\tan\theta = -\frac{2\sqrt{r_2(r_1+r_2)}}{r_1}$ is a constant.

Conversely, let $r_1 + 2r_2 = 1$ and $h = \frac{r_2^2}{r_1+r_2}$, then $\tan\theta$ is constant for the curve γ. Here $0 < r_1 < 1, 0 < h < r_2 < \frac{1}{2}$. □

As a result of lemma 3.9 we can give the following theorem.

Theorem 3.10. *Let $\gamma : I \to \mathbb{R}^3$ be an epitrochoid curve with*

$$\gamma(t) = ((r_1 + r_2)\sin t - h\sin((\frac{r_1 + r_2}{r_2})t), -(r_1 + r_2)\cos t + h\cos((\frac{r_1 + r_2}{r_2})t), 0),$$

where $r_1 + 2r_2 = 1$ and $h = \frac{r_2^2}{r_1+r_2}$. Space curve $\tilde{\gamma}$ generated by the epitrochoid curve is a slant helix.

For $r_1 + 2r_2 = 1$ and $h = \frac{r_2^2}{r_1+r_2}$, slant helix $\tilde{\gamma}$ generated by the epitrochoid curve is

$$\begin{aligned}\tilde{\gamma}(t) \;=\; & ((r_1 + r_2)\sin t - \frac{{r_2}^2}{r_1 + r_2}\sin((\frac{r_1 + r_2}{r_2})t), (r_1 + r_2)\cos t + \frac{r_2^2}{r_1 + r_2}\cos((\frac{r_1 + r_2}{r_2})t),\\ & - \frac{4r_2}{r_1}\sqrt{r_2(r_1 + r_2)}\sin((\frac{r_1}{2r_2})t))\end{aligned}$$

and it is on hyperboloid of one sheet with equation

$$\frac{x^2}{\frac{r_1^2}{(r_1+r_2)^2}} + \frac{y^2}{\frac{r_1^2}{(r_1+r_2)^2}} - \frac{z^2}{\frac{4r_2}{r_1+r_2}} = 1. \tag{3.25}$$

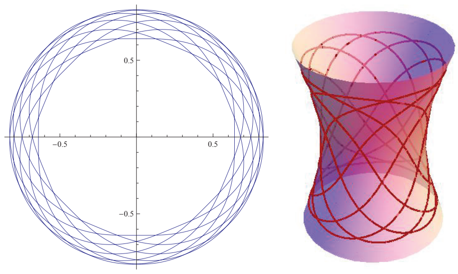

Figure 1. The epitrochoid curve and the slant helix generated by epitrochoid for $r_1 = \frac{16}{34}$, $r_2 = \frac{9}{34}$.

Example 3.11. For $r_1 = \frac{16}{34}$, $r_2 = \frac{9}{34}$ and $h = \frac{81}{250}$, the equation of curve γ is

$$\gamma(t) = (\frac{25}{34}\sin t - \frac{81}{850}\sin((\frac{25}{9})t), -\frac{25}{34}\cos t + \frac{81}{850}\cos((\frac{25}{9})t), 0)$$

and it is rendered in figure 9

If we calculate $\tan\theta$, we find as $\tan\theta = -\frac{15}{8}$ i.e. γ satisfies equation (3.13). Therefore, space curve generated by the epitrochoid is a slant helix, which is rendered in figure 9 and its equation is

$$\tilde{\gamma}(t) = (\frac{25}{34}\sin t - \frac{81}{850}\sin((\frac{25}{9})t), -\frac{25}{34}\cos t + \frac{81}{850}\cos((\frac{25}{9})t), -\frac{135}{136}\sin((\frac{8}{9})t)).$$

Moreover, the geodesic curvature of the principal normal indicatrix (N) of $\tilde{\gamma}$ is $\cot\theta = -\frac{8}{15}$

Corollary 3.12. *Frenet Frame $\{T, N, B\}$, curvature κ and torsion τ of slant helix $\tilde{\gamma}$ generated by the epitrochoid curve are*

$$\begin{aligned}
T(t) &= ((r_1 + r_2)\cos t - r_2\cos((\frac{r_1 + r_2}{r_2})t), (r_1 + r_2)\sin t - r_2\sin((\frac{r_1 + r_2}{r_2})t), \\
&\quad -2\sqrt{r_2(r_1 + r_2)}\cos(\frac{r_1}{2r_2}t)), \\
N(t) &= (\sqrt{r_2(r_1 + r_2)}\csc(\frac{r_1}{2r_2}t)(-\sin t + \sin((\frac{r_1 + r_2}{r_2})t)), \\
&\quad \sqrt{r_2(r_1 + r_2)}\csc(\frac{r_1}{2r_2}t)(\cos t - \cos((\frac{r_1 + r_2}{r_2})t)), r_1), \\
B(t) &= ((r_1 + r_2)\sin t + r_2\sin((\frac{r_1 + r_2}{r_2})t), -(r_1 + r_2)\cos t - r_2\cos((\frac{r_1 + r_2}{r_2})t), \\
&\quad 2\sqrt{r_2(r_1 + r_2)}\sin(\frac{r_1}{2r_2}t)),
\end{aligned}$$

$$\begin{aligned}
\kappa(t) &= \frac{\sqrt{r_2(r_1 + r_2)}}{r_2}\sin(\frac{r_1}{2r_2}t), \\
\tau(t) &= \frac{\sqrt{r_2(r_1 + r_2)}}{r_2}\cos(\frac{r_1}{2r_2}t).
\end{aligned}$$

4 Applications

Let $n = \frac{r_1}{r_2}$,

(i) If n is an integer, then the curve is closed, and has n cusps.

(ii) If n is a rational number, say $n = \frac{p}{q}$ expressed in simplest terms, then the curve has p cusps.

(iii) If n is an irrational number, then the curve never closes.

(i) For $n = 1$ ($r_1 = \frac{1}{3}, r_2 = \frac{1}{3}$), Since n is an integer, epitrochoid is simple closed curve. This epitrochoid is special case of limacon curve and also it is speed vector of cardioid curve. Slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 2, respectively.

(ii) For $n = 2$ ($r_1 = \frac{1}{2}, r_2 = \frac{1}{4}$), Since n is an integer, epitrochoid is simple closed curve and also it is speed vector of nephroid curve. Slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 3, respectively.

(iii) For $n = 3$ ($r_1 = \frac{3}{5}, r_2 = \frac{1}{5}$), Since n is an integer, epitrochoid is simple closed curve. Slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 4, respectively.

(iv) For $n = 4$ ($r_1 = \frac{4}{6}, r_2 = \frac{1}{6}$), Since n is an integer, epitrochoid is simple closed curve. Slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 5, respectively.

(v) For $n = 5$ ($r_1 = \frac{5}{7}, r_2 = \frac{1}{5}$), Since n is an integer, epitrochoid is simple closed curve. Slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 6, respectively.

(vi) For $n = \frac{1}{2}$ ($r_1 = \frac{1}{5}, r_2 = \frac{2}{5}$), Since n is an rational, epitrochoid is closed curve. Slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 7, respectively.

(vii) For $n = \frac{1}{3}$ ($r_1 = \frac{1}{7}, r_2 = \frac{3}{7}$), Since n is an rational. epitrochoid is closed curve, slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 8, respectively.

(viii) For $n = \frac{1}{4}$ ($r_1 = \frac{1}{9}, r_2 = \frac{4}{9}$), Since n is an rational. epitrochoid is closed curve, slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and closed curve. Spherical indicatricies of the slant helix lie on the unit sphere and they are also closed curves, which is rendered in Figure 9, respectively.

(ix) For $n = \frac{2\sqrt{5}-4}{3-\sqrt{5}}$ ($r_1 = \sqrt{5} - 2, r_2 = \frac{3-\sqrt{5}}{2}$), Since n is an irrational number, epitrochoid never closes, slant helix generated by the epitrochoid curve is on hyperboloid of one sheet and spherical indicatricies of the slant helix lie on the unit sphere, which is rendered in Figure 10, respectively.

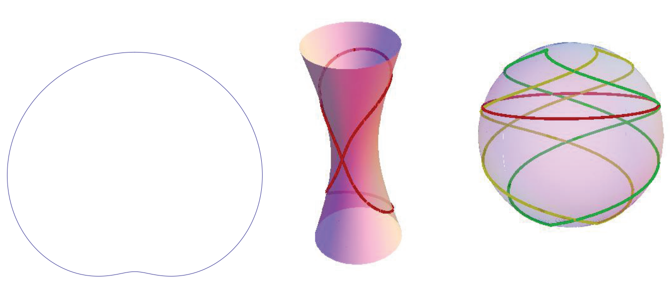

Figure 2. For $n = 1$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

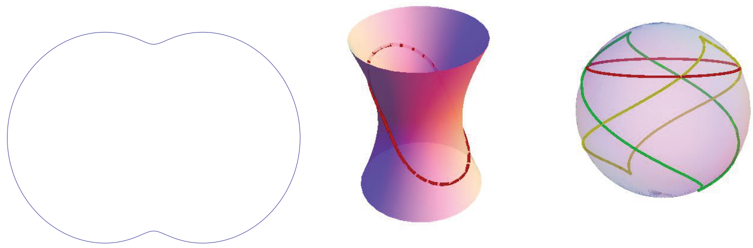

Figure 3. For $n = 2$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

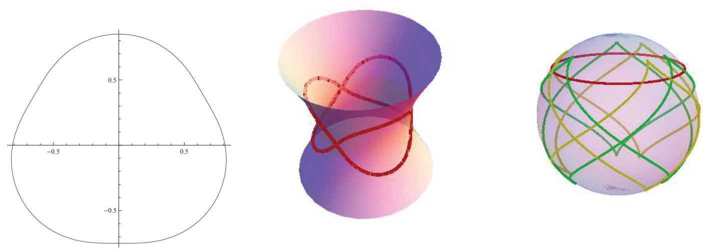

Figure 4. For $n = 3$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

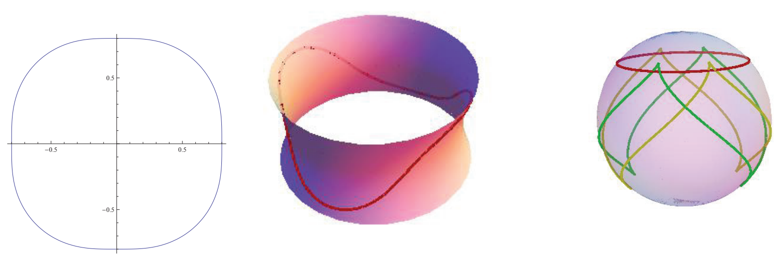

Figure 5. For $n = 4$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

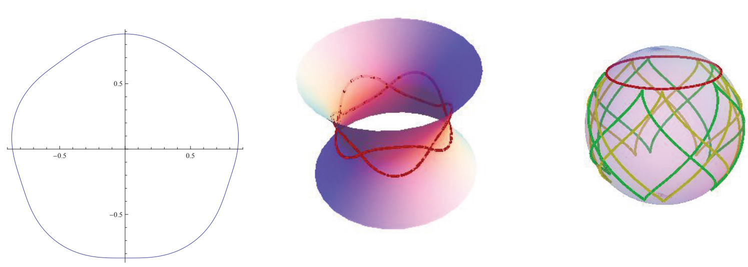

Figure 6. For $n = 5$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

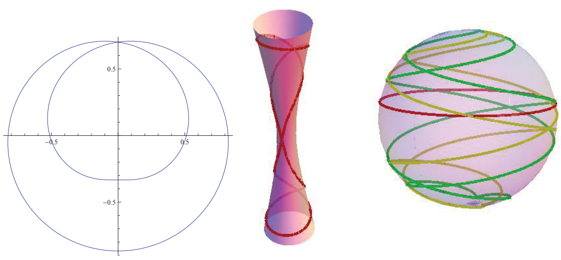

Figure 7. For $n = \frac{1}{2}$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

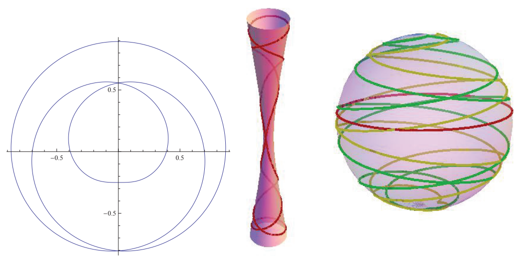

Figure 8. For $n = \frac{1}{3}$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

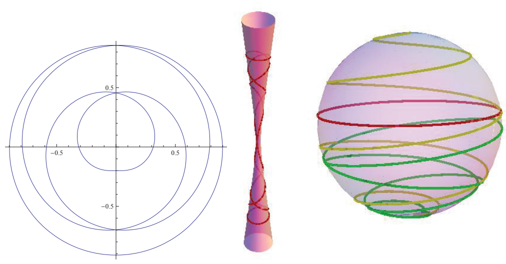

Figure 9. For $n = \frac{1}{4}$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

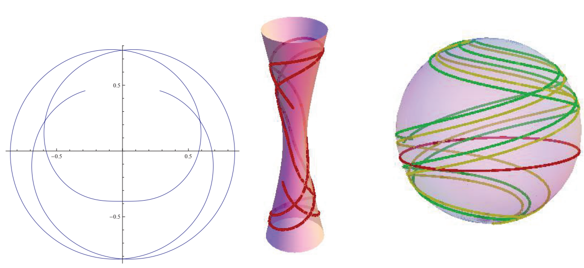

Figure 10. For $n = \frac{2\sqrt{5}-4}{3-\sqrt{5}}$, the epitrochoid, the slant helix generated by the epitrochoid curve and its spherical indicatricies (T, N, B) on unit sphere.

References

[1] Altınok, M., The characterization of some special space curves with plane curves, Master thesis, Ahi Evran University, Graduate School of Natural and Applied Sciences, (2011).

[2] Izumiya, S. and Takeuchi, N., Generic properties of helices and Bertrand curves, Mathematics Subject

classification, 74(2000), 97-109.

[3] Izumiya, S. and Tkeuchi, N., New special curves and developable surfaces, Turk J. Math 28 (2004), 153-163.

[4] Kula, L., Ekmekci, N., Yayli, Y. and Ilarslan, K., Characterizations of slant helices in Euclidean 3-space. Turkish J. Math., 34(2010), 261-274.

[5] Kula, L. and Yayli, Y., On slant helix and its spherical indicatrix. Applied Mathematics and Computation, 169(2005), 600-607.

[6] Lawrence, J. D., A catalog of special plane curves, Dover Publications, 1972.

Author information

Mesut Altınok and Levent Kula, Ahi Evran University, Faculty of Sciences and Arts
Department of Mathematics, Kirsehir, Turkey.
E-mail: altnokmesut@gmail.com lkula@ahievran.edu.tr

Received: December 22, 2014.

Accepted: June 6, 2015.