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PERMUTABILITY GRAPH OF CYCLIC SUBGROUPS

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Abstract. Let G be a group. *The permutability graph of cyclic subgroups of* G, denoted by $\Gamma_c(G)$, is a graph with all the proper cyclic subgroups of G as its vertices and two distinct vertices in $\Gamma_c(G)$ are adjacent if and only if the corresponding subgroups permute in G. In this paper, we classify the finite groups whose permutability graph of cyclic subgroups belongs to one of the following: bipartite, tree, star graph, triangle-free, complete bipartite, P_n, C_n, K_4, $K_{1,3}$-free, unicyclic. We classify abelian groups whose permutability graph of cyclic subgroups are planar. Also we investigate the connectedness, diameter, girth, totally disconnectedness, completeness and regularity of these graphs.

1 Introduction

The properties of a group can be studied by assigning a suitable graph to it and by analyzing the properties of the associated graphs using the tools of graph theory. The Cayley graph is a well known example of a graph associated to a group, which have been studied extensively in the literature (see, for example, [9, 14]). In the past twenty five years many authors have assigned various graphs to study some specific properties of groups . For instance, see [1, 8, 12, 16].

Recall that two subgroups H and K of a group G are said to *permute* if $HK = KH$; equivalently HK is a subgroup of G. In [2], Aschbacher defined a graph corresponding to a group G and for a fixed prime p, having all the subgroups of order p as its vertices and two vertices are adjacent if they permute. To study the transitivity of permutability of subgroups, Bianchi, Gillio and Verardi in [3], defined a graph corresponding to a group G, called the *permutability graph of non-normal subgroups of* G, having all the proper non-normal subgroups of G as its vertices and two vertices are adjacent if they permute (see, also in [4, 10]). In [19], the authors considered the generalized case of this graph, called *the permutability graph of subgroups of* G, denoted by $\Gamma(G)$, having the vertex set consisting of all proper subgroups of G and two vertices are adjacent if they permute.

In [5, p.14], Ballester-Bolinches *et al* introduced a graph corresponding to a group G, having all the cyclic subgroups of G as it vertices and two vertices are adjacent if they permute. In this paper, as a particular case, we consider a graph, denoted by $\Gamma_c(G)$ with vertex set consists of all proper cyclic subgroups of G and two vertices are adjacent if they permute. We will call this graph as *the permutability graph of cyclic subgroups of* G. By investigating the properties of this graph, we study the permutability of cyclic subgroups of the corresponding group. Especially, Theorems 3.9, 3.14 and 4.6, Corollaries 3.10 and 3.11 in this paper are some of the main applications for group theory.

Now we introduce some notion from graph theory that we will use in this article. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. G is said to be *complete* if any two of its vertices are adjacent. A complete graph with n vertices is denoted by K_n. G is *bipartite* if $V(G)$ is the union of two disjoint sets X and Y such that no two vertices in the same subset are adjacent. Here X and Y are called a *bipartition* of G. A bipartite graph G with bipartition X and Y is called *complete bipartite* if every vertex in X is adjacent with every vertex in Y. If $|X| = m$ and $|Y| = n$, then the corresponding graph is denoted by $K_{m,n}$. In particular, $K_{1,n}$ is called the *star graph* and $K_{1,3}$ is called the *claw graph*. A graph is *planar* if it can be drawn in a plane so that no two edges intersect except possibly at vertices. The *degree* of the vertex v in G is the number of edges incident with v and is denoted by $\deg_G(v)$. A graph is said to be *regular* if degrees of all the vertices are same. A *path* joining two vertices u and v in G is a finite sequence $(u =)v_0, v_1, \ldots, v_n(= v)$ of distinct vertices, except, possibly, u and v such that u_i is adjacent with u_{i+1}, for all $i = 0, 1, \ldots, n-1$. A path joining u and v is a cycle if $u = v$. The length of a path or cycle is the number of edges in it. A path or cycle of length n is denoted by P_n or C_n respectively. A graph with exactly one cycle is said to be *unicyclic*. A graph is a *tree* if it has no cycles. The *girth* of a graph G is the length of the smallest cycle in it and is denoted by $girth(G)$.

A graph is said to be *connected* if every pair of distinct vertices can be joined by a path. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the length of the shortest path between them, and $d(u, v) = 0$ if $u = v$. If there exists no path between them, then we define $d(u, v) = \infty$. The *diameter* of G, denoted by $diam(G)$ is the maximum distance between any two vertices in the graph. An *isomorphism* of graphs G_1 and G_2 is an edge-preserving bijection between the vertex sets of G_1 and G_2. G is said to be H*-free* if G has no subgraph isomorphic to H. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. Their *union* $G_1 \cup G_2$ is a graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. Their *join* $G_1 + G_2$ is a graph consist of $G_1 \cup G_2$ together with all the lines joining points of V_1 to points of V_2. For any connected graph G, we write nG for the graph with n components each isomorphic to G. For basic graph theory terminology, we refer to [11].

The dihedral group of order $2n$, $n \geq 3$ is defined by $D_{2n} = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle$. For any integer $n \geq 2$, the generalized Quaternion group of order $4n$ is given by $Q_{4n} = \big\langle a, b \mid a^{2n} = b^4 = 1, a^n = b^2 = 1, bab^{-1} = a^{-1} \big\rangle$. The modular group of order p^α, $\alpha \geq 3$ is given by $M_{p^\alpha} = \langle a, b \mid a^{p^{\alpha-1}} = b^p = 1, bab^{-1} = a^{p^{\alpha-2}+1} \rangle$. For an integer $n \geq 1$, S_n and A_n denotes the symmetric group and alternating group of degree n acting on $\{1, 2, \ldots, n\}$ respectively. If n is a any positive integer, then $\tau(n)$ denotes the number of positive divisors of n. We denote the order of an element $a \in \mathbb{Z}_n$ by $\mathrm{ord}_n(a)$. The number of Sylow p-subgroups of a group G is denoted by $n_p(G)$; or simply by n_p if there is no ambiguity.

The rest of the paper is arranged as follows: In Section 2, we study some basic properties of permutability graph of cyclic subgroups of groups.

Section 3 gives the classification of finite groups whose permutability graphs of cyclic subgroups are one of the following: bipartite, tree, star graph, triangle-free, complete bipartite, P_n, C_n, K_4, $K_{1,3}$-free, unicyclic. We estimate the girth of the permutability graphs of cyclic subgroups of finite groups. We also characterize the groups having totally disconnected permutability graphs of cyclic subgroups.

In Section 4, we investigate connectedness, diameter, regularity, completeness of the permutability graph of cyclic subgroups of a given group. Also we classify abelian groups whose permutability graph of cyclic subgroups are planar. We characterize the groups Q_8, S_3 and A_4 by using their permutability graph of cyclic subgroups. Moreover, we pose some open problems in this section.

We recall the following theorem, which we will use in the subsequent sections.

Theorem 1.1. ***([19, Corollary 5.1])*** *Let G be a finite group and p, q be distinct primes. Then*

(i) $\Gamma(G)$ is C_n if and only if $n = 3$ and G is either $\mathbb{Z}_{p^4}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$;

(ii) $\Gamma(G)$ is P_n if and only if $n = 1$ and G is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$;

(iii) $\Gamma(G)$ is claw-free if and only if G is either $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3, 4)$ or $\mathbb{Z}_{pq}$.

2 Some basic results

Note that the only groups having no proper cyclic subgroups are the trivial group, and the groups of prime order, so it follows that, we can define $\Gamma_c(G)$ only when the group G is not isomorphic to either of these groups.

In this section, we study some basic properties about permutability graph of cyclic subgroups of a given group. We start with the following result whose proof is immediate.

Lemma 2.1. *Let G be a group. If G has r proper cyclic subgroups, which are permutes with each other, then $\Gamma_c(G)$ has K_r as a subgraph.*

Theorem 2.2. *Let G_1 and G_2 be two groups. If $G_1 \cong G_2$, then $\Gamma_c(G_1) \cong \Gamma_c(G_2)$.*

Proof. Let $f : G_1 \to G_2$ be a group isomorphism. Define a map $\psi : V(\Gamma_c(G_1)) \to V(\Gamma_c(G_2))$ by $\psi(H) = f(H)$, for every $H \in V(\Gamma_c(G_1))$. Then it is easy to see that ψ is a graph isomorphism. □

Remark 2.3. The converse of Theorem 2.2 is not true. For example, consider the non-isomorphic groups $G_1 = \mathbb{Z}_{p^5}$, where p is a prime and $G_2 = \mathbb{Z}_3 \times \mathbb{Z}_3$. Here G_1 has subgroups $\mathbb{Z}_{p^i}$, $i = 1, 2, 3, 4$ and G_2 has proper cyclic subgroups $\langle(1, 0)\rangle$, $\langle(x, 1)\rangle$, $x = 0, 1, 2$. It follows that $\Gamma_c(G_1) \cong K_4 \cong \Gamma_c(G_2)$.

Theorem 2.4. *If G is a group and N is a subgroup of G, then $\Gamma_c(N)$ is a subgraph of $\Gamma_c(G)$.*

3 Some classification related results for $\Gamma_c(G)$

The aim of this section is to classify the solvable groups whose permutability graphs of cyclic subgroups are one of the following: bipartite, complete bipartite, tree, star graph, C_3-free, C_n, K_4, P_n, $K_{1,3}$-free, unicyclic. First we consider the finite groups and then we deal with the infinite groups.

3.1 Finite abelian groups

Proposition 3.1. *Let G be a finite abelian group and p, q be distinct primes. Then*

(i) $\Gamma_c(G)$ *is C_3-free if and only if G is either $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3)$ or $\mathbb{Z}_{pq}$;*

(ii) $\Gamma_c(G)$ *is bipartite if and only if it is C_3-free;*

(iii) $\Gamma_c(G)$ *is C_n if and only if $n = 3$ and G is either $\mathbb{Z}_{p^4}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$;*

(iv) $\Gamma_c(G)$ *is P_n if and only if $n = 1$ and G is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$;*

(v) $\Gamma_c(G)$ *is K_4 if and only if G is one of $\mathbb{Z}_{p^5}$, $\mathbb{Z}_{p^2q}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$;*

(vi) $\Gamma_c(G)$ *is claw-free if and only if G is one of $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3, 4)$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$;*

(vii) $\Gamma_c(G)$ *is unicyclic if and only if G is either $\mathbb{Z}_{p^4}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$ for every $i = 1, 2, \ldots k$. We divide the proof into two cases.
Case 1: If G is cyclic, then $\Gamma_c(G) \cong \Gamma(G)$. So in view of this fact and by the proof of [19, Theorem 3.1], we have

$$\Gamma_c(G) \cong K_r, \tag{3.1}$$

where r is the number of proper subgroups of G, which is given by $r = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1) - 2$. It follows that $\Gamma_c(G) \cong K_4$ if and only if G is one of $\mathbb{Z}_{p^5}$ or $\mathbb{Z}_{p^2q}$. Furthermore, $\Gamma_c(G)$ is bipartite or C_3-free if and only if G is either $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3)$ or $\mathbb{Z}_{pq}$. Note that the bipartiteness and C_3-freeness of permutability graphs of finite cyclic groups were proved in [20, Proposition 3.1 and corollary 3.1]. We repeated them here for the sake of completeness. Also by Theorem 1.1, we have

(i) $\Gamma_c(G)$ is C_n if and only if $n = 3$ and $G \cong \mathbb{Z}_{p^4}$.

(ii) $\Gamma_c(G)$ is P_n if and only if $n = 1$ and G is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$.

(iii) $\Gamma_c(G)$ is claw-free if and only if G is one of $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3, 4)$, $\mathbb{Z}_{pq}$.

Case 2: If G is non-cyclic, then we have the following cases to consider:
Subcase 2a: $k = 1$. If $\alpha_1 > 2$, then G has a subgroup isomorphic to either $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, for some prime p. It is easy to see that these groups have at least five proper cyclic subgroups, so they form K_5 as a subgraph of $\Gamma_c(G)$. If $\alpha_1 = 2$, then $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$, for some

prime p. But the number of nontrivial subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ is $p+1$; they are $\langle(1,0)\rangle$, $\langle(a,1)\rangle$, for each $a \in \{0,1,2,\ldots,p-1\}$. Thus, by Lemma 2.1,

$$\Gamma_c(G) \cong K_{p+1}. \tag{3.2}$$

Therefore, $\Gamma(G_1)$ contains C_3 as a subgraph; it is C_3 if and only if $p=2$; it is K_4 if and only if $p=3$; it is claw-free if and only if $p=2$.

Subcase 2b: $k>1$. If $\alpha_i>1$ for some i, then G has a subgroup H isomorphic to $\mathbb{Z}_{pq} \times \mathbb{Z}_p$, for some distinct primes p and q. It is easy to see that H has at least five proper cyclic subgroups, so they form K_5 as a subgraph of $\Gamma_c(G)$.

The proof follows by combining these cases. □

3.2 Finite non-abelian groups

Proposition 3.2. *Let G be a non-abelian of order p^α, where p is a prime and $\alpha \geq 3$. Then $\Gamma_c(G)$ contains C_3 and $K_{1,3}$ as proper subgraphs; $\Gamma_c(G) \cong K_4$ if and only if $G \cong Q_8$.*

Proof. We first prove this result when $\alpha=3$. According to the Burnside [7], up to isomorphism there are only four non-abelian groups of order p^3, where p is a prime, namely Q_8, M_8, M_{p^α} and $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $p>2$. If $G \cong Q_8$, then by [19, Theorem 4.3], we have

$$\Gamma_c(G) \cong K_4. \tag{3.3}$$

If $G \cong M_8$, then $H_1 := \langle a\rangle$, $H_2 := \langle a^2\rangle$, $H_3 := \langle b\rangle$, $H_4 := \langle ab\rangle$, $H_5 := \langle a^2b\rangle$ are proper cyclic subgroups of G, so $|V(\Gamma_c(G))| \geq 5$. Since H_1, H_2 are normal in G, they permutes with all the subgroups of G. Thus, $\Gamma_c(G)$ has C_3 as a subgraph induced by the vertices H_1, H_2, H_3; but it is not K_4 as it has five vertices. Also $K_{1,3}$ is a subgraph of $\Gamma_c(G)$ with bipartition $X := \{H_1\}$ and $Y := \{H_2, H_3, H_4\}$. If $G \cong M_{p^\alpha}$, where p is a prime and $p>2$, then $H_1 := \langle a\rangle$, $H_2 := \langle ab\rangle$, $H_3 := \langle ab^2\rangle$, $H_4 := \langle b\rangle$, $H_5 := \langle a^p\rangle$ are proper cyclic subgroups of G, so $|V(\Gamma_c(G))| \geq 5$. Here any two subgroups of G permutes, so K_5 is a subgraph of $\Gamma_c(G)$. If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, then $\mathbb{Z}_p \times \mathbb{Z}_p$ is a subgroup of G and since $p>2$, so by (3.2), $\Gamma_c(G)$ contains K_4 as a proper subgraph. Clearly $|V(\Gamma_c(G))| \geq 5$.

Now we prove this result when $\alpha \geq 4$. We need to consider the following two cases:

Case 1: $G \cong Q_{2^\alpha}$. Then G has two subgroups each isomorphic to Q_8, so in the view of (3.3), $\Gamma_c(G)$ contains C_3 and $K_{1,3}$ as proper subgraphs. Also G has at least five proper cyclic subgroups, so $|V(\Gamma_c(G))| \geq 5$.

Case 2: $G \ncong Q_{2^\alpha}$. By [22, Proposition 1.3], the number of subgroups of order p of G is not unique and so by [7, Theorem IV, p.129], G has at least three subgroups, say H_i, $i=1,2,3$ of order p; also it has a subgroup, say H of order p^3. Suppose $\Gamma_c(H)$ contains C_3 and $K_{1,3}$; also $|V(\Gamma_c(H))| \geq 5$, then $\Gamma_c(G)$ also has the same. So by Propositions 3.1 and 3.2, the only cases remains to check are $H \cong \mathbb{Z}_{p^3}$ or Q_8. If $H \cong \mathbb{Z}_{p^3}$, then by (3.2), $\Gamma_c(H) \cong K_2$, so H together with its subgroups forms C_3 as a subgraph of $\Gamma_c(G)$. The cyclic subgroups of H together with the subgroups H_i's make $|V(\Gamma_c(G))| \geq 5$. By [7, Corollary of Theorem IV, p.129], G has a normal subgroup of order p, without loss of generality, say H_1. Then $K_{1,3}$ is a subgraph of $\Gamma_c(G)$ with bipartition $X := \{H_1\}$ and $Y := \{H, H_2, H_3\}$. If $H \cong Q_8$, then by (3.3), $\Gamma_c(H) \cong K_4$. Also the cyclic subgroups of H together with H_i's also make $|V(\Gamma_c(G))| \geq 5$.

The proof follows by combining all the above cases. □

Proposition 3.3. *Let G be the non-abelian group of order pq, where p, q are distinct primes and $p<q$. Then $\Gamma_c(G) \cong K_{1,q}$.*

Proof. We have $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. Here every subgroup of G is cyclic, so $\Gamma_c(G) \cong \Gamma(G)$. By the proof of Theorem 4.4 in [19], we have

$$\Gamma_c(G) \cong K_{1,q}. \tag{3.4}$$

This completes the proof. □

Consider the semi-direct product $\mathbb{Z}_q \rtimes_t \mathbb{Z}_{p^\alpha} = \langle a, b | a^q = b^{p^\alpha} = 1, bab^{-1} = a^i, ord_q(i) = p^t \rangle$, where p and q are distinct primes with $p^t \mid (q-1)$, $t \geq 0$. Then every semi-direct product $Z_q \rtimes Z_{p^\alpha}$ is one of these types [6, Lemma 2.12]. In the future, when $t = 1$ we will suppress the subscript.

Proposition 3.4. *Let G be a non-abelian group of order p^2q, where p, q are distinct primes. Then $\Gamma_c(G)$ contains C_3 as a proper subgraph; it is $K_{1,3}$-free if and only if $G \cong A_4$; it has at least five vertices.*

Proof. Here we use the classification of groups of order p^2q given in [7, p. 76-80]. We have the following cases to consider:
Case 1: $p < q$:
Case 1a: $p \nmid (q-1)$. By Sylow's Theorem, it is easy to see that there is no non-abelian group in this case.
Case 1b: $p \mid (q-1)$, but $p^2 \nmid (q-1)$. In this case, there are two non-abelian groups.

The first group is $G_1 := \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, ord_q(i) = p \rangle$. It has $H_1 := \langle a \rangle$, $H_2 := \langle ab^p \rangle$, $H_3 := \langle b \rangle$, $H_4 := \langle b^p \rangle$, $H_5 := \langle ab \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_1))| \geq 5$. Here H_1 and H_2 are normal in G, so they permutes with all the subgroups of G; H_4 is a subgroup of H_3 and H_5. So K_4 is a subgraph of $\Gamma_c(G_1)$ induced by H_i, $i = 1, 2, 3, 4$.

The second group in this case is $G_2 := \langle a, b, c \mid a^q = b^p = c^p = 1, bab^{-1} = a^i, ca = ac, cb = bc, ord_q(i) = p \rangle$. It has $H_1 := \langle a \rangle$, $H_2 := \langle b \rangle$, $H_3 := \langle c \rangle$, $H_4 := \langle bc \rangle$, $H_5 := \langle ab \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_2))| \geq 5$. Here H_3 permutes with all the subgroups of G_2; H_2, H_3, H_4 permutes with each other. So C_3 is a subgraph of $\Gamma_c(G_2)$ induced by the vertices H_2, H_3, H_4; and $K_{1,3}$ is a subgraph of $\Gamma_c(G_2)$ with bipartition $X := \{H_3\}$ and $Y := \{H_1, H_2, H_3\}$.
Case 1c: $p^2 \mid (q-1)$. In this case, we have both groups G_1 and G_2 from Case 1b together with the group $G_3 := \mathbb{Z}_q \rtimes_2 \mathbb{Z}_p = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, ord_q(i) = p^2 \rangle$. But in Case 1b, we already dealt with G_1 and G_2. Now we consider G_3. It has $H_1 := \langle a \rangle$, $H_2 := \langle b \rangle$, $H_3 := \langle b^p \rangle$, $H_4 := \langle ab \rangle$, $H_5 := \langle a^2 b \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_3))| \geq 5$. Since H_1 is normal in G_3, it permutes with all the subgroups of G_3; H_3 is a subgroup of H_2. So C_3 is a subgraph of $\Gamma_c(G)$ induced by H_1, H_2, H_3 and $K_{1,3}$ is a subgraph of $\Gamma_c(G_3)$ with bipartition $X := \{H_1\}$ and $Y := \{H_2, H_3, H_4\}$.
Case 2: $p > q$:
Case 2a: $q \nmid (p^2 - 1)$. In this case there is no non-abelian group.
Case 2b: $q \mid (p-1)$. In this case there are two groups. The first one is $G_4 := \langle a, b \mid a^{p^2} = b^q = 1, bab^{-1} = a^i, ord_{p^2}(i) = q \rangle$. It has $H_1 := \langle a \rangle$, $H_2 := \langle a^p \rangle$, $H_3 := \langle a^p b \rangle$, $H_4 := \langle b \rangle$, $H_5 := \langle ab \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_4))| \geq 5$. Since H_1 is a normal subgroup of G_4, so it permutes with all the subgroup of G_4; $H_2H_3 = \langle a^p, b \rangle = H_2H_4$; $H_2H_5 = \langle a^p, ab \rangle$. So C_3 is a subgraph of $\Gamma_c(G_4)$ induced by H_1, H_2, H_4; $K_{1,3}$ is a subgraph of $\Gamma_c(G_4)$ with bipartition $X := \{H_1\}$ and $Y := \{H_2, H_3, H_4\}$.

Next, we have the family of groups $\langle a, b, c \mid a^p = b^p = c^q = 1, cac^{-1} = a^i, cbc^{-1} = b^{i^t}, ab = ba, ord_p(i) = q \rangle$. There are $(q+3)/2$ isomorphism types in this family (one for $t = 0$ and one for each pair $\{x, x^{-1}\}$ in $\mathbb{F}_p^\times$. We will refer to all of these groups as $G_{5(t)}$ of order p^2q. They have a subgroup H isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Since $p > 2$, so by (3.2), $\Gamma_c(G_{5(t)})$ contains K_4 as a subgraph. In addition to these four vertices, $\Gamma_c(G_{5(t)})$ have $\langle c \rangle$ as their vertex, so $|V(\Gamma_c(G_{5(t)}))| \geq 5$.
Case 2c: $q \mid (p+1)$. In this case, we have only one group of order p^2q, given by $G_6 := (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q = \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = a^i b^j, cbc^{-1} = a^k b^l \rangle$, where $\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right)$ has order q in $GL_2(p)$. It has a subgroup H isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Since $p > 2$, so by (3.2), $\Gamma_c(G_6)$ contains K_4 as a subgraph. In addition to these four vertices, $\Gamma_c(G_6)$ has $\langle c \rangle$ as its vertex, so $|V(\Gamma_c(G_6))| \geq 5$.

Note that if $(p, q) = (2, 3)$, the Cases 1 and 2 are not mutually exclusive. Up to isomorphism, there are three non-abelian groups of order 12: $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$, D_{12}, and A_4. In Case 1b we already dealt with $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ (the group G_1), and D_{12} (the group G_2). But for the case of A_4 (the group G_6), we can not use the argument as in Case 2c, since $p = 2$. So we now separately deal with this case. Note that $A_4 \cong (Z_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$. Here $H_1 := \mathbb{Z}_2 \times \mathbb{Z}_2$ is a subgroup of A_4 of order 4, and it has three nontrivial subgroups, say H_i, $i = 2, 3, 4$ each of order 2. Also A_4 has four subgroups of order 3, let them be H_j, $j = 5, 6, 7, 8$. These eight subgroups are the only proper subgroups of A_4, so $|V(\Gamma_c(G))| \geq 5$. Further, H_2, H_3 and H_4 permutes with each other, but no two subgroups H_5, H_6, H_7, H_8 permutes; for if they permutes, then G has a subgroup of order 9, which is not possible. Also, no H_i $(i = 2, 3, 4)$ permutes with H_j $(j = 5, 6, 7, 8)$; for if they permutes, then G has a subgroup of order 6, which is not possible. Thus,

$$\Gamma_c(G_6) \cong K_3 \cup \overline{K}_4. \tag{3.5}$$

The proof follows by combining all the cases. □

Proposition 3.5. *If G is a non-abelian group of order $p^\alpha q$, where p, q are two distinct primes with $\alpha \geq 3$, then $\Gamma_c(G)$ has C_3 and $K_{1,3}$ as proper subgraphs; it has at least five vertices.*

Proof. Let P denote a Sylow p-subgroup of G. We first prove this result for $\alpha = 3$. If $p > q$, then $n_p = 1$, by Sylow's Theorem and our group $G \cong P \rtimes \mathbb{Z}_q$. Suppose $\Gamma_c(P)$ contains C_3 and $K_{1,3}$; $|V(\Gamma_c(G))| \geq 5$, then $\Gamma_c(G)$ also has the same. So by Propositions 3.1 and 3.2, the only possibilities are $P \cong \mathbb{Z}_{p^3}$ or Q_8. If $P \cong \mathbb{Z}_{p^3}$, then $G \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q = \langle a, b \mid a^{p^3} = q = 1, bab^{-1} = a^i, ord_{p^3}(i) = q\rangle$ and it has $H_1 := \langle a\rangle$, $H_2 := \langle a^p\rangle$,$H_3 := \langle a^{p^2}\rangle$, $H_4 := \langle b\rangle$, $H_5 := \langle ab\rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G))| \geq 5$. Here H_1, H_2, H_3 are normal in G, so they permutes with all the subgroups of G. It follows that $\Gamma_c(G)$ contains K_4 as a proper subgraph. If $P \cong Q_8$, then by (3.3), $\Gamma_c(P) \cong K_4$. But this K_4 is a proper subgraph of $\Gamma_c(G)$, since G has a cyclic subgroup isomorphic to $\mathbb{Z}_q$, in addition and so $|V(\Gamma_c(G))| \geq 5$.

Now, let us consider the case $p < q$ and $(p, q) \neq (2, 3)$. Here $n_q = p$ is not possible. If $n_q = p^2$, then $q \mid (p + 1)(p - 1)$ which implies that $q|(p + 1)$ or $q \mid (p - 1)$. But this is impossible, since $q > p > 2$. If $n_q = p^3$, then there are $p^3(q - 1)$ elements of order q. But this only leaves $p^3q - p^3(q - 1) = p^3$ elements, and the Sylow p-subgroup must be normal, a case we already considered. Therefore, the only remaining possibility is that $G \cong \mathbb{Z}_q \rtimes P$. Suppose $\Gamma_c(P)$ contains C_3 and $K_{1,3}$; $|V(\Gamma_c(P))| \geq 5$, then $\Gamma_c(G)$ also has the same. So by Propositions 3.1 and 3.2, we have the only possibilities $P \cong \mathbb{Z}_{p^3}$ or Q_8. If $P \cong \mathbb{Z}_{p^3}$, then $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^3} = \langle a, b \mid a^q = b^{p^3} = 1, bab^{-1} = a^i, ord_q(i) = p^3\rangle$ and it has $H_1 := \langle a\rangle$, $H_2 := \langle b\rangle$, $H_3 := \langle b^p\rangle$, $H_4 := \langle b^{p^2}\rangle$, $H_5 := \langle ab^p\rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G))| \geq 5$. Here H_1, H_5 are normal in G, so they permutes with all the subgroups of G; H_3 is a subgroups of H_2. So K_4 is a subgraph of $\Gamma_c(G)$ induced by H_1, H_2, H_3, H_4. The case $P \cong Q_8$ is similar to the earlier case.

If $(p, q) = (2, 3)$, then $G \cong S_4$ and it has a subgroup H isomorphic to D_8. Therefore, by Theorem 3.2, $\Gamma_c(H)$ contains C_3 and $K_{1,3}$ as proper subgraphs. Also H has more than four cyclic subgroups, so $\Gamma_c(G)$ also has the same properties.

If $\alpha \geq 4$, then G has a subgroup, say H of order p^4. Suppose $\Gamma_c(H)$ contains C_3 and $K_{1,3}$; also $|V(\Gamma_c(H))| \geq 5$, then $\Gamma_c(G)$ also has the same properties. So by Propositions 3.1 and 3.2, we need to check when $H \cong \mathbb{Z}_{p^4}$. If $H \cong \mathbb{Z}_{p^4}$, then by (3.2), $\Gamma_c(H) \cong K_3$, so H together with its subgroups forms K_4 as a subgraph of $\Gamma_c(G)$. Also $|V(\Gamma_c(G))| \geq 5$, since G has a subgroup of order q in addition. □

Proposition 3.6. *If G is a non-abelian group of order p^2q^2, where p, q are two distinct primes, then $\Gamma_c(G)$ contains C_3 and $K_{1,3}$ as proper subgraphs; it has at least five vertices.*

Proof. We use the classification of groups of order p^2q^2 given in [15]. Let P and Q denote a Sylow p, q-subgroups of G respectively. Without loss of generality, we assume that $p > q$. By Sylow's Theorem, $n_p = 1, q, q^2$. But $n_p = q$ is not possible, since $p > q$. If $n_p = q^2$, then $p \mid (q + 1)(q - 1)$, this implies that $p \mid (q + 1)$, which is true only when $(p, q) = (3, 2)$. When $(p, q) \neq (3, 2)$, then $G \cong P \rtimes Q$. Now we have the following possibilities.

If $G \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{q^2} = \langle a, b \mid a^{p^2} = b^{q^2} = 1, bab^{-1} = a^i, i^{q^2} \equiv 1 \pmod{p^2}\rangle$, then $H_1 := \langle a\rangle$, $H_2 := \langle a^p\rangle$, $H_3 := \langle b\rangle$, $H_4 := \langle b^q\rangle$, $H_5 := \langle ab\rangle$ are proper cyclic subgroups of G, so $|V(\Gamma_c(G))| \geq 5$. Here H_1, H_2 are normal in G; H_3, H_4 permutes with each other. So K_4 is a proper subgraph of $\Gamma_c(G)$ induced by H_1, H_2, H_3, H_4.

If $G \cong \mathbb{Z}_{p^2} \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $H_1 := \langle a\rangle$, $H_2 := \langle a^p\rangle$, $H_3 := \langle b\rangle$, $H_4 := \langle c\rangle$, $H_5 := \langle bc\rangle$ are proper cyclic subgroups of G, so $|V(\Gamma_c(G))| \geq 5$. Here H_1 is a normal subgroup of G; H_3, H_4, H_5 permutes with each other. So K_4 is a proper subgraph of $\Gamma_c(G)$ induced by H_i, $i = 1, 3, 4, 5$.

If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$ or $(\mathbb{Z}_p \times Z_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $\mathbb{Z}_p \times \mathbb{Z}_p$ is a subgroup of G. Since $p > 2$, so by (3.2), $\Gamma_c(G)$ contains K_4 as a proper subgraph and so $|V(\Gamma_c(G))| \geq 5$.

Next, we consider the case when $(p, q) = (3, 2)$ and $n_p = 1$. Consider the Sylow 3-subgroup P and a Sylow 2-subgroup Q of G. Let H be a subgroup of Q of order 2. Since $|G|$ does not divide $[G : P]!$, so P contains a subgroup, say K of order 3, which is normal in G; $H_1 := QK$ is a subgroup of order 12. Suppose $\Gamma_c(H)$ contains C_3 and $K_{1,3}$; also $|V(\Gamma_c(H))| \geq 5$, then $\Gamma_c(G)$ also has the same. So by Propositions 3.1 and 3.4, the only cases remains to check is when $H \cong \mathbb{Z}_{p^2q}$ or A_4. If $H_1 \cong \mathbb{Z}_{p^2q}$, then by (3.2), $\Gamma_c(H_1) \cong K_4$, so H together with its subgroups forms K_5 as a proper subgraph of $\Gamma_c(G)$ and so $|V(\Gamma_c(G))| \geq 5$. If $H_1 \cong A_4$, then by (3.5), $\Gamma_c(H_1) \cong K_3 \cup \overline{K}_4$, so $|V(\Gamma_c(G))| \geq 5$. Also $K_{1,3}$ is a subgraph of $\Gamma_c(G)$ with bipartition $X := \{K\}$ and $Y := \{K_1, K_2, K_3\}$, where K_i's are the vertices of K_3 in $\Gamma_c(H_1)$. □

Proposition 3.7. *If G is a non-abelian group of order $p^\alpha q^\beta$, where p, q are distinct primes, and $\alpha,\ \beta \geq 2$, then $\Gamma_c(G)$ has C_3 and $K_{1,3}$ as proper subgraphs; it has at least five vertices.*

Proof. We prove the result by induction on $\alpha + \beta$. If $\alpha + \beta = 4$, then by Propositions 3.1 and 3.6, the result is true in the case. Assume that the result is true for all non-abelian groups of order $p^m q^n$ with $m, n \geq 2$, and $m + n < \alpha + \beta$. We prove the result when $\alpha + \beta > 4$. Since G is solvable, G has a subgroup H of prime index, with out loss of generality, say q. So $|H| = p^\alpha q^{\beta-1}$. If H is abelian, then by Proposition 3.1, the result is true. If H is non-abelian, then we have the following cases to consider:
Case 1: If $\beta = 2$, then $\alpha > 2$. So by Proposition 3.5, the result is true for $\Gamma_c(H)$.
Case 2: If $\beta > 2$, then by induction hypothesis, the result is true for $\Gamma_c(H)$.
Case 3: If $\alpha = 2$, then $\beta > 2$. So by Case 2, the result is true for $\Gamma_c(H)$.
Case 4: If $\alpha > 2$, then by induction hypothesis, the result is true for $\Gamma_c(H)$.
Then by Theorem 2.4, result is true for $\Gamma_c(G)$ also. □

Proposition 3.8. *Let G be a finite group of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, $k \geq 3$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma_c(G)$ contains C_3, $K_{1,3}$ and it has more than four vertices.*

Proof. If $\alpha_i = 1$, for every i, then G is solvable. We consider the following cases:
Case 1: $k = 3$. If $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then without loss of generality, we assume that $p_1 < p_2 < p_3$. Since G is solvable, it has a Sylow basis $\{P_1, P_2, P_3\}$, where P_i is the Sylow p_i-subgroup of G for every $i = 1, 2, 3$. Also $H_1 := \langle ab \rangle$ and $H_2 := \langle bc \rangle$ are proper cyclic subgroups of G, where a, b, c are generators of P_1, P_2, P_3 respectively, so we have $|V(\Gamma_c(G))| \geq 5$. Moreover, P_1, P_2, P_3 permutes with each other, so $\Gamma_c(G)$ contains C_3 as a proper subgraph. Further, G has a normal subgroup, say N of order p_3, so it follows that $\Gamma_c(G)$ contains $K_{1,3}$ as a subgraph with bipartition $X := \{N\}$ and $Y := \{P_1, P_2, H_1\}$.
Case 2: $k > 3$. Since G is solvable, it has a Sylow basis containing P_1, P_2, P_3 , where P_i is the Sylow p_i-subgroup of G for every $i = 1, 2, 3$. Then $H := P_1P_2P_3$ is a subgroup of G. So by Proposition 3.1 and by Case 1 of this proof, $\Gamma_c(H)$ contains C_3 and $K_{1,3}$ as subgraphs; also $|V(\Gamma_c(H))| \geq 5$. It follows that $\Gamma_c(G)$ also has the same properties.

If $\alpha_i > 1$, for some i, then without loss of generality, we assume that $\alpha_1 > 1$. By Sylow's theorem, G has a Sylow p_1-subgroup, say P and G has an element, say b of order p_2. If P is non-abelian, then by Proposition 3.2, $\Gamma_c(P)$ contains C_3, $K_{1,3}$ as a subgraph. By Theorem 3.2, taking the cyclic subgroups of P together with $\langle b \rangle$, we have $|V(\Gamma_c(G))| \geq 5$.

If P is abelian, then we consider the following cases:
Case 3: P is cyclic. Let $P := \langle a \rangle$. Now consider the subgroup $\langle a, b \rangle$ of G. Then by Propositions 3.1, 3.3, 3.4, 3.5, 3.6, and 3.7, we have $\Gamma_c(\langle a, b \rangle)$ contains C_3, $K_{1,3}$. Also by Propositions 3.3, 3.4, 3.5, 3.6 and 3.7, taking cyclic subgroups of $\langle a, b \rangle$ together with $\langle a, b \rangle$, we have $|V(\Gamma_c(G))| \geq 5$.
Case 4: P is non-cyclic. If $\alpha_1 = 2$, then $P \cong \mathbb{Z}_p \times \mathbb{Z}_p := \langle a_1, a_2 \rangle$.
subcase 4a: If $\langle a_1, a_2, b \rangle \ncong A_4$, then by Propositions 3.1, 3.3, 3.4, 3.5, 3.6 and 3.7, $\Gamma_c(\langle a_1, a_2, b \rangle)$ contains C_3, $K_{1,3}$ and $|V(\langle a_1, a_2, b \rangle)| \geq 5$.
subcase 4b: If $\langle a_1, a_2, b \rangle \cong A_4$. By (3.5), $\Gamma_c(\langle a_1, a_2, b \rangle)$ has C_3 as a subgraph. Let c be an element of G of order p_3. If $\langle c \rangle$ permutes with a cyclic subgroups of $\langle a_1, a_2 \rangle$, then $\Gamma_c(\langle a_1, a_2 \rangle) \cong$

C_3. So $\langle c\rangle$ together with cyclic subgroups of $\langle a_1, a_2\rangle$ forms $K_{1,3}$. If $\langle c\rangle$ does not permute with a subgroups of $\langle a_1, a_2\rangle$, the by Propositions 3.1, 3.3, 3.4, 3.5, 3.6 and 3.7, $\Gamma_c(\langle a_1, c\rangle)$ contains $K_{1,3}$ as a subgraph. Also $|V(\langle a_1, a_2, b\rangle)| \geq 5$, since by (3.5). If $\alpha_1 \geq 3$, then by Proposition 3.1, the result is true for $\Gamma_c(P)$, so it is true for $\Gamma_c(G)$ also.

The proof follows by combining all these cases. □

3.3 Main results for finite groups

Combining all the results obtained so-far in this section, we have the following main results which are applications for group theory.

Theorem 3.9. *Let G be a finite group and p, q be distinct primes. Then*

(i) $\Gamma_c(G)$ *is C_3-free if and only if G is one of* $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3)$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_q \rtimes \mathbb{Z}_p$;

(ii) $\Gamma_c(G)$ *is C_n if and only if $n = 3$ and G is either* $\mathbb{Z}_{p^4}$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$;

(iii) $\Gamma_c(G)$ *is P_n if and only if $n = 1$ and G is either* $\mathbb{Z}_{p^3}$ *or* $\mathbb{Z}_{pq}$;

(iv) $\Gamma_c(G)$ *is K_4 if and only if G is one of* $\mathbb{Z}_{p^5}$, $\mathbb{Z}_{p^2q}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, Q_8;

(v) $\Gamma_c(G)$ *is claw-free if and only if G is one of* $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3, 4)$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, A_4.

Corollary 3.10. *Let G be a finite group and p, q are distinct primes.*

(i) *The following are equivalent:*

  (a) $\Gamma_c(G)$ *is C_3-free;*

  (b) $\Gamma_c(G)$ *is bipartite;*

  (c) $\Gamma_c(G)$ *is complete bipartite;*

  (d) $\Gamma_c(G)$ *is tree;*

  (e) $\Gamma_c(G)$ *is star graph.*

(ii) $\Gamma_c(G)$ *is P_2-free if and only if G is either* $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3)$ *or* $\mathbb{Z}_{pq}$.

(iii) $girth(\Gamma_c(G))$ *is infinity if G is one of* $\mathbb{Z}_{p^\alpha}$ $(\alpha = 2, 3)$, $\mathbb{Z}_{pq}$ *or* $\mathbb{Z}_q \rtimes \mathbb{Z}_p$; *otherwise* $girth(\Gamma_c(G)) = 3$.

Proof. To classify the groups whose permutability graph is either bipartite or complete bipartite, it is enough to consider the groups whose permutability graph of cyclic subgroups are C_3-free. By Theorem 3.9(i) and (3.1), (3.4), we have $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. Now, to classify the groups whose permutability graphs of cyclic subgroups is one of tree, star graph or P_2-free, it is enough to consider the groups whose permutability graphs of cyclic subgroups are bipartite. So by the above argument and by Theorem 3.9(i), (3.1), (3.4), we have $(b) \Leftrightarrow (d) \Leftrightarrow (e)$ and $\Gamma_c(G)$ is P_2-free if and only if $\mathbb{Z}_{p^\alpha}(\alpha = 2, 3)$ or $\mathbb{Z}_{pq}$. This completes the proof of parts (i) and (ii). The proof of part (iii) follows by the part (i) of this corollary and by Theorem 3.9(i). □

Corollary 3.11. *Let G be a finite group. Then $\Gamma_c(G)$ is totally disconnected if and only if* $G \cong \mathbb{Z}_{p^2}$.

Proof. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_i's are distinct primes, $k \geq 1$ and $\alpha_i \geq 1$. If $\alpha_i = 1$, for every i, then G is solvable. Suppose $k = 1$, then G does not contains a proper subgroup. It follows that $k \geq 2$ and so any two subgroups in Sylow basis of G permutes with each other. Therefore, $\Gamma_c(G)$ is not totally disconnected. If $\alpha_i > 1$, for some i, then without loss of generality we assume that $\alpha_1 > 1$ and so by Sylow's Theorem, G has a Sylow p_1 subgroup, say P. Suppose $P \ncong \mathbb{Z}_{p^2}$, then by Propositions 3.1 and 3.2, $\Gamma_c(G)$ is not totally disconnected. If $P \cong \mathbb{Z}_{p^2}$, then P and its subgroup of order p permutes with each other. Thus $\Gamma_c(G)$ is not totally disconnected. □

Remark 3.12. Not every graph is a permutability graph of cyclic subgroups of some group. For example, by Theorem 3.9 (3), the graph C_n, $n \geq 4$ is not a permutability graph of cyclic subgroups of any group.

3.4 Infinite groups

We now investigate the of permutability graph of cyclic subgroups of infinite groups. It is well known that any infinite group has infinite number of subgroups. Let G be an infinite abelian group. If G is finitely generated, then by fundamental theorem of finitely generated abelian groups, $\mathbb{Z}$ is a subgroup of G. Since $\mathbb{Z}$ is cyclic, it follows that $\Gamma_c(\mathbb{Z})$ contains K_r as a proper subgraph for every positive integer r. Therefore, by Theorem 2.4, $\Gamma_c(G)$ also has the same property. If G is not finitely generated, then we can take the cyclic groups generated by each generating element and so $\Gamma_c(G)$ contains K_r as a proper subgraph, for every positive integer r. Thus we have the following result.

Theorem 3.13. *The permutability graph of cyclic subgroups of any infinite abelian group contains K_r as a subgraph, for every positive integer r.*

Next, we consider the infinite non-abelian groups. Recall that an infinite non-abelian group G in which every proper subgroups of G have order a fixed prime number p is called a *Tarski monster group*. Existence of such groups was given by Ol'shanskii in [17]. In general, the existence of infinite non-abelian groups in which the order of all proper subgroups are of prime order (primes not necessarily distinct) were also given by him in [18, Theorem 35.1]. Also M. Shahryari in [21, Theorem 5.2] give the existence of countable non-abelian simple groups with the property that their all non-trivial finite subgroups are cyclic of order a fixed prime p (of course, this existence can also be deduced from the results of [18]). It is easy to see that the permutability graph of cyclic subgroups of the above mentioned first two class of non-abelian groups are totally disconnected and for the third class of non-abelian groups, it is totally disconnected if that group does not have $\mathbb{Z}$ as a subgroup. In the next result, we characterize the infinite non-abelian groups whose permutability graph of cyclic subgroups is totally disconnected.

Theorem 3.14. *Let G be an infinite group. Then $\Gamma_c(G)$ is totally disconnected if and only if every non-trivial finite subgroup of G is of prime order (primes not necessarily distinct) and $\mathbb{Z}$ is not a subgroup of G.*

Proof. It is easy to see that if every proper subgroup of G is of prime order (primes not necessarily distinct) and $\mathbb{Z}$ is not a subgroup of G then $\Gamma_c(G)$ is totally disconnected. Conversely, suppose that $\Gamma_c(G)$ is totally disconnected. Then by Theorem 3.13, G must be non-abelian. Suppose not every proper subgroup of G is of prime order, then we have the following possibilities.

(i) G may have a subgroup whose order is a composite number; or

(ii) all the subgroups of G may have infinite order.

If G is of type (i), then let H be a subgroup of G of composite order. If $H \ncong Z_{p^2}$, then by Corollary 3.11, $\Gamma_c(H)$ is not totally disconnected. If $H \cong Z_{p^2}$, then H and its subgroup of order p permutes with each other. So it follows that $\Gamma_c(G)$ is not totally disconnected.

If G is of type (ii), then it must have $\mathbb{Z}$ as a subgroup and so by Theorem 3.13, $\Gamma_c(G)$ is not totally disconnected. Hence the proof. □

4 Further results on $\Gamma_c(G)$

Recall that a subgroup H of a group G is said to be *permutable* if it permutes with all the subgroups of G. In [13], Iwasawa characterized the groups whose subgroups are permutable.

Theorem 4.1. *([13]) A group whose subgroups are permutable is a nilpotent group in which for every Sylow p-subgroup P, either P is a direct product of a quaternion group and an elementary abelian 2-group, or P contains an abelian normal subgroup A and an element $b \in P$ such that $P = A\langle b\rangle$ and there exists a natural number s, with $s \geq 2$ if $p = 2$, such that $a^b = a^{1+p^s}$ for every $a \in A$.*

The next theorem classifies the groups whose permutability graph of cyclic subgroups are complete (see, also in [5, p.14]).

Theorem 4.2. *Let G be a group. Then $\Gamma_c(G)$ is complete if and only if G is one of the groups given in Theorem 4.1.*

Theorem 4.3. *Let G be a group with a permutable proper cyclic subgroup. Then $\Gamma_c(G)$ is regular if and only if $\Gamma_c(G)$ is complete.*

Proof. Let N be a permutable cyclic subgroup of G. Assume that $\Gamma_c(G)$ is regular. Since N permutes with all the cyclic subgroup of G, so from the regularity of $\Gamma_c(G)$, it follows that any two vertices in $\Gamma_c(G)$ are adjacent and hence $\Gamma_c(G)$ is complete. Converse of the result is obvious. □

Theorem 4.4. *Let G be a group with a permutable proper cyclic subgroup. Then $\Gamma_c(G)$ is connected and $diam(\Gamma_c(G)) \leq 2$.*

Proof. If every cyclic subgroups of G are permutable, then obviously $\Gamma_c(G)$ is connected and $diam(\Gamma_c(G)) = 1$. Let N be a permutable proper cyclic subgroup of G. Suppose H and K are two proper cyclic subgroups of G such that $HK \neq KH$. Then we have a path $H - N - K$ in $\Gamma_c(G)$ and so $\Gamma_c(G)$ is connected and $diam(\Gamma_c(G)) = 2$. □

Problem 4.1. Which groups have connected permutability graph of cyclic subgroups ? and estimate their diameter.

In the next result, we classify the abelian groups whose permutability graph of cyclic subgroups are planar.

Theorem 4.5. *Let G be an abelian group and p, q be distinct primes. Then $\Gamma_c(G)$ is planar if and only if G is isomorphic to one of the following: $\mathbb{Z}_{p^\alpha}(\alpha = 2, 3, 4, 5)$, $\mathbb{Z}_{pq}, \mathbb{Z}_{p^2q}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. If G is infinite abelian, then by Theorem 3.13, $\Gamma_c(G)$ is non-planar. So in the rest of the proof, we assume that G is finite.

Suppose G is cyclic, then with the notations used in the proof of Proposition 3.1 and by (3.1), we have $\Gamma_c(G) \cong K_r$. So $\Gamma_c(G)$ is planar if and only if $r \leq 4$. This is true only when one of the following holds:

(i) $k = 1$ with $\alpha_1 < 6$;

(ii) $k = 2$ with $\alpha_1 = 1, \alpha_2 = 1$;

(iii) $k = 2$ with $\alpha_1 = 2, \alpha_2 = 1$.

If G is non-cyclic, then we need to consider the following cases:
Case 1: $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then the number of proper subgroups of G is $p+1$; they are $\langle(1, 0)\rangle$, and $\langle x, 1\rangle$, $x \in \{0, 1, \ldots, p-1\}$. By (3.2), $\Gamma_c(G)$ is planar only when $p = 2, 3$.
Case 2: $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. Then $\langle(1, 0)\rangle$, $\langle(1, 1)\rangle$, $\langle(p, 0)\rangle$, $\langle(0, 1)\rangle$, $\langle(p, 1)\rangle$ are proper subgroups of G, so $\Gamma_c(G)$ contains K_5 as a subgraph.
Case 3: $G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p$. Then $\mathbb{Z}_{pq}$, $\mathbb{Z}_q$, $\mathbb{Z}_p \times \mathbb{Z}_p$ are proper subgroups of G. Here $\mathbb{Z}_p \times \mathbb{Z}_p$ has at least three proper subgroups of order p, so these three subgroups together with $\mathbb{Z}_{pq}$, $\mathbb{Z}_q$ forms K_5 as a subgraph of $\Gamma_c(G)$.
Case 4: $G \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^l}$, $k, l \geq 2$. Then $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ is a proper subgroup of G, so by Case 2 and by Theorem 2.4, $\Gamma_c(G)$ contains K_5 as a subgraph.
Case 5: $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. then G has two subgroups each isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. It follows that G has at least five subgroups of order p and so they form K_5 as a subgraph of $\Gamma_c(G)$.
Case 6: $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p_k^{\alpha_k}}$, where p_i's are primes and $\alpha_i \geq 1$. If $k = 2$ or 3, then $\alpha_i > 1$, for some i and if $k \geq 4$, then $\alpha_i \geq 1$. In either case, one of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_{pq} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ is a proper subgroup of G, so by Cases 2, 3 and 5, $\Gamma_c(G)$ contains K_5 as a subgraph. The result follows by combining all the above cases. □

Proposition 3.3 shows the existence of a finite non-abelian group whose permutability graph of cyclic subgroups is planar. Further, the Torski monster group is an example of an infinite non-abelian group whose permutability graph of cyclic subgroups is planar. Now we pose the following

Problem 4.2. Classify all non-abelian groups whose permutability graph of cyclic subgroups are planar.

The next result characterize some non-abelian groups by using their permutability graph of cyclic subgroups.

Theorem 4.6. *Let G be a finite group.*

(i) *If G is non-abelian and $\Gamma_c(G) \cong \Gamma_c(Q_8)$, then $G \cong Q_8$.*

(ii) *If $\Gamma_c(G) \cong \Gamma_c(S_3)$, then $G \cong S_3$.*

(iii) *If $\Gamma_c(G) \cong \Gamma_c(A_4)$, then $G \cong A_4$.*

Proof.

(i); By Theorem 3.9(5), Q_8 is the only non-abelian group such that $\Gamma_c(Q_8) = K_4$, so the result follows.

(ii): By Theorem 3.9(1) and (3.1), (3.4), S_3 is the only group such that $\Gamma_c(S_3) = K_{1,3}$, so the result follows.

(iii): By Theorem 3.9(6) and (3.1), (3.2), (3.5), A_4 is the only group such that $\Gamma_c(A_4) = K_3 \cup \overline{K}_4$, so the result follows.□

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