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Forgetful Lifetime distributions

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Abstract A lifetime distribution is forgetful at certain points if it obeys the classical lack-of-memory property at these points. In this paper, a new notion about the memory property of lifetime distributions is introduced in terms of what we call the ϕ-forgetful property. We prove a theorem for the existence of this property for all lifetime distributions. This theorem is used to characterize lifetime distributions such as the Weibull, Pareto, Gompertz, and extreme value.

1 Introduction

It is well known that the exponential distribution is the unique continuous distribution possessing the memoryless or the classical lack-of-memory property (LMP) (See, for example, Klein [3]). That is, if X is exponentially distributed, then

$$P(X > s + t | X > t) = P(X > s), \tag{1.1}$$

for all real numbers $t > 0$ and $s > 0$.

Equation 1.1 says that X fully forgets its elapsed history. In other words, it does not have a memory of its survival times. Other lifetime distributions also forget some of their survival times.

Many authors wrote about the classical LMP that characterizes the exponential distribution, but only few of them defined the notion of memory as a property of lifetime distributions. For example, Muth [5] defined the memory of a lifetime distribution, under general regulating conditions, in terms of the mean residual lifetime function (MRL), $r(t) = E(T - t|T > t)$, but he did not explicitly relate the memory property to the LMP.

Gertsbakh et al. [1] characterized the Weibull family by an information version of the classical lack-of-memory property related to the Fisher information.

The notion of the forgetful property for lifetime distributions was introduced in Marsaglia et al. [4] as in the following definition.

Definition 1.1. The non-negative random variable T is forgetful at a point u for which $S(u) > 0$ if

$$P(T > u + t|T > u) = P(T > t) \tag{1.2}$$

Marsaglia et al. [4] used the forgetful property to characterize the exponential distribution being the only occasionally forgetful continuous distribution.

We may extend Definition 1.1 as follows.

Definition 1.2. We say that T is ϕ-forgetful, or ϕ-memoryless, if

$$P(T > \phi(s + t)\big|T > \phi(t)) = P(T > \phi(s)), \tag{1.3}$$

for $s > 0$ and $t > 0$, where ϕ is a non-negative real-valued function defined on $(0, \infty)$.

Throughout this paper, we consider non-negative continuous random variables T with support $(0, \infty)$ and finite expectation $E(T)$. Moreover, we use $f(t)$ and $F(t)$ to denote pdf and cdf of T, respectively. The random variable T models time to a particular event such as the failure time, service or repair time, waiting time, etc. The survival function of T at time t, denoted $S(t)$, defines the probability of survival beyond time t; namely,

$$S(t) = P(T > t) = 1 - F(t).$$

The inverse of the survival function at t, denoted by $S^{-1}(u)$, for $0 < u < 1$, is defined by

$$Z(u) \stackrel{\text{def}}{=} S^{-1}(u) = \inf\{t : S(t) \le u\}.$$

In other words, $Z(u) = S^{-1}(u) = F^{-1}(1-u)$ is the smallest non-negative integer t for which $F(t) \ge 1 - u$.

2 Preliminary Results

If $h(t)$ is the hazard function of T, then the function $H(t) = \int_0^t h(x)dx$ is called the cumulative hazard function of T up to time t. The survival function $S(t)$ and the cumulative hazard function $H(t)$ relate to each other (See, for example, [2]) by the equation

$$S(t) = \exp[-\int_0^t h(x)dx] = \exp[-H(t)].$$

The following lemma is essential to our main results in this paper.

Lemma 2.1. *Let $S(t)$ be the survival function of T. If $X = \frac{1}{\lambda}H(T)$, then $X \sim Exp(\lambda)$; i.e., X is exponentially distributed with parameter λ.*

Proof. Since $S(T) = \exp[-H(T)]$, we have

$$\begin{aligned} P(X > x) &= P(H(T) > \lambda x) = P(e^{-H(T)} < e^{-\lambda x}) \\ &= P(S(T) < e^{-\lambda x}) = P(T > S^{-1}(e^{-\lambda x})) \\ &= e^{-\lambda x}, \end{aligned}$$

where we reversed the direction of the inequality due to the fact that $S(t)$ is non-increasing in t. □

It follows from the proof of Lemma 2.1 that $S(T)$ is uniformly distributed on $(0, 1)$.

Remark 2.2. By letting $\lambda = 1$ in the statement of Lemma 2.1, the cumulative hazard function $H(T) \sim \text{Exp}(1)$. This means that $H(T)$ has the LMP; i.e., $H(T)$ satisfies

$$P(H(T) > s + t | H(T) > t) = P(H(T) > s), \tag{2.1}$$

for $s > 0$ and $t > 0$.

3 Characterizing Lifetime Distributions

In this section, we state and proof a theorem that characterizes lifetime distributions with respect to the ϕ-forgetful property, where, for each distribution, ϕ is closely related to the inverse survival function of the distribution.

Theorem 3.1. *Let T be a non-negative random variable with support $(0, \infty)$ and survival function $S(t)$. Then T is ϕ-forgetful, where $\phi(u) = S^{-1}(e^{-u})$, for $t > 0$ and $s > 0$.*

Proof. Note that both of the survival function $S(t)$ and its inverse $S^{-1}(u)$ are non-increasing. Since $\exp(-t)$ is a non-increasing function of t, the function $\phi(t) = S^{-1}(e^{-t})$ is a non-decreasing function of t. Therefore, $\phi(s + t) > \phi(t)$, for all $s > 0$ and $t > 0$.
The proof follows from the equation $P(H(T) > u) = P(T > S^{-1}(e^{-u}))$. In fact,

$$\begin{aligned} P(T > \phi(s+t) \mid T > \phi(t)) &= P(T > s+t)/P(T > t) \\ &= P(H(T) > s+t)/P(H(T) > t) \\ &= P(H(T) > s) = P(T > \phi(s)). \end{aligned}$$

Hence, T satisfies Equation 1.3. That is T is ϕ-forgetful. □

Since the proof of Theorem 3.1 basically depends on transforming T into the exponential distribution of parameter 1, we may say that T partially inherits the LMP from the exponential distribution.

Remark 3.2. When T has an exponential distribution with parameter λ, $\phi(t) = t/\lambda$ and the ϕ-forgetful property becomes the classical LMP.

Corollary 3.3. *If T has a Weibull distribution with shape parameter β and scale parameter λ, then Equation 2.1 is equivalent to*

$$P(T > \lambda^{-1}(s+t)^{1/\beta} \mid T > \lambda^{-1}t^{1/\beta}) = P(T > \lambda^{-1}s^{1/\beta}) \tag{3.1}$$

And the Weibull distribution with parameters β and λ is characterized by the property of being ϕ-forgetful, where $\phi(t) = \lambda^{-1}t^{1\beta}$.

Remark 3.4. In Corollary 3.3, if we let $s = (\lambda u)^\beta$ and $t = (\lambda v)^\beta$, we get

$$P(T > (u^\beta + v^\beta)^{1/\beta} \mid T > v) = P(T > u) \tag{3.2}$$

Equation 3.2 characterizes the Weibull distribution.

Corollary 3.5. *If T has a Pareto distribution with minimum value parameter α and shape parameter β, then Equation 2.1 is equivalent to*

$$P(T > \alpha e^{(s+t)/\beta} \mid T > \alpha e^{t/\beta}) = P(T > \alpha e^{s/\beta}) \tag{3.3}$$

And the Pareto distribution with parameters α and β is characterized by the property of being ϕ-forgetful, where $\phi(t) = \alpha e^{t/\beta}$.

Remark 3.6. In Corollary 3.5, if we let $s = \beta \log(u/\alpha)$ and $t = \beta \log(v/\alpha)$, we get

$$P(T > \alpha^{-1}(uv) \mid T > v) = P(T > u) \tag{3.4}$$

Equation 3.5 characterizes the Pareto distribution.

Remark 3.7. When $u = v = 2\alpha$, we see that

$$P(T > 4\alpha \mid T > 2\alpha) = P(T > 2\alpha),$$

That is T is memoryless at the points $s = t = 2\alpha$.

Corollary 3.8. *If T has a Gompertz distribution with scale parameter α and frailty parameter β, then Equation 2.1 is equivalent to*

$$P(T > \phi(s+t) \mid T > \phi(t)) = P(T > \phi(s)), \tag{3.5}$$

where $\phi(u) = \alpha^{-1}\log(1 + \frac{u}{\beta})$. And the Gompertz distribution with parameters α and β is characterized by the property of being ϕ-forgetful.

Remark 3.9. In Corollary 3.8, if we let $s = \beta(e^{\alpha u} - 1)$ and $t = \beta(e^{\alpha v} - 1)$, for $u > 0$ and $v > 0$, we get

$$P(T > \frac{1}{\alpha}\log(e^{\alpha u} + e^{\alpha v} - 1) \mid T > v) = P(T > u) \tag{3.6}$$

Equation 3.6 characterizes the Gombertz distribution.

Corollary 3.10. *If T has an extreme value distribution with location parameter α and scale parameter $\beta > 0$, then Equation 2.1 is equivalent to*

$$P(T > \alpha + \beta\log(s+t) \mid T > \alpha + \beta\log(t)) = P(T > \alpha + \beta\log(s)) \tag{3.7}$$

And the extreme value distribution with parameters α and β is characterized by the property of being ϕ-forgetful, where $\phi(t) = \alpha + \beta\log(t)$.

Remark 3.11. In Corollary 3.10, if we let $s = e^{(u-\alpha)/\beta}$ and $t = e^{(v-\alpha)/\beta}$, for $u > 0$ and $v > 0$, we get

$$P(T > \beta\log(e^{u/\beta} + e^{v/\beta}) \mid T > v) = P(T > u) \tag{3.8}$$

Equation 3.8 characterizes the extreme value distribution.

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