Palestine Journal of Mathematics

Vol. 5(2) (2016) , 61–70

Approximation by Radial Type Multidimensional Singular Integral Operators

Gumrah Uysal, Mine Menekse Yilmaz and Ertan Ibikli

Communicated by Pradeep K. Banerji

MSC 2010 Classifications: Primary 41A35, 41A25; Secondary 41A63, 26B15

Keywords and phrases: generalized Lebesgue point; radial kernel; rate of convergence

Abstract In the present paper, we study the pointwise approximation of multidimensional singular integral operators with radial kernels such that $H_\lambda(t-x) = K_\lambda(|t-x|)$ of the form:

$$L_\lambda(f;x) = \int_D f(t)\, K_\lambda(|t-x|)\, dt,\ \ x \in D$$

where $D = \prod_{i=1}^{n} \langle a_i, b_i \rangle$ is open, semi-open or closed multidimensional arbitrary bounded box in $\mathbb{R}^n$ or $D = \mathbb{R}^n$, $\lambda \in \Lambda \subset \mathbb{R}_0^+$ and the symbol $|.|$ stands for multidimensional Euclidean distance, at a μ-generalized Lebesgue point of $f \in L_p(D)$. Also we investigate the rate of convergence at this point.

1 Introduction

Taberski [12] studied the pointwise approximation to functions $f \in L_1(\langle -\pi, \pi \rangle)$ and their derivatives by a family of convolution type singular integral operators depending on two parameters of the form:

$$U_\lambda(f;x) = \int_{-\pi}^{\pi} f(t)\, K_\lambda(t-x)\, dt,\ \ \ x \in (-\pi, \pi) \tag{1}$$

where $K_\lambda(t)$ is the kernel satisfying suitable assumptions. Based on Taberski's study [12], Gadjiev [3] and Rydzewska [7] obtained the pointwise convergence of the operators of type (1) at which the points are generalized Lebesgue points and $\mu-$generalized Lebesgue points of functions $f \in L_1(\langle -\pi, \pi \rangle)$, respectively. Further, in [5], Karsli and Ibikli extended the results of [12, 3] , and [7] by considering the more general integral operators.

In [13], Taberski explored the pointwise convergence of integrable functions in $L_1(Q)$ by a three-parameter family of convolution type singular integral operators of the form:

$$T_\lambda(f;x,y) = \iint_Q f(t,s)\, K_\lambda(t-x, s-y)\, dtds,\ \ (x,y) \in Q \tag{2}$$

where Q denotes a given rectangle. After Taberski's study [13], Siudut [9, 10] obtained significant results relating the pointwise convergence of singular integral operators by considering the operators of type (2).

In [14], Uysal and Yilmaz investigated the pointwise convergence of $L_\lambda(f;x,y)$ to $f(x_0, y_0)$ in the space $L_{1,\varphi}(\langle -\pi, \pi \rangle \times \langle -\pi, \pi \rangle)$, by the three parameter family of singular integral operators with radial kernels of the form:

$$L_\lambda(f;x,y) = \iint_R f(t,s)\, H_\lambda(t-x, s-y)\, dtds,\ \ (x,y) \in R \tag{3}$$

where $R = \langle -\pi, \pi \rangle \times \langle -\pi, \pi \rangle$ is open, semi-open or closed region and $(x_0, y_0) \in R$ is a generalized Lebesgue point of the function $f \in L_{1,\varphi}(\langle -\pi, \pi \rangle \times \langle -\pi, \pi \rangle)$.

The current manuscript presents a continuation and further generalization of [15]. The purpose of this paper is to investigate the pointwise approximation of multidimensional singular integral operators with radial kernels such that $H_\lambda(t-x) = K_\lambda(|t-x|)$ at μ-generalized Lebesgue point of the functions $f \in L_p(D)$ and the rate of pointwise convergence of these operators in the following form:

$$L_\lambda(f;x) = \int\limits_D f(t)\, K_\lambda(|t-x|)\, dt, \quad x \in D \tag{4}$$

where $D = \prod\limits_{i=1}^{n} \langle a_i, b_i \rangle$ is open, semi-open or closed arbitrary bounded box in $\mathbb{R}^n$ or $D = \mathbb{R}^n$, $\lambda \in \Lambda \subset \mathbb{R}_0^+$ and $|t-x| = \sqrt{(t_1-x_1)^2 + ... + (t_n-x_n)^2}$.

The paper is organized as follows: In Section 2, we introduce the fundamental definitions. In Section 3, we prove the existence of the operators of type (4). In Section 4, we present four theorems concerning the pointwise convergence of $L_\lambda(f;x)$ whenever x is a μ-generalized Lebesgue point of the function $f \in L_p(D)$. In Section 5, we establish the rate of pointwise convergence of operators of type (4).

2 Preliminaries

In [4, 8], denoting unit sphere by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, the polar coordinates transformation on $\mathbb{R}^n$ is given by $G : \mathbb{R}^n \to \mathbb{R}^n$, $G(r, \theta_1, ..., \theta_{n-1}) = (x_1, .., x_n)$ where

$$\begin{aligned} &x_1 = r\cos\theta_1, \quad x_2 = r\sin\theta_1\cos\theta_2, \quad x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, ..., \\ &x_k = r\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k, ..., x_n = r\sin\theta_1 ... \sin\theta_{n-2}\sin\theta_{n-1}. \end{aligned} \tag{5}$$

Here, $k = 2, ..., n-1$, $0 \le \theta_k \le \pi$, $0 \le \theta_{n-1} \le 2\pi$, $r = |x| \neq 0$, $x' = \frac{x}{r} \in S^{n-1}$ and the jacobian of the transformation is $J = r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}...(\sin\theta_{n-2})$. Therefore if f is either non-negative and measurable on $\mathbb{R}^n$ or $f \in L_1(\mathbb{R}^n)$ the following equality is obtained (see [4], p.78)

$$\begin{aligned} \int\limits_{\mathbb{R}^n} f(x)\, dx &= \int\limits_0^\infty \int\limits_0^\pi ... \int\limits_0^\pi \int\limits_0^{2\pi} f(rx') r^{n-1} (\sin\theta_1)^{n-2} (\sin\theta_2)^{n-3} ... (\sin\theta_{n-2}) d_{\theta_1} .. d_{\theta_{n-1}} dr \\ &= \int\limits_0^\infty \int\limits_{S^{n-1}} f(rx') r^{n-1} dx' dr. \end{aligned}$$

Now we introduce the basic definitions used in the paper.

Definition 2.1. A function $\Phi \in L_1(\mathbb{R}^n)$, is said to be radial, if there exists a function $\Psi(|t|)$, defined on $0 \le |t| < \infty$ such that $\Phi(t) = \Psi(|t|)$ a.e. [1].

Definition 2.2. $(Class\ A)$ Let $\Lambda \subset \mathbb{R}_0^+$ be an index set and λ_0 be an accumulation point of it. Let $H_\lambda : \mathbb{R}^n \to \mathbb{R}$ is non-negative and integrable function for each $\lambda \in \Lambda$ such that $H_\lambda(t) := K_\lambda(|t|)$ a.e. for the function $K_\lambda : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ for each $\lambda \in \Lambda$.

$H_\lambda(t) = K_\lambda(|t|)$ belongs to class A if the following conditions are satisfied:

a. $\|K_\lambda(.)\|_{L_1(\mathbb{R}^n)} \le M < \infty, \quad \forall \lambda \in \Lambda$.

b. $\lim\limits_{\lambda \to \lambda_0} \left[\int\limits_0^\infty \int\limits_{S^{n-1}} K_\lambda(r)\, r^{n-1} dt' dr \right] = 1.$

c. $\lim\limits_{\lambda \to \lambda_0} \left[\sup\limits_{\xi \le r < \infty} |K_\lambda(r)| \right] = 0\,, \quad \forall \xi > 0.$

d. $\lim\limits_{\lambda \to \lambda_0} \left[\int\limits_\xi^\infty \int\limits_{S^{n-1}} K_\lambda(r) r^{n-1} dt' dr \right] = 0, \quad \forall \xi > 0.$

e. $K_\lambda(r)$ is non-increasing w.r.t. r on $[0,\infty)$.

f. $\lim\limits_{\lambda\to\lambda_0} K_\lambda(0)=\infty$.

Throughout this paper K_λ belongs to class A.

3 Existence of Operators

Theorem 3.1. *If $f\in L_p(D)$, then $L_\lambda(f;x)$ defines a continuous transformation acting on $L_p(D)$.*

Proof. Let $p=1$ and $D=\prod\limits_{i=1}^{n}\langle a_i,b_i\rangle$. By the linearity of the operator $L_\lambda(f;x)$ defined by (4), it is sufficient to show that the expression given by

$$\|L_\lambda\|=\sup_{f\neq 0}\frac{\|L_\lambda(f;x)\|_{L_1(D)}}{\|f\|_{L_1(D)}}$$

is bounded. Using Fubini's theorem [2] and condition (a) of class A, we obtain

$$\|L_\lambda(f;x)\|_{L_1(D)}\leq M\|f\|_{L_1(D)}.$$

This proves the theorem for the case $p=1$.

Now, let $1<p<\infty$ and the extention of f to $\mathbb{R}^n$ is given by

$$g(t)=\begin{cases} f(t), & t\in D\\ 0, & t\in\mathbb{R}^n\backslash D.\end{cases}$$

We will show that the expression given by

$$\|L_\lambda\|=\sup_{f\neq 0}\frac{\|L_\lambda(f;x)\|_{L_p(D)}}{\|f\|_{L_p(D)}}$$

is bounded. Write

$$\begin{aligned}\|L_\lambda(f;x)\|_{L_p(D)} &= \left\|\int_D f(t)K_\lambda(|t-x|)\,dt\right\|_{L_p(D)}\\ &= \left\|\int_{\mathbb{R}^n} g(t+x)K_\lambda(|t|)\,dt\right\|_{L_p(D)}\\ &= \left(\int_D\left|\int_{\mathbb{R}^n} g(t+x)K_\lambda(|t|)\,dt\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

Using generalized Minkowski inequality [11] and condition (a) of class A, we have

$$\begin{aligned}\|L_\lambda(f;x)\|_{L_p(D)} &\leq \int_{\mathbb{R}^n}\left(\int_D |g(t+x)|^p|K_\lambda(|t|)|^p\,dx\right)^{\frac{1}{p}}dt\\ &= \int_{\mathbb{R}^n}|K_\lambda(|t|)|\,dt\left(\int_D|g(t+x)|^p\,dx\right)^{\frac{1}{p}}\\ &\leq M\|f\|_{L_p(D)}.\end{aligned}$$

This proves the theorem for the case $1<p<\infty$. The case $D=\mathbb{R}^n$ can be proved with the same method. Thus the proof is completed. □

4 Convergence at Characteristic Points

In Theorem 4.1, we prove the pointwise convergence of the operators of type (4) for the case $p = 1$ and $D = \prod_{i=1}^{n} \langle a_i, b_i\rangle$, where D is open, semi open or closed multidimensional arbitrary bounded box in $\mathbb{R}^n$.

Theorem 4.1. *Let $x \in D$ be a $\mu-$generalized Lebesgue point of the function $f \in L_1(D)$ such that the following equality holds:*

$$\lim_{h\to 0} \frac{1}{\mu(h)} \int_0^h \int_{S^{n-1}} |f(rt' + x) - f(x)|\, r^{n-1} dt' dr = 0 \tag{6}$$

where $\mu(|t|)$ is defined, increasing, absolutely continuous on $[0, b]$ for the finite real number b and $\mu(0) = 0$. Then one has

$$\lim_{\lambda\to\lambda_0} L_\lambda(f; x) = f(x)$$

on any set Z on which the function

$$\int_0^\delta \mu'(r) K_\lambda(r) dr, \quad 0 < \delta < b$$

is bounded as λ tends to λ_0.

Proof. Define the function g by

$$g(t) = \begin{cases} f(t), & t \in D \\ 0, & t \in \mathbb{R}^n \backslash D \end{cases}.$$

Let $x \in D$ be a fixed $\mu-$generalized Lebesgue point of $f \in L_1(D)$. Using condition (b) of class A, we have the following inequality

$$\begin{aligned} |L_\lambda(f; x) - f(x)| &\le \int_D |f(t) - f(x)|\, K_\lambda(|t - x|) dt \\ &+ |f(x)| \left| \int_{\mathbb{R}^n} K_\lambda(|t|) dt - 1 \right| \\ &+ \int_{\mathbb{R}^n\backslash D} |g(t) - f(x)|\, K_\lambda(|t - x|) dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Our aim is to show that $I_1 \to 0$, $I_2 \to 0$ and $I_3 \to 0$ as $\lambda \to \lambda_0$.

Without loss of generality let $B_\delta(x) \subset D$, where $B_\delta(x) = \{t \in D : |t - x| < \delta\}$. Using formula (5), we have the following equality for I_1

$$\begin{aligned} I_1 &= \int_{B_\delta(x)} |f(t) - f(x)|\, K_\lambda(|t - x|) dt + \int_{D\backslash B_\delta(x)} |f(t) - f(x)|\, K_\lambda(|t - x|) dt \\ &= \int_0^\delta \int_{S^{n-1}} |f(rt' + x) - f(x)|\, K_\lambda(r) r^{n-1} dt' dr + \int_{D\backslash B_\delta(x)} |f(t) - f(x)|\, K_\lambda(|t - x|) dt \\ &= I_{11} + I_{12}. \end{aligned}$$

Let us show that $I_{11} \to 0$ as $\lambda \to \lambda_0$. If $x \in D$ be a $\mu-$generalized Lebesgue point of the function $f \in L_1(D)$ then for every $\epsilon > 0$ there exists $\delta_0 > 0$ such that the following inequality is satisfied

$$\int_0^\delta \int_{S^{n-1}} |f(rt' + x) - f(x)| \, r^{n-1} dt' dr < \epsilon\mu(\delta)$$

where $0 < \delta \leq \delta_0$.

It is easy to see that the following inequality holds for I_{11}

$$I_{11} \leq \epsilon \int_0^\delta \left[\underset{r \leq s \leq \delta}{var} K_\lambda(s) + K_\lambda(\delta) \right] \mu'(r) dr$$
$$= \epsilon \int_0^\delta K_\lambda(r)\mu'(r)dr.$$

Since the following expression

$$\int_0^\delta \mu'(r) K_\lambda(r) dr,$$

remains bounded as $\lambda \to \lambda_0$ and $\epsilon > 0$ is arbitrarily small, $I_{11} \to 0$ as $\lambda \to \lambda_0$.

Let us show that $I_{12} \to 0$ as $\lambda \to \lambda_0$. Since the following inequality:

$$I_{12} \leq \sup_{\delta \leq r < \infty} K_\lambda(r) \left(\|f\|_{L_1(D)} \; + \; |f(x)| \int_D dt \right)$$

holds, in view of condition (c) of class A, $I_{12} \to 0$ as $\lambda \to \lambda_0$ and by condition (b) of class A, $I_2 \to 0$ as $\lambda \to \lambda_0$. Finally, since

$$I_3 \leq \; |f(x)| \int_\delta^\infty \int_{S^{n-1}} K_\lambda(r) r^{n-1} dt' dr,$$

by condition (d) of class A, $I_3 \to 0$ as $\lambda \to \lambda_0$. Thus the proof is completed. □

In Theorem 4.2, we prove the pointwise convergence of the operators of type (4) for the case $p = 1$ and $D = \mathbb{R}^n$.

Theorem 4.2. *Suppose that the hyphothesis of Theorem 4.1 is satisfied. Then one has*

$$\lim_{\lambda \to \lambda_0} L_\lambda(f; x) = f(x)$$

whenever $x \in \mathbb{R}^n$ *is a* $\mu-$*generalized Lebesgue point of the function* $f \in L_1(\mathbb{R}^n)$.

Proof. Following the same strategy as in Theorem 4.1, we have

$$|L_\lambda(f; x) - f(x)| \leq K_\lambda(\delta) \|f\|_{L_1(\mathbb{R}^n)} \; + \; |f(x)| \int_\delta^\infty \int_{S^{n-1}} K_\lambda(r) r^{n-1} dt' dr$$
$$+ \epsilon \int_0^\delta \mu'(r) K_\lambda(r) dr + |f(x)| \left| \int_{\mathbb{R}^n} K_\lambda(|t|) dt - 1 \right|.$$

Since K_λ belongs to class A, the remaining part of the proof is obvious. □

In Theorem 4.3, we prove the pointwise convergence of the operators of type (4) for the case $1 < p < \infty$ and $D = \prod_{i=1}^{n} \langle a_i, b_i \rangle$, where D is open, semi open or closed multidimensional arbitrary bounded box in $\mathbb{R}^n$.

Theorem 4.3. *Let $1 < p < \infty$ and $x \in D$ be a $\mu-$generalized Lebesgue point of the function $f \in L_p(D)$ such that the following equality holds:*

$$\lim_{h \to 0} \left(\frac{1}{\mu(h)} \int_0^h \int_{S^{n-1}} |f(rt' + x) - f(x)|^p \, r^{n-1} dt' dr \right)^{\frac{1}{p}} = 0 \tag{7}$$

where $\mu(|t|)$ is defined, increasing, absolutely continuous on $[0, b]$ for the finite real number b and $\mu(0) = 0$. Then one has

$$\lim_{\lambda \to \lambda_0} L_\lambda(f; x) = f(x)$$

on any set Z on which the function

$$\int_0^\delta \mu'(r) K_\lambda(r) dr, \quad 0 < \delta < b$$

is bounded as λ tends to λ_0.

Proof. Define the function g by

$$g(t) = \begin{cases} f(t), & t \in D \\ 0, & t \in \mathbb{R}^n \backslash D. \end{cases}$$

Let $x \in D$ be a fixed $\mu-$generalized Lebesgue point of $f \in L_p(D)$. Using condition (b) of class A, we have the following inequality

$$\begin{aligned} |L_\lambda(f; x) - f(x)| &\leq \int_D |f(t) - f(x)| \, K_\lambda(|t - x|) dt \\ &+ |f(x)| \left| \int_{\mathbb{R}^n} K_\lambda(|t|) dt - 1 \right| \\ &+ \int_{\mathbb{R}^n \backslash D} |g(t) - f(x)| \, K_\lambda(|t - x|) dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Since

$$I_3 \leq |f(x)| \int_\delta^\infty \int_{S^{n-1}} K_\lambda(r) r^{n-1} dt' dr,$$

by condition (d) of class A, $I_3 \to 0$ as $\lambda \to \lambda_0$. Next, using Hölder's inequality [6] for the integral I_1 we have the following:

$$\begin{aligned} I_1 + I_2 \leq & \left(\int_D |f(t) - f(x)|^p \, K_\lambda(|t - x|) dt \right)^{\frac{1}{p}} \times \left(\int_D K_\lambda(|t - x|) dt \right)^{\frac{1}{q}} \\ &+ |f(x)| \left| \int_{\mathbb{R}^n} K_\lambda(|t|) dt - 1 \right|. \end{aligned}$$

Since for m, n being positive numbers the inequality $(m+n)^p \leq 2^p(m^p+n^p)$ holds [6], by taking the $p-$th power of both sides we have

$$\begin{aligned}
(I_1+I_2)^p &\leq 2^p \int\limits_D |f(t)-f(x)|^p K_\lambda(|t-x|)dt \times \left(\int\limits_{\mathbb{R}^n} K_\lambda(|t|)dt\right)^{\frac{p}{q}} \\
&\quad +2^p |f(x)|^p \left|\int\limits_{\mathbb{R}^n} K_\lambda(|t|)dt - 1\right|^p \\
&= 2^p I^* II^* + 2^p III^*.
\end{aligned}$$

By condition (b) of class A, $II^* \to 1$ and $III^* \to 0$ as $\lambda \to \lambda_0$.

Without loss of generality let $B_\delta(x) \subset D$, where $B_\delta(x) = \{t \in D : |t-x| < \delta\}$. Using formula (5), we have the following equality:

$$\begin{aligned}
I^* &= \int\limits_{B_\delta(x)} |f(t)-f(x)|^p K_\lambda(|t-x|)dt + \int\limits_{D\setminus B_\delta(x)} |f(t)-f(x)|^p K_\lambda(|t-x|)dt \\
&= \int\limits_0^\delta \int\limits_{S^{n-1}} |f(rt'+x)-f(x)|^p K_\lambda(r) r^{n-1} dt' dr + \int\limits_{D\setminus B_\delta(x)} |f(t)-f(x)|^p K_\lambda(|t-x|)dt \\
&= I_{11} + I_{12}.
\end{aligned}$$

Let us show that $I_{11} \to 0$ as $\lambda \to \lambda_0$. If $x \in D$ be a $\mu-$generalized Lebesgue point of the function $f \in L_p(D)$ then for every $\epsilon > 0$ there exists $\delta_0 > 0$ such that the following inequality is satisfied

$$\int\limits_0^\delta \int\limits_{S^{n-1}} |f(rt'+x)-f(x)|^p r^{n-1} dt' dr < \epsilon^p \mu(\delta)$$

where $0 < \delta \leq \delta_0$.

It is easy to see that the following inequality holds for I_{11}

$$\begin{aligned}
I_{11} &\leq \epsilon^p \int\limits_0^\delta \left[\underset{r\leq s\leq \delta}{var} K_\lambda(s) + K_\lambda(\delta)\right] \mu'(r)dr \\
&= \epsilon^p \int\limits_0^\delta K_\lambda(r)\mu'(r)dr.
\end{aligned}$$

Since the following expression,

$$\int\limits_0^\delta \mu'(r) K_\lambda(r) dr$$

remains bounded as $\lambda \to \lambda_0$ and $\epsilon > 0$ is arbitrarily small, $I_{11} \to 0$ as $\lambda \to \lambda_0$.

Let us show that $I_{12} \to 0$ as $\lambda \to \lambda_0$. Since the following inequality holds

$$|I_{12}| \leq 2^p \sup_{\delta \leq r < \infty} K_\lambda(r) \left(\|f\|^p_{L_p(D)} + |f(x)|^p \int\limits_D dt \right)$$

and in view of condition (c) of class A, $I_{12} \to 0$ as $\lambda \to \lambda_0$.

Thus the proof is completed. □

In Theorem 4.4 , we prove the pointwise convergence of the operators of type (4) for the case $1 < p < \infty$ and $D = \mathbb{R}^n$.

Theorem 4.4. *Suppose that the hyphothesis of Theorem 4.3 is satisfied. Then one has*

$$\lim_{\lambda\to\lambda_0} L_\lambda(f;x) = f(x)$$

whenever $x \in \mathbb{R}^n$ is a $\mu-$generalized Lebesgue point of the function $f \in L_p(\mathbb{R}^n)$.

Proof. Following the same strategy as in Theorem 4.3, we have

$$|L_\lambda(f;x) - f(x)|^p \le 2^{2p}\left\{K_\lambda(\delta)\,\|f\|^p_{L_p(\mathbb{R}^n)} \;+\; |f(x)|^p \int_\delta^\infty \int_{S^{n-1}} K_\lambda(r) r^{n-1} dt' dr\right\}$$
$$\times \left(\int_{\mathbb{R}^n} K_\lambda(|t|)dt\right)^{\frac{p}{q}} + 2^p\epsilon^p \int_0^\delta \mu'(r)K_\lambda(r)dr \times \left(\int_{\mathbb{R}^n} K_\lambda(|t|)dt\right)^{\frac{p}{q}}$$
$$+\, 2^p\,|f(x)|^p \left|\int_{\mathbb{R}^n} K_\lambda(|t|)dt - 1\right|^p .$$

Since $K_\lambda(r)$ belongs to class A, the remaining part of the proof is obvious. □

Example 4.5. Consider the function $f \in L_1(\mathbb{R}^2)$ defined by

$$f(x,y) = \begin{cases} \frac{e^{-\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}, & \text{if } \sqrt{x^2+y^2} \in (0,a] \\ 0, & \text{otherwise.} \end{cases}$$

One can compute that $(0,0)$ is not a Lebesgue point of f. On the other hand by taking , $\mu(|t|) = \sqrt{|t|}$ we see that $(0,0)$ is a $\mu-$generalized Lebesgue point of f.

5 Rate of Convergence

Theorem 5.1. *Suppose that the hypotheses of Theorem 4.2 and Theorem 4.4 are satisfied. Let*

$$\Delta(\lambda,\delta) = \int_0^\delta \mu'(r)K_\lambda(r)dr$$

where $0 < \delta \le \delta_0$, and the following assumptions are satisfied:
(i) $\Delta(\lambda,\delta) \to 0$ *as* $\lambda \to \lambda_0$ *for some* $\delta > 0$.
(ii) *For every* $\xi > 0$

$$K_\lambda(\xi) = o(\Delta(\lambda,\delta))$$

as $\lambda \to \lambda_0$.
(iii) *For every* $\xi > 0$

$$\int_\xi^\infty \int_{S^{n-1}} K_\lambda(r) r^{n-1} dt' dr = o(\Delta(\lambda,\delta))$$

as $\lambda \to \lambda_0$.
Then at each point for which (6) and (7) hold we have as $\lambda \to \lambda_0$

$$|L_\lambda(f;x) - f(x)| = o(\Delta(\lambda,\delta)^{\frac{1}{p}}).$$

Proof. Using Theorem 4.4. we may write

$$\left|L_{\lambda}\left(f;x\right)-f\left(x\right)\right|^{p}\leq 2^{2p}\left\{K_{\lambda}(\delta)\left\|f\right\|^{p}_{L_{p}(\mathbb{R}^{n})}\ +\ \left|f(x)\right|^{p}\int\limits_{\delta}^{\infty}\int\limits_{S^{n-1}}K_{\lambda}(r)r^{n-1}dt'dr\right\}$$

$$\times\left(\int\limits_{\mathbb{R}^{n}}K_{\lambda}(|t|)dt\right)^{\frac{p}{q}}+2^{p}\epsilon^{p}\int\limits_{0}^{\delta}\mu'(r)K_{\lambda}(r)dr\times\left(\int\limits_{\mathbb{R}^{n}}K_{\lambda}(|t|)dt\right)^{\frac{p}{q}}$$

$$+\,2^{p}\left|f(x)\right|^{p}\left|\int\limits_{\mathbb{R}^{n}}K_{\lambda}(|t|)dt-1\right|^{p}.$$

From (i)-(iii) and using class A conditions, we get the desired result:

$$\left|L_{\lambda}\left(f;x\right)-f\left(x\right)\right|=o(\Delta(\lambda,\delta)^{\frac{1}{p}}).$$

Note that, using Theorem 4.2. the similar result can be obtained for $p=1$. Thus the proof is completed. □

References

[1] S. Bochner and K. Chandrasekharan, Fourier Transforms, Annals. of Mathematics Studies, no. 19, Princeton University Press, Princeton, N. J.; Oxford University Press, London (1949).

[2] P. L. Butzer and R. J. Nessel, Fourier Analysis and Approximation, Academic Press, New York, London (1971).

[3] A. D. Gadjiev, The order of convergence of singular integrals which depend on two parameters, in: Special Problems of Functional Analysis and their Appl. to the Theory of Diff. Eq. and the Theory of Func., Izdat. Akad. Nauk, Azerbaĭdažan, SSR., 40–44 (1968).

[4] G. B. Folland, Real Analysis: Modern Techniques and Their applications, Pure and Applied Mathematics, Wiley, New York (1999).

[5] H. Karsli and E. Ibikli, On convergence of convolution type singular integral operators depending on two parameters, *Fasc. Math.*, **38**, 25-39 (2007).

[6] W. Rudin, Real and Complex Analysis, Mc-Graw Hill Book Co., London (1987).

[7] B. Rydzewska, Approximation des fonctions par des intégrales singulières ordinaires, *Fasc. Math.*, **7**, 71-81 (1973).

[8] C. Sadosky, Interpolation of Operators and Singular Integrals: An Introduction to Harmonic Analysis, Marcel Dekker, Basel, New York (1979).

[9] S. Siudut, A theorem of Romanovski type for double singular integrals, *Comment. Math. Prace Mat.*, **28**(2), 355-359 (1989).

[10] S. Siudut, On the convergence of double singular integrals, *Comment. Math. Prace Mat.*, **28**(1), 143-146 (1988).

[11] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, New Jersey (1970).

[12] R. Taberski, Singular integrals depending on two parameters, *Prace Mat.*, **7** (1962).

[13] R. Taberski, On double integrals and Fourier Series, *Ann. Polon. Math.*, **15**, 97–115 (1964).

[14] G. Uysal and M. M. Yilmaz, Some theorems on the approximation of non-integrable functions via singular integral operators, *Proc. Jangjeon Math. Soc.*, **18**(2), 241-251 (2015).

[15] G. Uysal, M. M. Yilmaz and E. Ibikli, A study on pointwise approximation by double singular integral operators, *J. Inequal. Appl.*, **2015**:94, 1-10 (2015).

Author information

Gumrah Uysal, Karabuk University, Faculty of Science,Department of Mathematics, Karabuk, Turkey.
E-mail: guysal@karabuk.edu.tr

Mine Menekse Yilmaz, Gaziantep University, Faculty of Arts and Science, Department of Mathematics, Gaziantep, Turkey.

Ertan Ibikli, Ankara University, Faculty of Science, Department of Mathematics, Ankara, Turkey

Received: March 23, 2015.

Accepted: May 22, 2015.