Palestine Journal of Mathematics

Vol. 11 (Special Issue III)(2022) , 113–119

FIXED POINT THEOREM FOR MULTIVALUED QUASI-CONTRACTION MAPS IN AN M-MENGER SPACE

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: Multivalued quasi-contraction, Menger space, Fixed point theorem.

Abstract Traditional concepts such as a metric space is fascinating since it facilitates a notion that measures distance between two points. Recently, many mathematicians have been interested in generalizing the notion of various spaces by extending it from two points to n points, $n > 2$. These new n-dimensional generalized spaces leave room for further development in fixed point theory and allow for new fixed point theorems to emerge. In this paper we introduce M-menger spaces defined on $n > 2$ points and a fixed point theorem in an M-menger space is also established and validated.

1 Introduction

Menger spaces are probabilistic metric spaces equipped with a t-norm that associates a pair of points with a distribution function. The presumption of Menger was to create a metric by substituting real numbers in the definition of metric spaces with distribution functions. More precisely, in place of distance between two points, Menger proposed a distribution function $F_{ab}(\alpha)$ that can be understood as probability that the distance or length between the pair of points a and b is less than some positive value α. Menger initially called this new space a statistical metric space [1]. Shortly after, Wald suggested minor improvements to statistical metric spaces [2]. A statistical metric space with Wald improvements began to be referred to as a Menger space by subsequent authors including Schweizer and Sklar who released a book that details probabilistic metric spaces [3].

In 2016, Gupta and Kanwar introduced V-fuzzy metric spaces [4]. A V-fuzzy metric space as a generalized version of a fuzzy metric space. In order to achieve this generalization they built upon the existing literature and extended the concepts further.

We begin the same approach as Gupta and Kanwar and extend the concepts involving menger spaces in order to introduce a generalized version of the menger space which we shall call an M-menger space.

2 Menger Space

Definition 2.1. [3] A t-norm is a function $*: [0,1] \times [0,1] \to [0,1]$ such that the following are satisfied for all $p, q, r, s \in [0,1]$,

(i) $p * 1 = p$ (1 acts as the identity element)

(ii) $p * q = q * p$ (symmetry)

(iii) $p * q \leq r * s$ whenever $p \leq r$ and $q \leq s$ (non-decreasing)

(iv) $p * (q * r) = (p * q) * r$. (associative).

Additionally, we say that a t-norm $*$ is a continuous if for every sequence $\{x_n\}$ and $\{y_n\}$ in $[0,1]$ whose limit exist,

$$\lim_{n\to\infty}(x_n * y_n) = \lim_{n\to\infty} x_n * \lim_{n\to\infty} y_n, \text{ for all } n \in \mathbb{N}.$$

Definition 2.2. [3] $F : [-\infty, \infty] \to [0, 1]$ is said to be a distribution function or simply a distribution provided that it is left continuous, non-decreasing, $F(-\infty) = 0$ and $F(\infty) = 1$.

Example 2.3. Define $H : [-\infty, \infty] \to [0, 1]$ by,

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} .$$

H is called the Heaviside function and it is a distribution function.

Definition 2.4. [3] A function $F : X \times X \to S$ is called a probabilistic distance on X where X be a non-empty set and S be the set of all distribution functions on $[-\infty, \infty]$. $F(x, y)$ is usually denoted by F_{xy} for all $x, y \in X$.

Definition 2.5. [3] A probabilistic metric space is a pair (X, F) where X is a non-empty set and F is a probabilistic distance such that following conditions holds for all $x, y, z \in X$,

(i) $F_{xy}(t) = 1$, for all $t > 0$ if and only if $x = y$

(ii) $F_{xy}(0) = 0$

(iii) $F_{xy}(t) = F_{yx}(t)$ for all $t \geq 0$

(iv) If $F_{xy}(t) = 1$, $F_{yz}(s) = 1$ then $F_{xz}(t + s) = 1$ for all $t, s > 0$.

Remark 2.6. [1] $F_{ab}(t)$ can be interpreted as probability of the distance between a and b is less than t

Definition 2.7. [3] Suppose (X, F) is a probabilistic metric space and $*$ is a continuous t-norm. $(X, F, *)$ is a Menger space if

$$F_{xy}(t + s) \geq F_{xz}(t) * F_{zy}(s),$$

where $x, y, z \in X$ with $t, s \geq 0$.

3 Convergence, Cauchy Sequences and Completeness in a Menger Space

In this section $(X, F, *)$ denotes a Menger space and * to mean a continuous t-norm, X a non-empty set and $F : X \times X \to S$ where S is the set of all distribution functions.

Definition 3.1. A sequence $\{x_n\}$ in $(X, F, *)$ is said to be convergent and converges to $x \in X$ if and only if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $N = N(\epsilon, \lambda)$ such that, $F_{x_n x}(\epsilon) > 1 - \lambda$ for $n \geq N$ and we write, $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Definition 3.2. A sequence $\{x_n\}$ in $(X, F, *)$ is Cauchy sequence if for every $0 < \lambda < 0$ and $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $F_{x_n x_m}(\epsilon) > 1 - \lambda$ for all for $n, m \geq N$.

Definition 3.3. A Menger space is complete if every sequence that is Cauchy is also convergent.

4 M-Menger Space

Definition 4.1. Suppose that $*$ a continuous t-norm, X a non-empty set and $F : X^n \to S$, where S is the set of all distribution functions. Then the triple $(X, F, *)$ is an M-menger space provided that for all $x_i \in X, i = 1, 2, ..., n$,

(i) $F_{x_1 x_2 \ldots x_n}(t) = 1$ for all $t > 0$ if and only if $x_1 = x_2 = \cdots = x_n$,

(ii) $F_{x_1 x_1 \ldots x_1 x_2}(t) \geq F_{x_1 x_2 \ldots x_n}(t)$ with $x_2 \neq x_3 \neq \cdots \neq x_n$, where $t \geq 0$,

(iii) $F_{x_1 x_2 \ldots x_n}(0) = 0$,

(iv) $F_{x_1 x_2 \ldots x_n}(t) = F_{p(x_1 x_2 \ldots x_n)}(t)$ where $p(x_1 x_2 \ldots x_n)$ is a permutation of $\{x_1 x_2 \ldots x_n\}$ for all $t \geq 0$,

(v) If $F_{x_1x_2...x_{n-1}a}(t) = 1, F_{aa...ax_n}(t) = 1$ then $F_{x_1x_2...x_{n-1}x_n}(t) = 1$, where $t > 0$,

(vi) $F_{x_1x_2...x_n}(t) = 1$ as $t \to \infty$,

(vii) $F_{x_1x_2...x_{n-1}x_n}(t + s) \geq F_{x_1x_2...x_{n-1}z}(t) * F_{zz...zx_n}(s)$, where $t, s \geq 0$.

Remark 4.2. $F_{x_1x_2...x_n}(t)$ can be interpreted as probability of the distance between the points $x_1, x_2, ..., x_n$ is less than t.

5 Convergence, Cauchy Sequences and Completeness in an M-Menger Space

In this section $(X, F, *)$ will be understood to be an M-menger space where $*$ a continuous t-norm, X a non-empty set and $F : X^n \to S$, where S is the set of all distribution functions.

Definition 5.1. A sequence $\{x_n\}$ is convergent and converges to $x \in X$ if for all $t > 0$ and $0 < \lambda < 1$, there exist $N \in \mathbb{N}$ such that,

$$F_{x_nx_n...x_nx}(t) > 1 - \lambda,$$

for all $n \geq N$. That is $F_{x_nx_n...x_nx}(t) \to 1$ as $n \to \infty$.

Definition 5.2. A sequence $\{x_n\}$ is Cauchy if for all $t > 0$ and $0 < \lambda < 1$, there is an $N \in \mathbb{N}$ such that

$$F_{x_nx_n...x_nx_m}(t) > 1 - \lambda,$$

for all $n, m \geq N$. That is $F_{x_nx_n...x_nx_m}(t) \to 1 \text{ as } n, m \to \infty$.

Definition 5.3. If every sequence that is Cauchy is also convergent then the M-menger space is complete.

Lemma 5.4. *$F_{x_1x_2...x_n}(\cdot)$ is non-decreasing. That is for all $0 < r < t$,*

$$F_{x_1x_2...x_n}(r) \leq F_{x_1x_2...x_n}(t).$$

Proof. Since $r < t$ we have that $t - r > 0$. Now

$$F_{x_1x_2x_3...x_n}(r) * F_{x_nx_nx_n...x_n}(t - r) \leq F_{x_1x_2x_3...x_n}(t).$$

Hence for all $0 < r < t$ we have

$$F_{x_1x_2...x_n}(r) \leq F_{x_1x_2...x_n}(t).$$

□

Lemma 5.5. *If for all $t > 0$ and $x_1, x_2, ..., x_n \in X$ there exist $0 < k < 1$ such that*

$$F_{x_1x_2...x_n}(kt) \geq F_{x_1x_2...x_n}(t),$$

then $x_1 = x_2 = \cdots = x_n$.

Proof. Since $kt < t$, by the previous lemma and our hypothesis, we have $F_{x_1x_2...x_n}(kt) \leq F_{x_1x_2...x_n}(t) \leq F_{x_1x_2...x_n}(kt)$.

This implies $F_{x_1x_2...x_n}(kt) = F_{x_1x_2...x_n}(t)$. In a similar manner since $t < \frac{t}{k} < \frac{t}{k^2} < ...$, we get

$$F_{x_1x_2...x_n}(kt) = F_{x_1x_2...x_n}(t) = F_{x_1x_2...x_n}\left(\frac{t}{k}\right) = \cdots = \to 1.$$

Hence $x_1 = x_2 = \cdots = x_n$.

□

We denote the set of closed, bounded and non-empty subsets of X by $CB_M(X)$.

Lemma 5.6. *If for every $t > 0$ and $x \in X$ with $k \in (0,1)$ and $A \subseteq CB_V(X)$ we have,*

$$F_{x,A,\ldots,A}(kt) \geq F_{x,A,\ldots,A}(t),$$

then $x \in A$.

Proof. Assume for a contraction that

$$x \notin A. \tag{5.1}$$

Let $a \in A$. Then $F_{x,a,\ldots,a}(kt) \geq F_{x,a,\ldots,a}(t)$. This implies $x = a \in A$ by Lemma 5.5. This contradicts (5.1). Hence $x \in A$.

□

Definition 5.7. Let $A_1, A_2, ..., A_n \subseteq CB_M(X)$ and $t > 0$. The Hausdorff M-menger space distance we denoted by $H_{A_1A_2\ldots A_n}(t)$ and defined it as

$$H_{A_1A_2\ldots A_n}(t) = \max \left\{ \begin{array}{c} Sup_{x\in A_1} F_{xA_2A_3\ldots A_n}(t), \\ Sup_{x\in A_2} F_{A_1xA_3\ldots A_n}(t), \\ \vdots, \\ Sup_{x\in A_n} F_{A_1A_2\ldots A_{n-1}x}(t) \end{array} \right\},$$

where

$$F_{xA_2\ldots A_n}(t) = \inf\{F_{xa_2a_3\ldots a_n}(t) : a_2 \in A_2, a_3 \in A_3, ..., a_n \in A_n\},$$

$$\vdots,$$

$$F_{A_1A_2\ldots A_{n-1}x}(t) = \inf\{F_{a_1a_2\ldots a_{n-1}x}(t) : a_1 \in A_1, a_2 \in A_2, ..., a_{n-1} \in A_{n-1}\},$$

Definition 5.8. $\Gamma : X \to CB_M(X)$ is called a q multivalued quasi-contraction mapping provided that there exist $0 \leq q < 1$ such that

$$H_{\Gamma a_1\Gamma a_2\ldots\Gamma a_n}(t) \leq r.\max \left\{ \begin{array}{c} F_{a_1a_2\ldots a_n}(t), \\ F_{a_1\Gamma a_1\Gamma a_1\ldots\Gamma a_1}(t), \\ F_{a_1\Gamma a_2\Gamma a_3\ldots\Gamma a_n}(t), \\ F_{a_2\Gamma a_2\Gamma a_2\ldots\Gamma a_2}(t), \\ F_{a_2\Gamma a_1\Gamma a_3\ldots\Gamma a_n}(t), \\ \vdots, \\ F_{a_n\Gamma a_n\Gamma a_n\ldots\Gamma a_n}(t), \\ F_{a_n\Gamma a_1\Gamma a_2\ldots\Gamma a_{n-1}}(t) \end{array} \right\}.$$

for all $a_i \in A_i, i = 1, 2, ..., n$,

6 Fixed Point Theorem in an M-Menger Space

Theorem 6.1. *Suppose $(X, F, *)$ is an M-menger space that is complete and $\Gamma : X \to CB_M(X)$ is a q-multivalued quasi-contraction. Then there exist $u \in X$ with $u \in \Gamma u$. That is Γ admits a fixed point.*

Proof. By definition of a q-multivalued quasi-contraction, there exist $0 \leq q < 1$ such that for all $a_i \in X, i = 1, 2, ..., n$,

$$H_{\Gamma a_1,\Gamma a_2,\ldots,\Gamma a_n}(t) \leq q.\max \left\{ \begin{array}{c} F_{a_1,a_2,\ldots,a_n}(t), \\ F_{a_1,\Gamma a_1,\Gamma a_1,\ldots,\Gamma a_1}(t), \\ F_{a_1,\Gamma a_2,\Gamma a_3,\ldots,\Gamma a_n}(t), \\ F_{a_2,\Gamma a_2,\Gamma a_2,\ldots,\Gamma a_2}(t), \\ F_{a_2,\Gamma a_1,\Gamma a_3,\ldots,\Gamma a_n}(t), \\ \vdots, \\ F_{a_n,\Gamma a_n,\Gamma a_n,\ldots,\Gamma a_n}(t), \\ F_{a_n,\Gamma a_1,\Gamma a_2,\ldots,\Gamma a_{n-1}}(t) \end{array} \right\}. \tag{6.1}$$

It is clear that for some $a_1 \in A_1$, with $a_2 \in A_2, a_3 \in A_3, ..., a_n \in A_n$ we have

$$F_{a_1,a_2,\dots,a_n}(t) \leq H_{A_1,A_2,\dots,A_n}(t).$$

Using this fact and setting $x_1 \in \Gamma x_0$ with $x_2 \in \Gamma x_1, \cdots, x_n \in \Gamma x_{n-1}$, Inequality 6.1 becomes,

$$\begin{aligned} F_{x_1x_2\dots x_n}(t) &\leq H_{\Gamma x_0,\Gamma x_1,\dots,\Gamma x_{n-1}}(t) \\ &\leq q.\max\left\{\begin{array}{c} F_{x_0,x_1,x_2,\dots,x_{n-1}}(t), \\ F_{x_0,\Gamma x_0,\Gamma x_0,\dots,\Gamma x_0}(t), \\ F_{x_0,\Gamma x_1,\Gamma x_2,\dots,\Gamma x_{n-1}}(t), \\ F_{x_1,\Gamma x_1,\Gamma x_1,\dots,\Gamma x_1}(t), \\ F_{x_1,\Gamma x_0,\Gamma x_2,\dots,\Gamma x_{n-1}}(t), \\ \vdots, \\ F_{x_{n-1},\Gamma x_{n-1},\Gamma x_{n-1},\dots,\Gamma x_{n-1}}(t), \\ F_{x_{n-1},\Gamma x_0,\Gamma x_1,\dots,\Gamma x_{n-2}}(t) \end{array}\right\}. \end{aligned}$$

Similarly setting $x_2 \in \Gamma x_1$, with $x_3 \in \Gamma x_2, \cdots, x_{n+1} \in \Gamma x_n$, Inequality 6.1 becomes,

$$\begin{aligned} F_{x_2,x_3,\dots,x_{n+1}}(t) &\leq H_{\Gamma x_1,\Gamma x_2,\dots,\Gamma x_n}(t) \\ &\leq q.\max\left\{\begin{array}{c} F_{x_1,x_2,x_3,\dots,x_n}(t), \\ F_{x_1,\Gamma x_1,\Gamma x_1,\dots,\Gamma x_1}(t), \\ F_{x_1,\Gamma x_2,\Gamma x_3,\dots,\Gamma x_n}(t), \\ F_{x_2,\Gamma x_2,\Gamma x_2,\dots,\Gamma x_2}(t), \\ F_{x_2,\Gamma x_1,\Gamma x_3,\dots,\Gamma x_n}(t), \\ \vdots, \\ F_{x_n,\Gamma x_n,\Gamma x_n,\dots,\Gamma x_n}(t), \\ F_{x_n,\Gamma x_1,\Gamma x_1,\dots,\Gamma x_{n-1}}(t) \end{array}\right\}. \end{aligned}$$

Continuing in a similar fashion by Mathematical Induction we get a sequence $\{x_k\}_{k=0}^{\infty}$ such that

$$\begin{aligned} F_{x_k,x_{k+1},\dots,x_{k+n-1}}(t) &\leq H_{\Gamma x_{k-1},\Gamma x_k,\dots,\Gamma x_{k+n-2}}(t) \\ &\leq q.\max\left\{\begin{array}{c} F_{x_{k-1},x_k,x_{k+1},\dots,x_{k+n-2}}(t), \\ F_{x_{k-1},\Gamma x_{k-1},\Gamma x_{k-1},\dots,\Gamma x_{k-1}}(t), \\ F_{x_{k-1},\Gamma x_k,\Gamma s_{k+1},\dots,\Gamma x_{k+n-2}}(t), \\ F_{x_k,\Gamma x_k,\Gamma x_k,\dots,\Gamma x_k}(t), \\ F_{x_k,\Gamma x_{k-1},\Gamma x_{k+1},\dots,\Gamma x_{k+n-2}}(t), \\ \vdots, \\ F_{x_{k+n-2},\Gamma x_{k+n-2},\Gamma x_{k+n-2},\dots,\Gamma x_{k+n-2}}(t), \\ F_{x_{k+n-2},\Gamma x_{k-1},\Gamma x_k,\dots,\Gamma x_{k+n-3}}(t) \end{array}\right\}. \end{aligned}$$

We now show that $\{x_k\}_{k=0}^{\infty}$ is Cauchy. If $a = b$ (trivial case) we get $F_{x_a,x_a,\dots,x_a,x_b}(t) = 1 > 1-\epsilon$ where $\epsilon \in (0, 1)$ and therefore $\{x_k\}$ is Cauchy. Assume $a < b$ and $a \neq b$. We have,

$$F_{x_a,x_a,\ldots,x_a,x_b}(t) \leq H_{\Gamma x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{b-1}}(t)$$

$$\leq q.\max\left\{\begin{array}{c} F_{x_{a-1},x_{a-1},\ldots,x_{a-1},x_{b-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{b-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{b-1}}(t), \\ \vdots, \\ F_{x_{b-1},\Gamma x_{b-1},\ldots,\Gamma x_{b-1},\Gamma x_{b-1}}(t), \\ F_{x_{b-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t) \end{array}\right\}$$

$$= q.\max\left\{\begin{array}{c} F_{x_{a-1},x_{a-1},\ldots,x_{a-1},x_{b-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{b-1}}(t), \\ F_{x_{b-1},\Gamma x_{b-1},\ldots,\Gamma x_{b-1},\Gamma x_{b-1}}(t), \\ F_{x_{b-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t) \end{array}\right\}.$$

Now we consider the five cases:

Case I: If

$$\max\left\{\begin{array}{c} F_{x_{a-1},x_{a-1},\ldots,x_{a-1},x_{b-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{b-1}}(t), \\ F_{x_{b-1},\Gamma x_{b-1},\ldots,\Gamma x_{b-1},\Gamma x_{b-1}}(t), \\ F_{x_{b-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t) \end{array}\right\} = F_{x_{a-1},x_{a-1},\ldots,x_{a-1},x_{b-1}}(t).$$

Then as $a, b \to \infty$ and using the fact that $q \in (0,1)$ we have,

$$\begin{aligned} 1 \geq F_{x_{a-1},x_{a-1},\ldots,x_{a-1},x_{b-1}}(t) &\geq \frac{1}{q}F_{x_a,x_a,\ldots,x_a,x_b}(t) \\ &\geq \frac{1}{q^2}F_{x_{a+1},x_{a+1},\ldots,x_{a+1},x_{b+1}}(t) \\ &\geq \cdots \\ &\geq \frac{1}{q^{s+1}}F_{x_{a+s},x_{a+s},\ldots,x_{a+s},x_{b+s}}(t), s \in \mathbb{N} \\ &\geq \cdots \\ &\geq 1. \end{aligned}$$

This implies that $F_{x_a,x_a,\ldots,x_a,x_b}(t) \to 1$ as $a, b \to \infty$. Therefore $\{x_k\}_{k=0}^{\infty}$ is Cauchy.

Case II: If

$$\max\left\{\begin{array}{c} F_{x_{a-1},x_{a-1},\ldots,x_{a-1},x_{b-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t), \\ F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{b-1}}(t), \\ F_{x_{b-1},\Gamma x_{b-1},\ldots,\Gamma x_{b-1},\Gamma x_{b-1}}(t), \\ F_{x_{b-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t) \end{array}\right\} = F_{x_{a-1},\Gamma x_{a-1},\ldots,\Gamma x_{a-1},\Gamma x_{a-1}}(t).$$

Then as $a, b \to \infty$ and using the fact that $q \in (0,1)$ we have,

$$\begin{aligned}
1 \geq F_{x_{a-1}, Tx_{a-1}, \ldots, Tx_{a-1}, Tx_{a-1}}(t) &\geq \frac{1}{q} F_{x_a, x_a, \ldots, x_a, x_b}(t) \\
&\geq \frac{1}{q^2} F_{x_{a+1}, x_{a+1}, \ldots, x_{a+1}, x_{b+1}}(t) \\
&\geq \cdots \\
&\geq \frac{1}{q^{s+1}} F_{x_{a+s}, x_{a+s}, \ldots, x_{a+s}, x_{b+s}}(t), s \in \mathbb{N} \\
&\geq \cdots \\
&\geq 1.
\end{aligned}$$

This implies that $F_{x_a, x_a, \ldots, x_a, x_b}(t) \to 1$ as $a, b \to \infty$. Therefore $\{x_k\}_{k=0}^{\infty}$ is Cauchy. There are three more cases that can be done similarly. In all five cases $\{x_k\}_{k=0}^{\infty}$ is Cauchy. By the completeness property, there exist $u \in X$ such that

$$F_{x_n, x_n, \ldots, x_n, u}(t) \to 1 \text{ as } n \to \infty.$$

That is $x_n \to u$ as $n \to \infty$. Now let $p \in (0,1)$. Then

$$\begin{aligned}
F_{u, \Gamma u, \ldots, \Gamma u}(t) \leq 1 &= F_{u,u,,\ldots,u}(pt) \\
&= \lim_{n \to \infty} F_{x_n, x_n, \ldots, x_n}(pt) \\
&= \lim_{n \to \infty} F_{x_n, \Gamma x_{n-1}, \ldots, \Gamma x_{n-1}}(pt) \text{ since } x_n \in \Gamma x_{n-1}. \\
&= F_{u, \Gamma u, \ldots, \Gamma u}(pt).
\end{aligned}$$

Hence using Lemma 5.6, $u \in \Gamma u$. □

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