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SOME NEW φ-FIXED POINT THEOREMS VIA b-SIMULATION FUNCTIONS AND THEIR APPLICATION ON QUANTUM STRUCTURES

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Abstract In this paper, we introduce a descendant of ϕ-fixed points, naturally termed as ϕ-coincidence points. In follow, we present a ϕ-fixed point theorem and a ϕ-coincidence point theorem, via b-simulation functions, in the turf of b-metric spaces; subsequently, we provide some interesting examples using MATLAB. As a closure, we give an application, in the field of quantum mechanics, to ensure the unique existence of a coincidence quantum state for two quantum operations on the Bloch sphere.

1 Introduction

In recent decades, many fixed point results are obtained, by extending Banach contraction, using control functions, to a general setup. Khojasteh et al.[13] posted the notion of simulation functions, which is one of the renowned class of control functions; the notion is later modified by Argoubi et al.[1] in 2015. Bakhtin[4] developed the notion of b-metric space to investigate pattern matching problems; the first ever fixed point theorem, in this setup is proved by Czerwik[9].

Bota et al.[7], Demmaa et al.[10], Babu and Mosissa[3], and Zada et al.[17] are some others, who posted certain significant works in the context of b-metric spaces. In 2013, Samet et al.[16] established that fixed point results on partial metric spaces can be derived directly from the results in metric spaces, using a new lower semi-continuous mapping $\varphi : X \to [0, \infty)$. Later, Jleli et al.[12] extended the results of Samet et al., by proposing the concept of φ-fixed points. The notion of φ-coupled fixed point is defined and discussed by Fan et al.[11].

Quantum mechanics plays a vital role in cryptography; it enables two communicating parties to detect whether the transmitted message has been intercepted by an eavesdropper. In the domain of quantum theory, a unit of quantum information is known as a qubit, which can be represented by a point on a sphere of unit radius called the Bloch sphere; a self mapping on the Bloch sphere is often referred as a quantum operation. Many significant works are carried out, in finding the conditions that ensures the existence of fixed states of certain quantum operations (see [2, 5, 6, 8, 15, 18, 19]).

In section 2, we give all important prerequisites to go through the theory. In section 3, we define the notion of φ-coincidence point; after that we present a φ-fixed point theorem and a φ-coincidence point theorem via b-simulation functions in the domain of b-metric spaces; further we present some interesting examples, using MATLAB, in order to validate our results. In section 4, we apply the theory, to attest the unique existence of a coincidence quantum state of two quantum operations on the Bloch sphere.

2 Preliminaries

We start with the definition of a φ-fixed point. Let f be a self mapping on S and let $\varphi : S \to [0, \infty)$. An element $u \in S$ is said to be a φ-fixed point[12] of f if $f(u) = u$ and $\varphi(u) = 0$.

Definition 2.1. [9] Let S be a nonempty set and $b \geq 1$. A mapping $\rho : S^2 \to [0, \infty)$ is called as a b-metric if it satisfies

(B1) $\rho(u, v) = 0 \Leftrightarrow u = v$;
(B2) $\rho(u, v) = \rho(v, u)$;
(B3) $\rho(u, w) \leq b[\rho(u, v) + \rho(v, w)]$,

for all $u, v, w \in S$. The pair (S, ρ) is called a b-metric space.

Definition 2.2. [1] A simulation function is a mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ that satisfies:

(1) $\zeta(\mu, \nu) < \nu - \mu$, for all $\mu, \nu > 0$;
(2) if $\{\mu_n\}, \{\nu_n\}$ are two sequences in $(0, \infty)$ so that

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \nu_n = l > 0,$$

then $\limsup_{n\to\infty} \zeta(\mu_n, \nu_n) < 0$.

Definition 2.3. [10] Let $b \geq 1$. A b-simulation function is a mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ that satisfies the following conditions:

(ζ1) $\zeta(\mu, \nu) < \nu - \mu$, for all $\mu, \nu > 0$;
(ζ2) if $\{\mu_n\}, \{\nu_n\}$ are two sequences in $(0, \infty)$ so that

$$0 < \lim_{n\to\infty} \mu_n \leq \liminf_{n\to\infty} \nu_n \leq \limsup_{n\to\infty} \nu_n \leq b \lim_{n\to\infty} \mu_n < \infty,$$

then $\limsup_{n\to\infty} \zeta(b\mu_n, \nu_n) < 0$.

Example 2.4. Let $b = 2$ and let $\zeta : [0, \infty)^2 \to \mathbb{R}$ be defined as

$$\zeta(\mu, \nu) = \begin{cases} \frac{\nu}{2} - \mu & \text{if } (\mu, \nu) \in [0, 1] \times [0, 1]; \\ \frac{\nu}{2\nu+1} - \mu & \text{otherwise.} \end{cases}$$

Then clearly, $\zeta(\mu, \nu) < \nu - \mu$. We wish to show that, ζ is a 2-simulation function. For, suppose $\{\mu_n\}, \{\nu_n\}$ are two sequences in $(0, \infty)$ so that

$$0 < \lim_{n\to\infty} \mu_n \leq \liminf_{n\to\infty} \nu_n \leq \limsup_{n\to\infty} \nu_n \leq 2 \lim_{n\to\infty} \mu_n < \infty.$$

If $\zeta(2\mu_n, \nu_n) \in [0, 1] \times [0, 1]$ except for finitely many n, then

$$\begin{aligned} \limsup_{n\to\infty} \zeta(2\mu_n, \nu_n) &= \limsup_{n\to\infty} \left(\frac{\nu_n}{2} - 2\mu_n\right) \\ &= \limsup_{n\to\infty} \frac{\nu_n}{2} - 2 \liminf_{n\to\infty} \mu_n \\ &< 0. \end{aligned}$$

On the other hand, if $\zeta(2\mu_n, \nu_n) \notin [0, 1] \times [0, 1]$, except for finitely many n, then

$$\begin{aligned} \limsup_{n\to\infty} \zeta(2\mu_n, \nu_n) &= \limsup_{n\to\infty} \left(\frac{\nu_n}{2\nu_n + 1} - 2\mu_n\right) \\ &= \limsup_{n\to\infty} \frac{\nu_n}{2\nu_n + 1} - 2 \liminf_{n\to\infty} \mu_n \\ &< 0. \end{aligned}$$

Finally, suppose there exist subsequences $\{\zeta(2\mu_{n_i}, \nu_{n_i})\}$ and $\{\zeta(2\mu_{n_j}, \nu_{n_j})\}$ such that $\{\zeta(2\mu_{n_i}, \nu_{n_i})\} \in [0, 1] \times [0, 1]$ and $\zeta(2\mu_{n_j}, \nu_{n_j}) \notin [0, 1] \times [0, 1]$. Then we have

$$\limsup_{i\to\infty} \zeta(2\mu_{n_i}, \nu_{n_i}) < 0 \text{ and } \limsup_{j\to\infty} \zeta(2\mu_{n_j}, \nu_{n_j}) < 0,$$

which in turn implies that $\limsup_{n\to\infty} \zeta(2\mu_n, \nu_n) < 0$, as desired.

3 Main results

We start this section, by proving a φ-fixed point theorem involving b-simulation functions, in the context of b-metric space.

Theorem 3.1. *Let f be a self map on S. Let $\varphi : S \to [0, \infty)$ be bijective and ζ be a b-simulation function so that*

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) \geq 0, \text{ for all } u, v \in S. \tag{3.1}$$

Then f has a unique φ-fixed point.

Proof. Let $u_0 \in S$. Then $\rho(u_0, f(u_0)) \in [0, \infty)$. Since φ is onto, there exists $u_1 \in S$ such that $\varphi(u_1) = \rho(u_0, f(u_0))$. Likewise construct a sequence $\{u_n\}$ so that $\varphi(u_n) = \rho(u_{n-1}, f(u_{n-1}))$. Then from the contractive condition (3.1), we have

$$0 \leq \zeta(b\rho(u_n, f(u_n)), \max\{\varphi(u_n), \varphi(u_n)\}) = \zeta(b\varphi(u_{n+1}), \varphi(u_n)).$$

But by $(\zeta 1)$, we have $\zeta(b\varphi(u_{n+1}), \varphi(u_n)) < \varphi(u_n) - b\varphi(u_{n+1})$, which in turn results that $b\varphi(u_{n+1}) < \varphi(u_n)$. Also since $b \geq 1$, we have

$$\varphi(u_{n+1}) < b\varphi(u_{n+1}) < \varphi(u_n).$$

Thus it is clear to observe that, $\{\varphi(u_n)\}$ is a decreasing sequence that is bounded below by 0 and hence it has to converges to some limit l(say).

We wish to show that $l = 0$. Suppose not, that is $l \neq 0$, then by the contractive condition (3.1), we have

$$0 \leq \zeta(b\rho(u_{n-1}, f(u_{n-1})), \max\{\varphi(u_{n-1}), \varphi(u_{n-1})\}) = \zeta(b\varphi(u_n), \varphi(u_{n-1})).$$

Now by taking $\limsup\limits_{n\to\infty}$ on both sides, we get

$$0 \leq \limsup_{n\to\infty} \zeta(b\varphi(u_n), \varphi(u_{n-1})),$$

that contradicts $(\zeta 2)$ and hence $l = 0$. Therefore

$$\lim_{n\to\infty} \varphi(u_n) = 0.$$

Sequentially, since φ is onto, there must exists $u \in S$ such that $\varphi(u) = 0$. Thus all that it remains to prove that $f(u) = u$. Now by using the contractive condition (3.1), we get that

$$0 \leq \zeta(b\rho(u, f(u)), \max\{\varphi(u), \varphi(u_n)\}) = \zeta(b\rho(u, f(u)), \varphi(u_n)).$$

Further, by using $(\zeta 1)$, we have $b\rho(u, f(u)) < \varphi(u_n)$, for all n. Hence by letting $n \to \infty$ on both sides, it is easy to see that, $b\rho(u, f(u)) = 0$ which in turn implies that $f(u) = u$. Now suppose u' is an other φ-fixed point of f, then we have $\varphi(u') = 0$ and $f(u') = u'$. But since φ is one-one, we have $u = u'$ as desired. □

Corollary 3.2. *Let f be a self map on S. Let $\varphi : S \to [0, \infty)$ be bijective and ζ be a simulation function such that*

$$\zeta\left(\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) \geq 0, \text{ for all } u, v \in S.$$

Then f has a unique φ-fixed point.

Proof. The proof follows trivially, by letting $b = 1$ in Theorem 3.1. □

Corollary 3.3. *Let f be a self map on ρ. Let $\varphi : S \to [0, \infty)$ be bijective such that*

$$\int_0^{b\rho(u,f(u))} \psi(x)dx \leq \max\{\varphi(u), \varphi(v)\}, \text{ for all } u, v \in S,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a mapping so that for every $\epsilon > 0$, $\int_0^{\epsilon} \psi(x)dx$ exists and $\int_0^{\epsilon} \psi(x)dx > \epsilon$. Then f has a unique φ-fixed point.

Proof. The proof follows obviously, if we let $\zeta(\mu,\nu) = \nu - \int_0^{\mu} \psi(x)dx$ in Theorem 3.1. □

Example 3.4. Let $S = [0, 1)$ and let

$$\rho(u, v) = \frac{|u - v|^2}{|u - v|^2 + 1}$$

be a b-metric. Then clearly (S, ρ) is a b-metric space, where $b = 2$. Let $f : S \to S$ be a mapping defined by

$$f(u) = u^2, \text{ for all } u \in S$$

and $\varphi : S \to [0, \infty)$ be a mapping defined by

$$\varphi(u) = \frac{20u}{1 - u}, \text{ for all } u \in S.$$

Then φ is bijective. Now if we let

$$\zeta(\mu, \nu) = \begin{cases} 1 & \text{if } (\mu, \nu) = (0, 0); \\ \frac{\nu}{2} - \mu & \text{otherwise,} \end{cases}$$

then clearly ζ is a simulation function. We claim that f, φ and ζ satisfies the contractive condition (3.1). Suppose both u and v are equal to zero, then we have

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) = \zeta(0, 0) = 1 \geq 0.$$

If both u and v are not equal to zero, then we have

$$\begin{aligned} \zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) &= \zeta\left(\frac{2|u - u^2|^2}{|u - u^2|^2 + 1}, \max\left\{\frac{20u}{1 - u}, \frac{20v}{1 - v}\right\}\right) \\ &= \frac{1}{2}\max\left\{\frac{20u}{1 - u}, \frac{20v}{1 - v}\right\} - \frac{2|u - u^2|^2}{|u - u^2|^2 + 1}. \end{aligned}$$

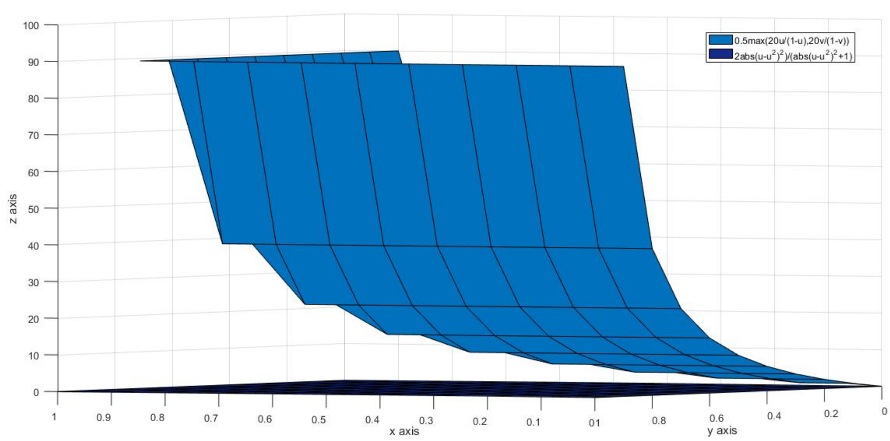

Figure 1.

Therefore, from Fig. 1, we can conclude that

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) \geq 0.$$

Sequentially, if we let $u = 0$ and $v \neq 0$, then we have

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) = \zeta\left(0, \frac{20v}{1 - v}\right) = \frac{10v}{1 - v} \geq 0.$$

Similarly, suppose we let $u \neq 0$ and $v = 0$, then

$$\begin{aligned}\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) &= \zeta\left(\frac{2|u-u^2|^2}{|u-u^2|^2+1}, \frac{20u}{1-u}\right) \\ &= \frac{10u}{1-u} - \frac{2|u-u^2|^2}{|u-u^2|^2+1} \\ &\geq 0\end{aligned}$$

as desired. Therefore, by Theorem 3.1, f has a unique φ-fixed point $0 \in S$.

Note that, if we exclude the condition that φ is one-one in the hypothesis of Theorem 3.1, then the inference that " there exist a unique φ-fixed point" in the theorem becomes questionable. We justify our claim through the following example.

Example 3.5. Let $S = (-\infty, -1] \cup \{0\}$ and let $\rho(u, v) = |u - v|^2$ be a b-metric. Then clearly (S, ρ) is a b-metric space, with $b = 2$. Let $f : S \to S$ and $\varphi : S \to [0, \infty)$ be the mappings defined by

$$f(u) = u \text{ and } \varphi(u) = \begin{cases} 0 & \text{if } u = 0 \\ u^4 - 1 & \text{otherwise.} \end{cases}$$

Then φ is not one-one. If we let $\zeta(\mu, \nu) = \frac{\nu}{\nu+1} - t$, then clearly ζ is a simulation function.

We claim that f, φ and ζ satisfies the contractive condition (3.1). Now suppose $u = 0$ and $v = 0$, then

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) = \zeta(0, 0) = 0.$$

If $u \neq 0$ and $v = 0$, then

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) = \zeta(0, u^4 - 1) = \frac{u^4 - 1}{u^4} \geq 0.$$

Suppose $u = 0$ and $v \neq 0$, then

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) = \zeta(0, v^4 - 1) = \frac{v^4 - 1}{v^4} \geq 0.$$

Finally, If $u \neq 0$ and $v \neq 0$, then

$$\begin{aligned}\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) &= \zeta\left(0, \max\{u^4 - 1, v^4 - 1\}\right) \\ &= \frac{\max\{u^4 - 1, v^4 - 1\}}{\max\{u^4 - 1, v^4 - 1\} + 1} \\ &\geq 0.\end{aligned}$$

Thus we have

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) \geq 0$$

for all $u, v \in S$ as desired. But it is easy to see that -1 and 0 are φ-fixed points of f.

Further, we note that, if we exclude the condition that "φ is onto" in the hypothesis of Theorem 3.1, then the existence of a φ-fixed point becomes questionable. We justify our claim through the forthcoming example.

Example 3.6. Let $S = [0, 1]$ and let $\rho(u, v) = |u - v|^2$ be a b-metric. Then clearly (S, ρ) is a b-metric space with $b = 2$. Let $f : S \to S$ and $\varphi : S \to [0, \infty)$ be the functions defined by

$$f(u) = \frac{u}{2} \text{ and } \varphi(u) = u + 1,$$

then clearly φ is not onto. If we let

$$\zeta(\mu, \nu) = \frac{\nu}{2} - \mu, \text{for all } \mu, \nu \in [0, \infty)$$

to be the simulation function, then

$$\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) = \frac{1}{2}\max\{u+1, v+1\} - 2\left|\frac{u}{2}\right|^2,$$

for all $\mu, \nu \in S$.

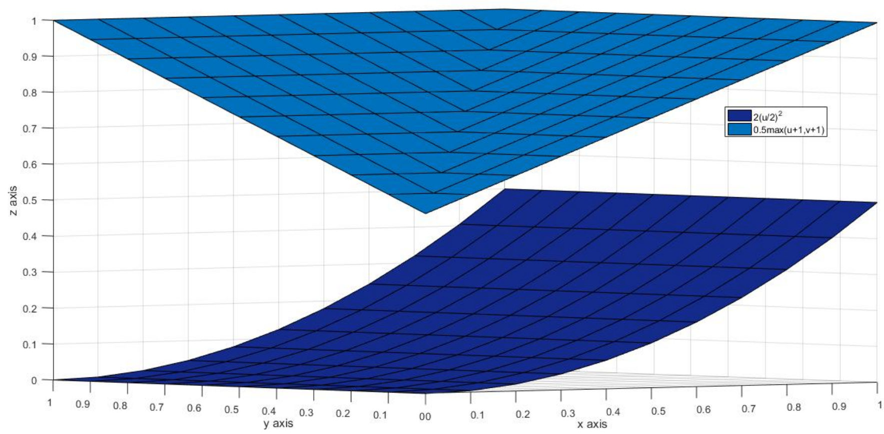

Figure 2.

Therefore, from Fig. 2, it is visible that $\zeta\left(b\rho(u, f(u)), \max\{\varphi(u), \varphi(v)\}\right) \geq 0$ for all $u, v \in S$, whereas f has no $\varphi-$fixed point.

Definition 3.7. Let f, g be two self maps on S and let φ be a mapping from S to $[0, \infty)$. An element $u \in S$ is said to be a φ-coincidence point if $f(u) = g(u)$ and $\varphi(u) = 0$.

Theorem 3.8. *Let f, g be two self maps on S. Let $\varphi : S \to [0, \infty)$ be onto and ζ be a b-simulation function such that*

$$\zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) \geq 0 \text{ for all } u, v \in S. \tag{3.2}$$

Then f and g have a φ-coincidence point. In addition, if either f or g is injective, then the existence is unique.

Proof. Let $u_0 \in S$, then $\rho(f(u_0), g(u_0)) \in [0, \infty)$; also as φ is onto, there exists $u_1 \in S$ such that $\varphi(u_1) = \rho(f(u_0), g(u_0))$. By continuing the argument repeatedly, it is easy to construct a sequence $\{u_n\}$ such that $\varphi(u_n) = \rho(f(u_{n-1}), g(u_{n-1}))$. Now by using the contractive condition (3.2), we have

$$0 \leq \zeta(b\rho(f(u_n), g(u_n)), \max\{\varphi(u_n), \varphi(u_n)\}) = \zeta(b\varphi(u_{n+1}), \varphi(u_n)).$$

Further, by using (ζ1), we get that $0 \leq \zeta(b\varphi(u_{n+1}), \varphi(u_n)) < \varphi(u_n) - b\varphi(u_{n+1})$, which implies $b\varphi(u_{n+1}) < \varphi(u_n)$. Also since $b \geq 1$, we have

$$\varphi(u_{n+1}) < b\varphi(u_{n+1}) < \varphi(u_n).$$

Thus it results that $\{\varphi(u_n)\}$ is a decreasing sequence of real numbers that bounded below by 0 and hence it converges to some limit l(say).

We wish to show that $l = 0$. Suppose not, that is $l \neq 0$, then by the contractive condition (3.2), we have

$$0 \leq \zeta(b\rho(f(u_{n-1}), g(u_{n-1})), \max\{\varphi(u_{n-1}), \varphi(u_{n-1})\}) = \zeta(b\varphi(u_n), \varphi(u_{n-1})).$$

Sequentially, by taking $\limsup_{n\to\infty}$ on both sides, we get

$$0 \leq \limsup_{n\to\infty} \zeta(b\varphi(u_n), \varphi(u_{n-1})),$$

which contradicts $(\zeta 2)$ and therefore $l = 0$. That is,

$$\lim_{n\to\infty} \varphi(u_n) = 0.$$

Now since φ is onto, there exists $u \in S$ such that $\varphi(u) = 0$. Here we claim that $f(u) = g(u)$. By using the contractive condition (3.2) and $(\zeta 1)$ consecutively, we have

$$0 \le \zeta(b\rho(f(u), g(u_n)), \max\{\varphi(u), \varphi(u_n)\}) = \zeta(b\rho(f(u), g(u_n)), \varphi(u_n)\})$$

and

$$b\rho(f(u), g(u_n)) < \varphi(u_n).$$

Further, since $b \ge 1$ and $\lim_{n\to\infty} \varphi(u_n) = 0$, we get that

$$\rho(f(u), g(u_n)) < \varphi(u_n) \text{ and } \lim_{n\to\infty} \rho(f(u), g(u_n)) = 0.$$

Similarly, we can prove that $\lim_{n\to\infty} \rho(g(u), f(u_n)) = 0$. Therefore

$$\rho(f(u), g(u)) = \lim_{n\to\infty} \rho(f(u_n), g(u_n)) = \lim_{n\to\infty} \varphi(u_{n+1}) = 0$$

which in turn implies that $f(u) = g(u)$.

Now all that remains to prove is the uniqueness of u. Without loss of generality, let us assume that f is one-one. Suppose u' is an other φ-coincidence point of f and g, then $f(u') = g(u')$ and $\varphi(u') = 0$. By the contractive condition (3.2), we have

$$0 \le \zeta(b\rho(f(u), g(u')), \max\{\varphi(u), \varphi(u')\}) = \zeta(b\rho(f(u), g(u')), 0)$$

and therefore by $(\zeta 1)$, we get $\rho(f(u), g(u')) = 0$. Thus it results that $f(u) = g(u') = f(u')$, which in turn implies $u = u'$ as desired. □

Corollary 3.9. *Let f, g be two self maps on S, $\varphi : S \to [0, \infty)$ be onto and ζ be a simulation function which satisfies the contractive condition*

$$\zeta(\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) \ge 0 \textit{ for all } u, v \in S.$$

Then f and g have a φ-coincidence point. Further, the existence is unique whenever either f or g is injective

Proof. The proof follows, if we let $b = 1$ in Theorem 3.8. □

Corollary 3.10. *Let f be a self maps on S, $\varphi : S \to [0, \infty)$ be onto and ζ be a b-simulation function which satisfies the contractive condition*

$$\zeta(b\rho(f(u), v), \max\{\varphi(u), \varphi(v)\}) \ge 0 \textit{ for all } u, v \in S.$$

Then f have a unique φ-fixed point.

Proof. We get the proof, by letting $g(u) = u$ in Theorem 3.8. □

Corollary 3.11. *Let f, g be two self maps on S, $\varphi : S \to [0, \infty)$ be onto which satisfies the contractive condition*

$$\int_0^{b\rho(f(u),g(v))} \psi(x)dx \le \max\{\varphi(u), \varphi(v)\}, \textit{ for all } u, v \in S,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a function so that $\int_0^{\epsilon} \psi(x)dx$ exists and $\int_0^{\epsilon} \psi(x)dx > \epsilon$, for each $\epsilon > 0$. Then f and g have a φ-coincidence point. Moreover, if any one of the self maps is injective, then the existence is unique.

Proof. The proof follows, if we let $\zeta(\mu, \nu) = \nu - \int_0^\mu \psi(x)dx$ in Theorem 3.8. □

Here we give an example, to show that the extra condition that " either f or g is injective" is not mandatory in Theorem 3.8.

Example 3.12. Let $S = \mathbb{R}$ be a b-metric with $\rho(u, v) = (u - v)^2$, where $b = 2$. Let $f, g : S \to S$ be the self maps defined by

$$f(u) = \begin{cases} 0 & \text{if } u \in [-2, 2] \\ |\frac{u}{2}| & \text{otherwise} \end{cases} \text{ and } g(u) = \begin{cases} 0 & \text{if } u \in [-2, 2] \\ |\frac{u}{4}| & \text{otherwise.} \end{cases}$$

Let $\varphi : S \to [0, \infty)$ be the mapping defined by $\varphi(u) = u^2$ for all $u \in S$ and let $\zeta(\mu, \nu) = \frac{\nu}{2} - \mu$ be the simulation function.

We claim that, $\zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) \geq 0$ for all $u, v \in S$. For, if $u, v \in [-2, 2]$, then we have

$$\begin{aligned} \zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) &= \zeta(0, \max\{u^2, v^2\}) \\ &= \frac{1}{2}\max\{u^2, v^2\} \geq 0. \end{aligned}$$

Suppose $u, v \notin [-2, 2]$, then

$$\begin{aligned} \zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) &= \zeta\left(2\Big(\left|\frac{u}{2}\right| - \left|\frac{v}{4}\right|\Big)^2, \max\{u^2, v^2\}\right) \\ &= \frac{1}{2}\max\{u^2, v^2\} - 2\Big(\left|\frac{u}{2}\right| - \left|\frac{v}{4}\right|\Big)^2. \end{aligned}$$

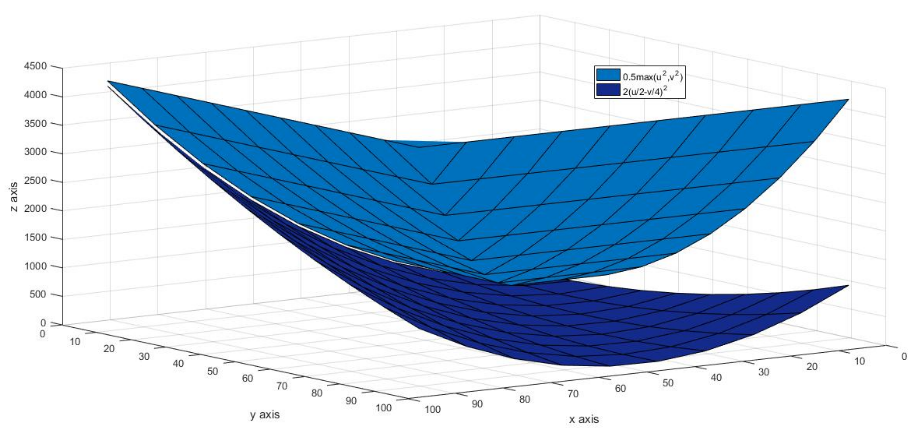

Figure 3.

Therefore, from Fig. 3, it is easy to observe that $\zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) \geq 0$. Sequentially, if we let $u \in [-2, 2]$ and $v \notin [-2, 2]$, then we have

$$\begin{aligned} \zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) &= \zeta\left(2\left|\frac{v}{4}\right|^2, \max\{u^2, v^2\}\right) \\ &= \frac{1}{2}\max\{u^2, v^2\} - 2\left|\frac{v}{4}\right|^2 \\ &= \frac{v^2}{2} - \frac{v^2}{8} = \frac{3v^2}{8} \\ &\geq 0. \end{aligned}$$

Finally, suppose $u \notin [-2, 2]$ and $v \in [-2, 2]$, then

$$\begin{aligned}\zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) &= \zeta\left(2\left|\frac{u}{2}\right|^2, \max\{u^2, v^2\}\right)\\ &= \frac{1}{2}\max\{u^2, v^2\} - 2\left|\frac{u}{2}\right|^2\\ &= \frac{u^2}{2} - \frac{u^2}{2}\\ &= 0\end{aligned}$$

as desired. Also it can be seen clearly that, $0 \in S$ is the only φ-coincidence point of f and g.

Here note that, the condition that "φ is onto" in the hypothesis of Theorem 3.8, is mandatory. Indeed, in Example 3.12, if we let $\varphi(u) = u^2 + 1$, then clearly φ will not be onto, whereas f will be satisfying the condition $\zeta(b\rho(f(u), g(v)), \max\{\varphi(u), \varphi(v)\}) \geq 0$, for all $u, v \in S$. But in fact, it is easy to observe that, f and g have no φ-coincidence point.

4 Application

The Bloch sphere is a geometrical representation of a two dimensional Hilbert space and a qubit or a quantum state is a unit vector in that space. A positive linear map on the Bloch sphere is often referred as a quantum operation. In our work, we consider the geometrical representation of the Hilbert space $\mathbb{C}^2$, consider as a vector space over the field of complex numbers endowed with standard inner product; the north and south poles of the Bloch sphere are typically chosen to correspond to the standard basis vectors $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In general, a qubit $|\psi\rangle$ in the Bloch sphere, can be written as a linear combination of $|0\rangle$ and $|1\rangle$ as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + (\cos\phi + i\sin\phi)\,\sin\left(\frac{\theta}{2}\right)|1\rangle$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Moreover, a qubit $|\psi\rangle$ in the Bloch sphere can be represented as a vector in the unit sphere as

$$(u, v, w) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Here we apply our theory to find whether there is any coincidence between any two given quantum operations.

For consider the Bloch sphere $\mathscr{B} = \{(u, v, w) : \|(u, v, w)\| \leq 1\}$. Let ρ be a metric on $\mathscr{B}$ defined by

$$\rho(\mathbf{x}, \mathbf{y}) = |u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|,$$

where $\mathbf{x} = (u_1, v_1, w_1)$ and $\mathbf{y} = (u_2, v_2, w_2)$.

Let $f, g : \mathscr{B} \to \mathscr{B}$ be the quantum operations defined by

$$f(\mathbf{x}) = (u, v, w)\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} + \left(0, 0, \frac{1}{4}\right) = \left(\frac{u}{2}, \frac{v}{2}, \frac{w}{4} + \frac{1}{4}\right)$$

and

$$g(\mathbf{x}) = (u, v, w)\begin{pmatrix} \frac{2}{5} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \\ 0 & 0 & \frac{-1}{4} \end{pmatrix} + \left(0, 0, \frac{1}{4}\right) = \left(\frac{2u}{5}, \frac{2v}{5}, \frac{1}{4} - \frac{w}{4}\right)$$

for all $\mathbf{x} = (u, v, w) \in \mathscr{B}$, then both f, g are one-one and for any $\mathbf{x} = (u_1, v_1, w_1)$, $\mathbf{y} = (u_2, v_2, w_2)$ belongs to $\mathscr{B}$, we have

$$\rho(f(\mathbf{x}), g(\mathbf{y})) = \left|\frac{u_1}{2} - \frac{2u_2}{5}\right| + \left|\frac{v_1}{2} - \frac{2v_2}{5}\right| + \left|\frac{w_1}{4} + \frac{w_2}{4}\right|.$$

We wish to show that there exist a unique φ-coincidence qubit of f and g.

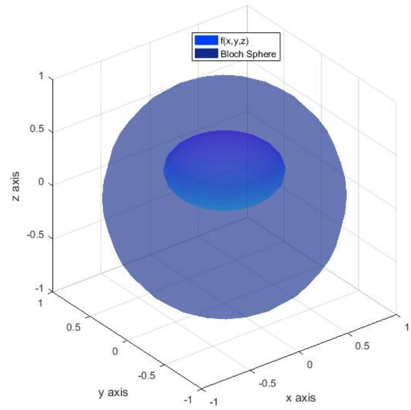

Figure 4.

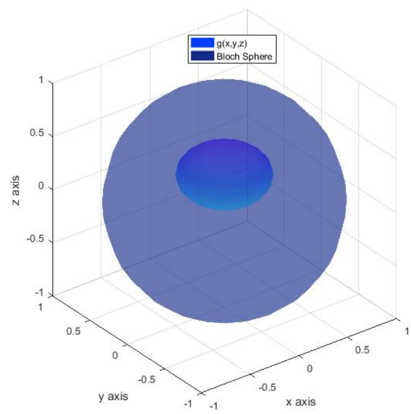

Figure 5.

From Fig. 4 and Fig. 5, it is easy to note that the quantum operations f and g transform the sphere into two ellipsoids:

$$\frac{u^2}{(\frac{1}{2})^2}+\frac{v^2}{(\frac{1}{2})^2}+\frac{(w-\frac{1}{4})^2}{(\frac{1}{4})^2}\leq 1 \text{ and } \frac{u^2}{(\frac{2}{5})^2}+\frac{v^2}{(\frac{2}{5})^2}+\frac{(w-\frac{1}{4})^2}{(\frac{1}{4})^2}\leq 1.$$

Now let $\varphi:\mathscr{B}\to[0,\infty)$ be the map defined by

$$\varphi(\mathbf{x})=\begin{cases} u+v+w & \text{if } u,v,w\geq 0;\\ \frac{1}{|u|} & \text{if } u<0, v=w=0;\\ \frac{1}{|v|} & \text{if } v<0, u=w=0;\\ \frac{1}{|w|} & \text{if } z<0, u=v=0;\\ \frac{1}{|u|}+\frac{1}{|v|} & \text{if } u<0, v\neq 0 \text{ and } w=0 \text{ or } u\neq 0, v<0 \text{ and } w=0;\\ \frac{1}{|u|}+\frac{1}{|w|} & \text{if } u<0, w\neq 0 \text{ and } v=0 \text{ or } u\neq 0, w<0 \text{ and } v=0;\\ \frac{1}{|v|}+\frac{1}{|w|} & \text{if } u<0, w\neq 0 \text{ and } u=0 \text{ or } v\neq 0, w<0 \text{ and } u=0;\\ \frac{1}{|u|}+\frac{1}{|v|}+\frac{1}{|w|} & \text{otherwise.} \end{cases}$$

Then clearly φ is onto. Let ζ be the simulation function defined by $\zeta(\mu,\nu) = \frac{9\nu}{10} - \mu$. We wish to show that $\zeta(\rho(f(\mathbf{x}), g(\mathbf{y})), \max\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in \mathscr{B}$. For, let $\mathbf{x} = (u_1, v_1, w_1)$, $\mathbf{y} = (u_2, v_2, w_2)$ belongs to $\mathscr{B}$. Suppose $u_1, u_2, v_1, v_2, w_1, w_2 \geq 0$, then we have

$$\max\{\varphi(u_1, v_1, w_1), \varphi(u_2, v_2, w_2)\} = \max\{u_1 + v_1 + w_1, u_2 + v_2 + w_2\}$$

and therefore

$$\zeta(\rho(f\mathbf{x}, g\mathbf{y}), \max\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\}) \geq 0.$$

Similarly, we can prove all the other cases. Therefore by corollary 3.9, f and g have a unique $\varphi-$coincidence qubit in $\mathscr{B}$. More precisely, $(0,0,0)$ is the unique $\varphi-$coincidence point. Further, it is easy to note that the point $(0,0,0)$ is nothing but the vector representation of the qubit $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Thus the quantum operations f and g have a coincidence qubit $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

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