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SOLUTION OF FRACTIONAL ORDER SIR EPIDEMIC MODEL USING RESIDUAL POWER SERIES METHOD

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Abstract This study presents the dynamics of a fraction-order SIR epidemic model using the residual power series (RPS) method. The proposed SIR model is described by coupled non-linear ordinary differential equations (NLDEs) with fractional order, where the fractional-order derivative is defined in the Caputo sense. The RPS method is a semi-analytical method based on the generalized Taylor series, which provides the approximate solution in the form of a convergent series that usually converges to the exact solution. The solution obtained using the RPS method is compared with the solution obtained by the fourth-order Runge-Kutta method for $\alpha = 1$ to demonstrate its accuracy, speed, and high order of convergence. The graphical and numerical results indicate that the RPS method can be applied as a semi-analytical technique to get the convergent series solutions of the proposed SIR model and some other coupled NLDEs of fractional order in epidemiology.

1 Introduction

An epidemic is the rapid spread of infectious disease to a large number of people in a given population of a particular area within a short period. Generally, an epidemic occurs when host immunity to either a newly emerging novel pathogen or an established pathogen is suddenly reduced.

Epidemiology is a branch of biology that studies the distribution and determinates of health-related issues in specified communities or populations.It is applicable to control the health problems in communities. The components of epidemiology are frequency, distribution, and the determinants of disease. Epidemiology aims to provide the data necessary for the planning, implementation, and evaluation of services to prevent, control, and treat disease by setting up the priorities among those services. The use of epidemiology is to study the rise and fall of the disease historically in the population, community diagnosis, planning and evaluation, evaluation of individual's risks and chances, syndrome identification, and completing the natural history of the disease.

The fractional calculus is about the theory of derivatives and integrals of arbitrary order, which unite and generalize the concept of integer-order differentiation and integration [1, 2]. Also, it has many more applications in the field of sciences and engineering [3, 4, 5]. The fractional order-based model can provide a more realistic interpretation for the real-world problem [6, 7, 8, 9]. In the present paper, we use Caputo fractional derivative due to its advantage in applied mathematics. The initial conditions for integer-order ordinary differential equations take on the same form as fractional-order differential equations (FODEs) with Caputo derivatives which avoids solvability issues.

Mathematical modeling is a process that uses the language of mathematics to analyze, make predictions, and provide insight into real-world problems. It is helpful because technical, ecological, economic, and other systems investigated by modern science can't be studied adequately using regular theoretical methods. Many real-world problems have been modeled in the form of FODEs, and the system of FODEs [10].

It is necessary to understand the mechanism of disease transmissions to control its spread [6, 9, 11]. Various studies are available on the mathematical modeling in epidemiology. In 1927, Kermack and McKendrick [10] introduced a fractional-order model to understand the dynamic behavior of the infectious disease. Anderson and May [12] discussed the infectious diseases of humans dynamics and control in 1992. In 2009, an analytical approximate solution of a SIR epidemic model with a constant vaccination strategy was studied by Yildrim and Cherruault [13]. In 2011, Yildirim and Kocak [14] discussed an analytical approach to transmission dynamics of infectious diseases with waning immunity. In 2019, the fractional-order Susceptible-Infected-Recovered model has been developed and solved with constant population size[15]. To know more about the epidemic models, their history, and methods to solve them, readers can refer to the references [16, 17, 18, 19, 20].

In this paper, fractional-order Susceptible-Infected-Removed (SIR) epidemic model is formulated with the impact of susceptible, infectious, and removal effects on the transmission dynamics of influenza epidemiology with the application of Caputo derivative for fractional-order $\alpha \in (0, 1]$. To solve the mathematical model, a semi-analytical method, the residual power series method, is used to obtain the approximate solution in the form of the infinite series. The obtained results are compared with the fourth-order Runge-Kutta (RK4) method to validate the accuracy and efficiency of the RPS method. The convergence of the infinite series is also shown in the result.

2 Preliminaries

This section presents the definition of Riemann-Liouville, Riesz, Grunwald-Letnikov, and Caputo derivatives and their properties.

Definition 2.1. [2] For t, $\alpha \in \mathbb{R}^+$, the Riemann-Liouville (RL) fractional derivative of order α of function $p(t)$ is defined as

$$^{RL}D_0^{\alpha} p(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \dfrac{p(y)}{(t-y)^{\alpha+1-m}} dy, & \text{if } (m-1) < \alpha < m \in \mathbb{N}, \\ \dfrac{d^m}{dt^m} p(t), & \text{if } \alpha = m \in \mathbb{N}. \end{cases}$$

Definition 2.2. [2] For t, $\alpha \in \mathbb{R}^+$, the Caputo fractional derivative of order α of function $p(t)$ is defined as

$$D_0^{\alpha} p(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{p^m(y)}{(t-y)^{\alpha-m+1}} dy, & \text{if } (m-1) < \alpha < m \in \mathbb{N}, \\ \dfrac{d^m}{dt^m} p(t), & \text{if } \alpha = m \in \mathbb{N}. \end{cases}$$

Definition 2.3. [21] The fractional power series about $t = t_0$ can be defined as

$$\sum_{r=0}^{\infty} d_r (t-t_0)^{r\alpha} = d_0 + d_1 (t-t_0)^{\alpha} + d_2 (t-t_0)^{2\alpha} + \dots,$$

where $(m-1) < \alpha \leq m,\ m \in N,\ t \geq t_0$. The constants $d_r,\ r = 0, 1, 2, \dots$ are called the coefficients of the power series.

Theorem 2.4. *[21] Let $p(t)$ has a fractional power series representation at $t = t_0$ of the form*

$$p(t) = \sum_{r=0}^{\infty} d_r (t-t_0)^{r\alpha},$$

$t_0 \leq t < (t_0 + \rho)$. If $D_t^{r\alpha} p(t_0), \forall r = 0, 1, 2, \dots$ are continuous on $(t_0, t_0 + \rho)$, then $d_r = \dfrac{D_{t_0}^{r} p(t_0)}{\Gamma(1 + r\alpha)}$, where $D_{t_0}^{r\alpha} = D_{t_0}^{\alpha} D_{t_0}^{\alpha} \dots . D_{t_0}^{\alpha}$ (r-times) and ρ is the radius of convergence.

Property 2.5. [2] Let $p(t) = t^m$, $t \geq 0$. The Caputo derivative of order α for function $p(t)$ can be defined as

$$D_0^\alpha t^m = \begin{cases} \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha} & \text{if } m \geq \lceil \alpha \rceil, \\ 0, & \text{if } m < \lceil \alpha \rceil. \end{cases}$$

Property 2.6. [2] Let $p_1(t)$ and $p_2(t)$ be the continuous functions. For $(m-1) < \alpha \leq m$, and $t \geq 0$, we have,

$$D_0^\alpha (a_1.p_1(t) + a_2.p_2(t)) = a_1.D_0^\alpha p_1(t) + a_2.D_0^\alpha p_2(t).$$

where a_1 and a_2 are constants.

3 The Fractional Order SIR Mathematical Model

We consider the SIR model of a short-termed influenza epidemic, assuming that

(i) the population is large and closed, i.e., there is no immigration, emigration, birth, and death during the influenza epidemic,

(ii) the recovery does not guarantee immunity, i.e., recovered individuals may be reinfected, and

(iii) the parameters do not vary seasonally and are fixed.

The total population N of a particular region consists of three groups, susceptible individuals $(S(t))$, infected individuals $(I(t))$, and removed individuals $(R(t))$ at time t. Let a susceptible individual becomes infected from the disease through contact with infected individuals at a rate of p_1, and an infected individual suffers from the disease through contact with removed and susceptible individuals at a rate of p_2 and p_3, respectively.

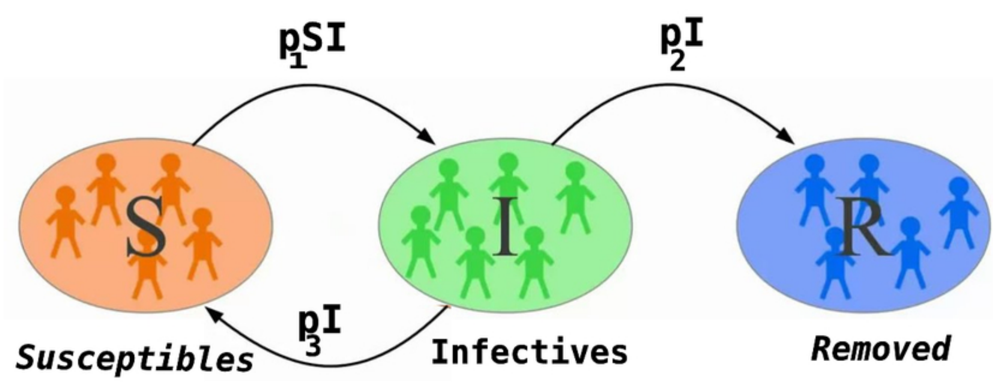

Figure 1. SIR Epidemic Model.

Using the compartments mathematical model approach [22, 23, 24, 25, 26, 27], the proposed SIR epidemic mathematical model of integer-order is formulated as

$$\frac{d}{dt}S(t) = -p_1 S(t)I(t) + p_3 I(t), \tag{3.1a}$$

$$\frac{d}{dt}I(t) = p_1 S(t)I(t) - p_3 I(t) - p_2 I(t), \tag{3.1b}$$

$$\frac{d}{dt}R(t) = p_2 I(t). \tag{3.1c}$$

Similarly, the SIR mathematical model with fractional-order can be written as

$$D_0^{\alpha_1} S(t) = -p_1 S(t)I(t) + p_3 I(t), \tag{3.2a}$$

$$D_0^{\alpha_2} I(t) = p_1 S(t)I(t) - p_3 I(t) - p_2 I(t), \tag{3.2b}$$

$$D_0^{\alpha_3} R(t) = p_2 I(t). \tag{3.2c}$$

Where $\alpha_i \in (0,1]$, $\forall\, i = 1,2,3$. p_1, p_2, and p_3 are positive constants known as the infection, removal, and recovery rates, respectively. Initial conditions for Eqs. (3.1a) to (3.1c) and (3.2a) to (3.2c) are

$$S(0) = S_0,\ I(0) = I_0,\ \text{and } R(0) = R_0. \tag{3.3}$$

Since

$$S(t) + I(t) + R(t) = N,$$

we have

$$D_0^{\alpha_1} S(t) + D_0^{\alpha_2} I(t) + D_0^{\alpha_3} R(t) = 0.$$

The above relation implies that one needs only to study the equations for two of the three variables. Here, $D_0^{\alpha_1} S(t)$, $D_0^{\alpha_2} I(t)$, and $D_0^{\alpha_3} R(t)$ are the Caputo derivatives of order $\alpha_i \in (0,1]$, $i = 1,2,3$ for $S(t)$, $I(t)$, and $R(t)$, respectively.

4 Solution Using Residual Power Series Method

We perform the following steps of the RPS method [28, 29, 30, 31, 32, 33] to construct the approximate solution of the proposed fractional SIR model, which is described in Eqs. (3.2a) to (3.2c) and (3.3).

Step (1): Suppose that $S(t)$, $I(t)$, and $R(t)$ have the fractional power series (FPS) about $t_0 = 0$ as

$$S(t) = \sum_{k=0}^{\infty} \frac{a_k t^{k\alpha_1}}{\Gamma(1+k\alpha_1)},\ I(t) = \sum_{k=0}^{\infty} \frac{b_k t^{k\alpha_2}}{\Gamma(1+k\alpha_2)},\ R(t) = \sum_{k=0}^{\infty} \frac{c_k t^{k\alpha_3}}{\Gamma(1+k\alpha_3)}, \tag{4.1}$$

where $t \in [0,\rho)$, $\rho > 0$ being the radius of convergence. The n^{th} truncated series $S_n(t)$, $I_n(t)$, and $R_n(t)$ of $S(t)$, $I(t)$, and $R(t)$, respectively, are defined as

$$S_n(t) = \sum_{k=0}^{n} \frac{a_k t^{k\alpha_1}}{\Gamma(1+k\alpha_1)},\ I_n(t) = \sum_{k=0}^{n} \frac{b_k t^{k\alpha_2}}{\Gamma(1+k\alpha_2)},\ R_n(t) = \sum_{k=0}^{n} \frac{c_k t^{k\alpha_3}}{\Gamma(1+k\alpha_3)}. \tag{4.2}$$

For $n = 0$, using Eq. (3.3), we have

$$\begin{aligned} S_0(t) &= a_0 = S_0(0) = S_0, \\ I_0(t) &= b_0 = I_0(0) = I_0, \\ R_0(t) &= c_0 = R_0(0) = R_0. \end{aligned}$$

So, the n^{th} truncated series in Eq. (4.2) can be written as

$$S_n(t) = S_0 + \sum_{k=1}^{n} \frac{a_k t^{k\alpha_1}}{\Gamma(1+k\alpha_1)}, \tag{4.3a}$$

$$I_n(t) = I_0 + \sum_{k=1}^{n} \frac{b_k t^{k\alpha_2}}{\Gamma(1+k\alpha_2)}, \tag{4.3b}$$

$$R_n(t) = R_0 + \sum_{k=1}^{n} \frac{c_k t^{k\alpha_3}}{\Gamma(1+k\alpha_3)}. \tag{4.3c}$$

Step (2): We define the residual functions for the model (Eqs. (3.2a) to (3.2c)) as

$$ResS(t) = D_0^{\alpha_1} S(t) + p_1 S(t) I(t) - p_3 I(t), \tag{4.4a}$$

$$ResI(t) = D_0^{\alpha_2} I(t) - p_1 S(t) I(t) + p_3 I(t) + p_2 I(t), \tag{4.4b}$$

$$ResR(t) = D_0^{\alpha_3} R(t) - p_2 I(t). \tag{4.4c}$$

Hence, the n^{th} residual functions of $S(t)$, $I(t)$, and $R(t)$, respectively are

$$ResS_n(t) = D_0^{\alpha_1} S_n(t) + p_1 S_n(t) I_n(t) - p_3 I_n(t), \tag{4.5a}$$

$$ResI_n(t) = D_0^{\alpha_2} I_n(t) - p_1 S_n(t) I_n(t) + p_3 I_n(t) + p_2 I_n(t), \tag{4.5b}$$

$$ResR_n(t) = D_0^{\alpha_3} R_n(t) - p_2 I_n(t). \tag{4.5c}$$

For approximate solution, $ResS(t) = ResI(t) = ResR(t) = 0, \ \ \forall t \geq 0$ and

$$\lim_{n\to\infty} ResS_n(t) = ResS(t), \ \lim_{n\to\infty} ResI_n(t) = ResI(t), \ \lim_{n\to\infty} ResR_n(t) = ResR(t).$$

Since the Caputo derivative of any constant is zero, we have

$$D_0^{(k-1)\alpha_1} ResS(0) = D_0^{(k-1)\alpha_1} ResS_k(0),$$
$$D_0^{(k-1)\alpha_2} ResI(0) = D_0^{(k-1)\alpha_2} ResI_k(0),$$
$$D_0^{(k-1)\alpha_3} ResR(0) = D_0^{(k-1)\alpha_3} ResR_k(0).$$

for $k = 1, \ldots, n$ [28].

Step (3): To obtain the coefficients a_k , b_k, and c_k for $k = 1, 2, 3, \ldots, n$. We substitute the n^{th} truncated series of $S(t)$, $I(t)$, and $R(t)$ into Eqs. (4.5a) to (4.5c) and then apply the Caputo fractional derivative operators$D_0^{(n-1)\alpha_1}$, $D_0^{(n-2)\alpha_2}$, and $D_0^{(n-3)\alpha_3}$ on $ResS(t)$, $ResI(t)$, and $ResR(t)$, respectively. Consequently, we have the equations

$$D_0^{(n-1)\alpha_1} ResS_n(0) = 0, \tag{4.6a}$$

$$D_0^{(n-1)\alpha_2} ResI_n(0) = 0, \tag{4.6b}$$

$$D_0^{(n-1)\alpha_3} ResR_n(0) = 0. \tag{4.6c}$$

for $n = 1, 2, 3, \ldots$.

Step (4): Solve the given algebraic Eqs. (4.6a) to (4.6c) to obtain the values of a_k, b_k, and c_k for $k = 1, 2, 3, \ldots, n$ to get the n^{th} residual power series. i.e., approximate solution of Eqs. (3.2a) to (3.2c) and (3.3).

Step (5): Repeat the procedure to obtain a sufficient number of terms in series Eqs. (4.3a) to (4.3c). Higher accuracy in the solution can be achieved by evaluating more terms in the series solution.

5 Numerical Solution

Consider a situation in which a small group of people is present and that group is suffering from an infectious disease, which is kept into a large population capable of being infected with $p_1 = 0.001$, $p_2 = 0.072$, and $p_3 = 0.005$. Let $S_0 = 620$, $I_0 = 10$, and $R_0 = 70$, so, total human population is $(P) = 620 + 10 + 70 = 700$, and let fractional order $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \in (0, 1]$.

Several steps of the RPS method to solve the fractional SIR epidemic model have been discussed in the previous section. The 1^{st} truncated power series approximations from Eqs. (4.3a) to (4.3c) are

$$S_1(t) = 620 + \frac{a_1 t^\alpha}{\Gamma(1+\alpha)}, \ I_1(t) = 10 + \frac{b_1 t^\alpha}{\Gamma(1+\alpha)}, \text{ and } R_1(t) = 70 + \frac{c_1 t^\alpha}{\Gamma(1+\alpha)}.$$

From Step (3), 1^{st} residual functions of $S(t)$, $I(t)$, and $R(t)$, respectively, are

$$ResS_1(t) = D_0^\alpha S_1(t) + 0.001 S_1(t) I_1(t) - 0.005 I_1(t),$$
$$ResI_1(t) = D_0^\alpha I_1(t) - 0.001 S_1(t) I_1(t) + 0.005 I_1(t) + 0.072 I_1(t),$$
$$ResR_1(t) = D_0^\alpha R_1(t) - 0.072 I_1(t).$$

Substituting $S_1(t)$, $I_1(t)$, and $R_1(t)$ in above expression and equating $ResS_1(0)$, $ResI_1(0)$, and $ResR_1(0)$ to zero give the values of a_1, b_1, and c_1 as

$$a_1 = -6.15, \quad b_1 = 5.43, \quad \text{and} \;\; c_1 = 0.72.$$

Hence, $S_1(t)$, $I_1(t)$, and $R_1(t)$ are

$$S_1(t) = 620 - \frac{6.15t^{\alpha}}{\Gamma(1+\alpha)}, \; I_1(t) = 10 + \frac{5.43t^{\alpha}}{\Gamma(1+\alpha)}, \; R_1(t) = 70 + \frac{0.72t^{\alpha}}{\Gamma(1+\alpha)}.$$

The 2^{nd} truncated power series approximation from Eqs. (4.3a) to (4.3c) are

$$S_2(t) = 620 - \frac{6.15t^{\alpha}}{\Gamma(1+\alpha)} + \frac{a_2 t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad I_2(t) = 10 + \frac{5.43t^{\alpha}}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$R_2(t) = 70 + \frac{0.72t^{\alpha}}{\Gamma(1+\alpha)} + \frac{c_2 t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

From Step (3), the 2^{nd} residual functions of $S(t)$, $I(t)$, and $R(t)$, respectively, are

$$\begin{aligned} ResS_2(t) &= D_0^{\alpha} S_2(t) + 0.001 S_2(t) I_2(t) - 0.005 I_2(t), \\ ResI_2(t) &= D_0^{\alpha} I_2(t) - 0.001 S_2(t) I_2(t) + 0.005 I_2(t) + 0.072 I_2(t), \\ ResR_2(t) &= D_0^{\alpha} R_2(t) - 0.072 I_2(t). \end{aligned}$$

Applying the operator D_0^{α} on $ResS_2(t)$, $ResI_2(t)$, and $ResR_2(t)$, we have

$$\begin{aligned} D_0^{\alpha} ResS_2(t) &= D_0^{2\alpha} S_2(t) + 0.001 D_0^{\alpha} S_2(t) I_2(t) - 0.005 D_0^{\alpha} I_2(t), \\ D_0^{\alpha} ResI_2(t) &= D_0^{2\alpha} I_2(t) - 0.001 D_0^{\alpha} S_2(t) I_2(t) + 0.077 D_0^{\alpha} I_2(t), \\ D_0^{\alpha} ResR_2(t) &= D_0^{2\alpha} R_2(t) - 0.072 D_0^{\alpha} I_2(t). \end{aligned}$$

Substitute $S_2(t)$, $I_2(t)$, and $R_2(t)$ in above expression and equating $D_0^{\alpha} ResS_2(0)$, $D_0^{\alpha} ResI_2(0)$, and $D_0^{\alpha} ResR_2(0)$ with zero, give the values of a_2, b_2, and c_2 as

$$a_2 = -3.27795, \quad b_2 = 2.88699, \quad \text{and} \quad c_2 = 0.39096.$$

Hence, $S_2(t)$, $I_2(t)$, and $R_2(t)$ are

$$S_2(t) = 620 - \frac{6.15t^{\alpha}}{\Gamma(1+\alpha)} - \frac{3.27795t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad I_2(t) = 10 + \frac{5.43t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2.88699t^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$R_2(t) = 70 + \frac{0.72t^{\alpha}}{\Gamma(1+\alpha)} + \frac{0.39096t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

The coefficients of $(k+1)^{th}$ truncations can be found using the following recurrence relation between the coefficients

$$\begin{aligned} a_{k+1} &= -p_1 \sum_{j=0}^{k} \frac{a_j b_{k-j} \Gamma(1+k\alpha)}{\Gamma(1+j\alpha)\Gamma(1+(k-j)\alpha)} + p_3 b_k, \\ b_{k+1} &= p_1 \sum_{j=0}^{k} \frac{a_j b_{k-j} \Gamma(1+(k)\alpha)}{\Gamma(1+(j)\alpha)\Gamma(1+(k-j)\alpha)} - (p_2+p_3) b_k = -a_{k+1} - p_2 b_k, \\ c_{k+1} &= p_2 b_k. \end{aligned}$$

for $k = 0, 1, 2, 3, \ldots$.

6 Result and Discussion

To check the convergence of the RPS solution, absolute errors $|S_{n+1}(t)-S_n(t)|$, $|I_{n+1}(t)-I_n(t)|$, and $|R_{n+1}(t)-R_n(t)|$ are plotted in Figs. 2 to 4, respectively, for for $n = 15, 16, 17, 18,$ and 19, $\alpha = 1$, and $t \in [0, 2]$. All the three absolute errors are $\mathcal{O}(10^{-12})$ or less for $n = 19$ up to $t = 2$. For the lesser value of t, the absolute error is even less with smaller n. Higher accuracy can be achieved by considering more terms in the RPS solution. The results reported in these figures confirm the effectiveness of the RPS method.

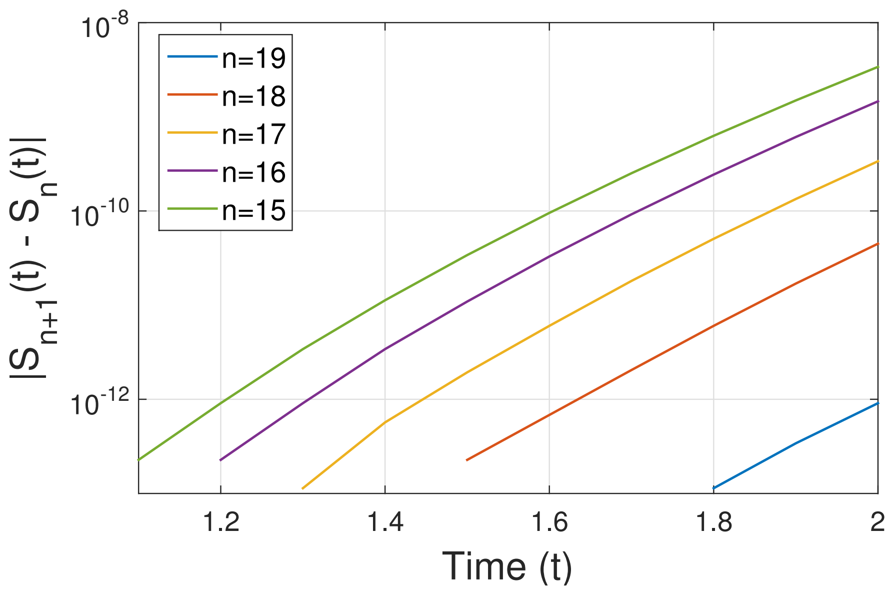

Figure 2. The absolute error between truncations ($n = 15, 16, 17, 18, \&19$) for $S(t)$ using RPS Methods for $\alpha = 1$ in the interval $t \in [0, 2]$.

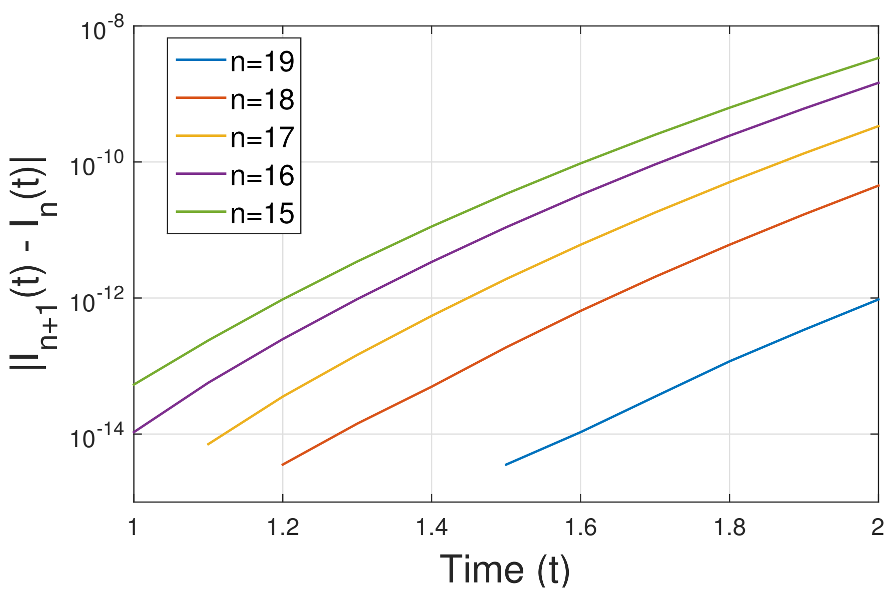

Figure 3. The absolute error between truncations ($n = 15, 16, 17, 18, \&19$) for $I(t)$ using RPS Methods for $\alpha = 1$ in the interval $t \in [0, 2]$.

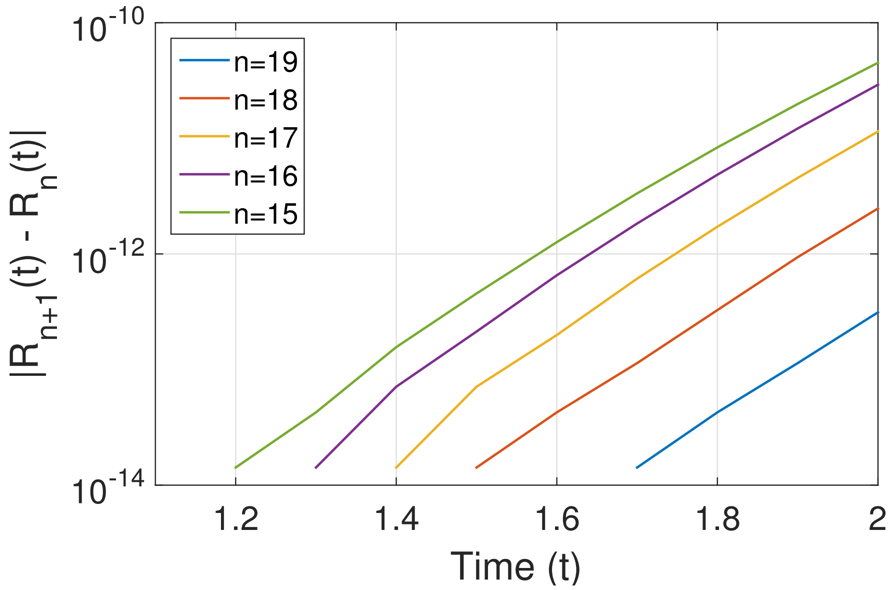

Figure 4. The absolute error between truncations ($n = 15, 16, 17, 18, \& 19$) for $R(t)$ using RPS Methods for $\alpha = 1$ in the interval $t \in [0, 2]$.

Now, we consider the 20^{th}-truncation of the RPS solution, which is close enough to the actual values of $S(t)$, $I(t)$, and $R(t)$, in subsequent study.

$$S(t) \approx S_{20}(t), \quad I(t) \approx I_{20}(t), \quad \& \quad R(t) \approx R_{20}(t).$$

Thus, we can consider $S(t)$, $I(t)$, and $R(t)$ as

$$S(t) = \sum_{k=0}^{20} \frac{a_k t^{k\alpha}}{\Gamma(1 + k\alpha)}, \quad I(t) = \sum_{k=0}^{20} \frac{b_k t^{k\alpha}}{\Gamma(1 + k\alpha)}, \quad R(t) = \sum_{k=0}^{20} \frac{c_k t^{k\alpha}}{\Gamma(1 + k\alpha)}.$$

Table 1. Approximate value of $S(t)$ using RK4 and RPS Methods for $\alpha = 1$.

t_i	RK4 method	RPS method	AbsEr(S(t_i))	RelEr(S(t_i))
0.00	620.000000000000000	620.000000000000000	0	0
0.10	619.368327701129830	619.368327657510350	4.36195E-08	7.04258E-11
0.20	618.702153816378770	618.702153724652930	9.17258E-08	1.48255E-10
0.30	617.999682039910570	617.999681895280220	1.4463E-07	2.3403E-10
0.40	617.259032603941480	617.259032401282640	2.02659E-07	3.28321E-10
0.50	616.478239561945540	616.478239295794650	2.66151E-07	4.31728E-10
0.60	615.655248125904110	615.655247790443130	3.35461E-07	5.44884E-10
0.70	614.787912080138200	614.787911669181200	4.10957E-07	6.68453E-10
0.80	613.873991297021010	613.873990803999960	4.93021E-07	8.03131E-10
0.90	612.911149382834760	612.911148800787370	5.82047E-07	9.49644E-10
1.00	611.896951485213090	611.896950806769950	6.78443E-07	1.10875E-09

This gives the approximate solution for $S(t)$, $I(t)$, and $R(t)$ in terms of fractional powers of time t for fractional order derivative α. For $\alpha = 1$, the solutions in terms of polynomials in time

t can be written as

$$\begin{aligned} S(t) =& 620 - 6.15t - 1.638975t^2 - 0.279321725t^3 - 0.03247663223125t^4 - 0.002310488164925t^5 \\ & + 6.6300959583333 \times 10^{-7}t^6 + 2.85674678775794 \times 10^{-5}t^7 + 4.90413546036706 \times 10^{-6}t^8 \\ & + 4.9453805972773 \times 10^{-7}t^9 + 2.534703453373 \times 10^{-8}t^{10} - 1.65160080322 \times 10^{-9}t^{11} \\ & - 5.9173164612 \times 10^{-10}t^{12} - 8.164527064 \times 10^{-11}t^{13} - 6.84286824 \times 10^{-12}t^{14} \\ & - 1.8357466 \times 10^{-13}t^{15} + 5.1703 \times 10^{-14}t^{16} + 1.11303 \times 10^{-14}t^{17} + 1.28567 \times 10^{-15}t^{18} \\ & + 8.57039 \times 10^{-17}t^{19} - 9.08862 \times 10^{-19}t^{20}, \end{aligned}$$

Table 2. Approximate value of $I(t)$ using RK4 and RPS Methods for $\alpha = 1$.

t_i	RK4 method	RPS method	AbsEr(I(t_i))	RelEr(I(t_i))
0.00	10.000000000000000	10.000000000000000	0	0
0.10	10.557682415377140	10.557682454124135	3.8747E-08	3.67003E-09
0.20	11.145742665297318	11.145742746766953	8.14696E-08	7.30948E-09
0.30	11.765752718571312	11.765752847013921	1.28443E-07	1.09167E-08
0.40	12.419356430322715	12.419356610275818	1.79953E-07	1.44897E-08
0.50	13.108271733277171	13.108271969577906	2.36301E-07	1.80268E-08
0.60	13.834292759467655	13.834293057265034	2.97797E-07	2.1526E-08
0.70	14.599291870396474	14.599292235163372	3.64767E-07	2.49852E-08
0.80	14.599291870396474	15.405222008647254	4.37545E-07	2.84024E-08
0.90	16.254116280801693	16.254116797279011	5.16477E-07	3.17752E-08
1.00	17.148093929799238	17.148094531720652	6.01921E-07	3.51014E-08

Table 3. Approximate value of $R(t)$ using RK4 and RPS Methods for $\alpha = 1$.

t_i	RK4 method	RPS method	AbsEr(R(t_i))	RelEr(R(t_i))
0.00	70.000000000000000	70.000000000000000	0	0
0.10	70.073989883493013	70.073989888365404	4.87239E-09	6.95321E-11
0.20	70.152103518323841	70.152103528579929	1.02561E-08	1.46198E-10
0.30	70.234565241518069	70.234565257705697	1.61876E-08	2.30479E-10
0.40	70.321610965735758	70.321610988441307	2.27055E-08	3.22882E-10
0.50	70.413488704777279	70.413488734627379	2.98501E-08	4.23926E-10
0.60	70.510459114628276	70.510459152291887	3.76636E-08	5.34156E-10
0.70	70.612796049465331	70.612796095655639	4.61903E-08	6.54135E-10
0.80	70.720787131876406	70.720787187352826	5.54764E-08	7.84443E-10
0.90	70.834734336363510	70.834734401933645	6.55701E-08	9.25678E-10
1.00	70.954954584987661	70.954954661509220	7.65216E-08	1.07845E-09

$$\begin{aligned} I(t) =& 10 + 5.43t + 1.443495t^2 + 0.244677845t^3 + 0.02807243102125t^4 + 0.00190624515821667t^5 \\ & - 2.35379514944444 \times 10^{-5}t^6 - 2.83253632335317 \times 10^{-5}t^7 - 4.64920719124504 \times 10^{-6}t^8 \\ & - 4.5734440219632 \times 10^{-7}t^9 - 2.205415483769 \times 10^{-8}t^{10} + 1.79595527124 \times 10^{-9}t^{11} \\ & + 5.8095591449 \times 10^{-10}t^{12} + 7.842766865 \times 10^{-11}t^{13} + 6.43952595 \times 10^{-12}t^{14} \\ & + 1.5266494 \times 10^{-13}t^{15} - 5.239 \times 10^{-14}t^{16} - 1.090842 \times 10^{-14}t^{17} - 1.24203 \times 10^{-15}t^{18} \\ & - 8.09972 \times 10^{-17}t^{19} + 1.20045 \times 10^{-18}t^{20}, \end{aligned}$$

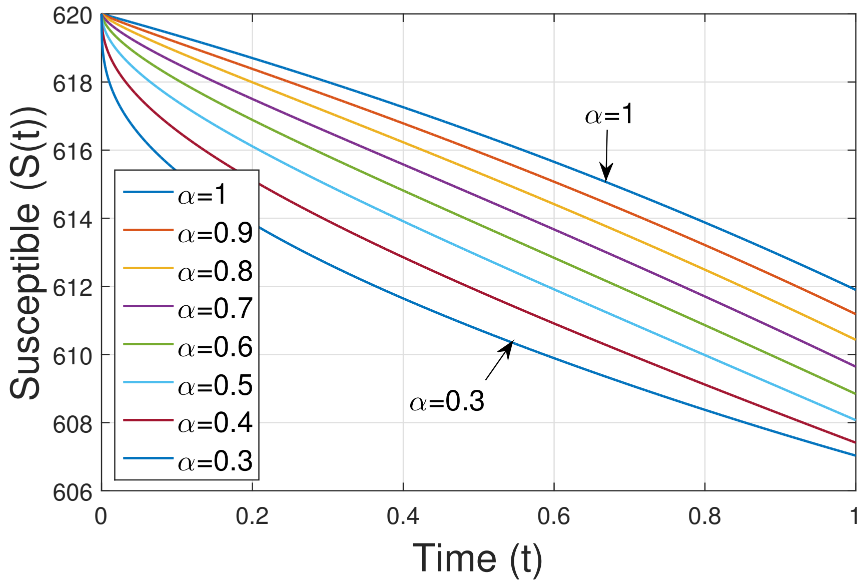

Figure 5. The value of $S(t)$ for distinct value of $\alpha \in (0,1]$ using the RPS method.

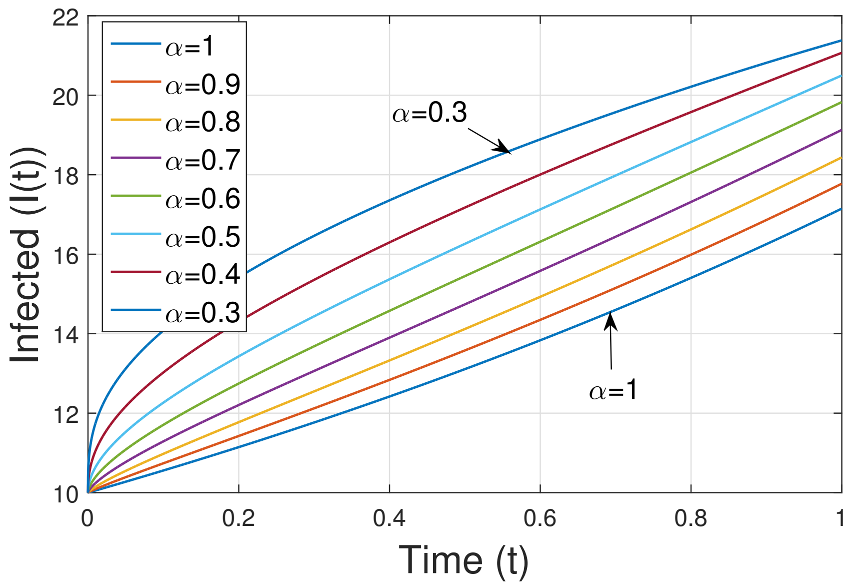

Figure 6. The value of $I(t)$ for distinct value of $\alpha \in (0,1]$ using the RPS method.

$$\begin{aligned}
R(t) =& 70 + 0.72t + 0.19548t^2 + 0.03464388t^3 + 0.00440420121t^4 + 0.000404243006708333t^5 \\
& + 2.28749418986111 \times 10^{-5}t^6 - 2.4210464384921 \times 10^{-7}t^7 - 2.5492826909722 \times 10^{-7}t^8 \\
& - 3.719365753142 \times 10^{-8}t^9 - 3.29287969577 \times 10^{-9}t^{10} - 1.4435446802 \times 10^{-10}t^{11} \\
& + 1.077573163 \times 10^{-11}t^{12} + 3.21760199 \times 10^{-12}t^{13} + 4.0334230 \times 10^{-13}t^{14} \\
& + 3.090972 \times 10^{-14}t^{15} + 6.86992 \times 10^{-16}t^{16} - 2.21887 \times 10^{-16}t^{17} - 4.36337 \times 10^{-17}t^{18} \\
& - 4.70665 \times 10^{-18}t^{19} - 2.9159 \times 10^{-19}t^{20}.
\end{aligned}$$

We also compare the obtained results using the RPS method with the RK4 method. The absolute and relative error of $S(t)$ are defined as

$$AbsEr(S(t)) = |RK4(S(t)) - RPS(S(t))|,$$
$$RelEr(S(t)) = \left|\frac{RK4(S(t)) - RPS(S(t))}{RK4(S(t))}\right|,$$

respectively. Similarly, the absolute and relative errors of $I(t)$ and $R(t)$ can be defined.

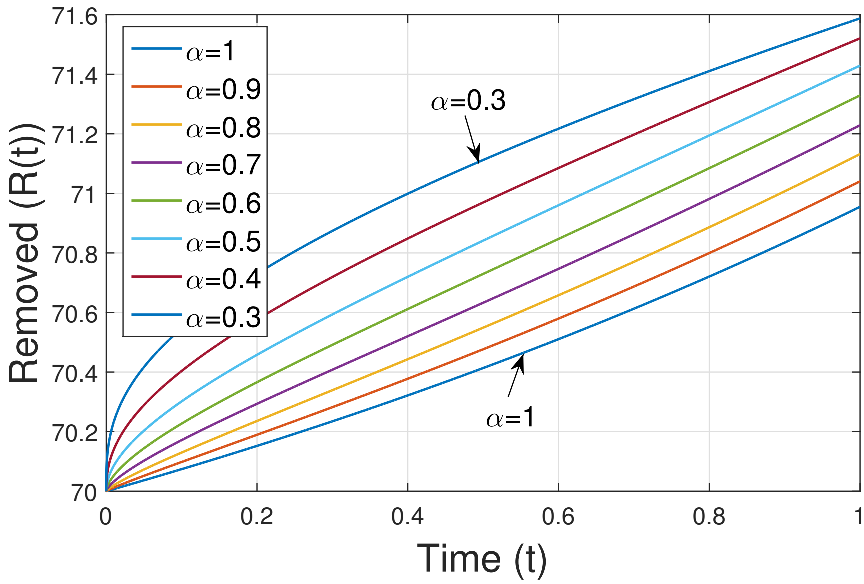

Figure 7. The value of $R(t)$ for distinct value of $\alpha \in (0, 1]$ using the RPS method.

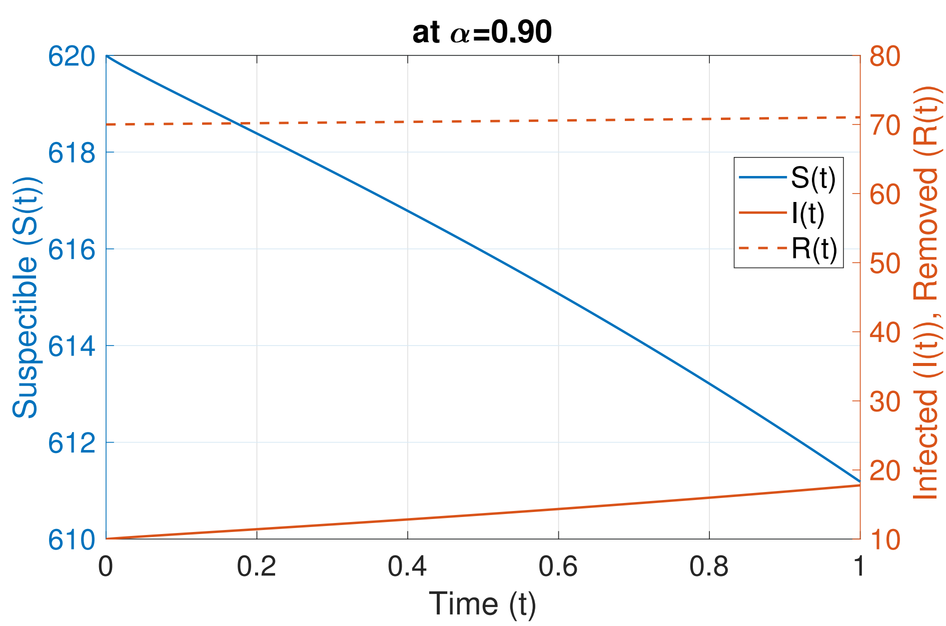

Figure 8. The values of $S(t)$, $I(t)$, and $R(t)$ using RPS solution for $\alpha = 0.90$.

The comparison between the 20^{th}-RPS solution and the RK4 solution with $\alpha = 1$ is presented in Tables 1 to 3 for $S(t)$, $I(t)$, and $R(t)$, respectively. Here, we observe an absolute error

$\mathcal{O}(10^{-6})$ in $S(t)$ and $I(t)$ for $t \in [0,1]$, while error is $\mathcal{O}(10^{-7})$ in $R(t)$. The results presented in Tables 1 to 3 are sufficient enough to approve the efficiency, accuracy of the RPS method for SIR model.

Moreover, to exhibit the effect of the fractional-order derivative present n the SIR model, $S(t)$, $I(t)$, and $R(t)$ for different values of $\alpha \in (0,1]$ are plotted in Figs. 5 to 7 over the interval $t \in [0,1]$.

Here, it is observed that the value of $S(t)$ is increasing with an increase in α, while the values of $I(t)$ and $R(t)$ are decreasing with an increase in α. Further, as $\alpha \to 1$, the solution for $S(t)$, $I(I)$, and $R(t)$ is approaching the solution for $\alpha = 1$.

Fig. 8, shows the behavior of $S(t)$, $I(t)$, and $R(t)$, respectively, using the RPS method for the fractional-order $\alpha = 0.90$, and $n = 20$ over the interval $t \in [0,1]$. It is observed that at any time t, the total population, i.e., sum of $S(t)$, $I(t)$, and $R(t)$ is constant.

7 Conclusion

A mathematical model is an essential tool to describe the transmission dynamics of any disease. In this paper, the SIR epidemic model of fractional order is defined as the generalization of the SIR model of integer order. RPS method is a semi-analytical technique in which we get the approximate solution of any linear and nonlinear differential equations in terms of a convergent series. The results obtained for the proposed model for different values of $\alpha \in (0,1]$ are shown graphically. It is observed that the proposed SIR epidemic model of fractional-order provides a more realistic way to understand the dynamic of epidemiology.

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