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ON ESSENTIALITY AND IRREDUCIBILITY IN A LATTICE

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Abstract We consider a bounded lattice $(L, \wedge, \vee)$ with the smallest element 0 and the greatest element 1. In this paper, we deal with the essentiality concepts associated with a lattice. For an arbitrary element θ of L, we define a θ-e-irreducible element in L, which is an analogy to the concept of the e-irreducible submodule in a module over a ring. It is well known that e-irreducible submodules have no proper essential extension. Indeed, we prove this remains true for elements in a bounded lattice. We establish a relation between the θ-complement and θ-e-irreducible element with suitable examples. We define the notion θ-socle and prove several properties when a lattice is compactly generated. Further, we construct a generalized complement graph of a distributive lattice and relate the properties such as connectedness, diameter, and cut vertices to atoms in a lattice.

1 Introduction

Many lattice theoretical aspects originated from the study of ideal theory in rings or associated submodules in modules over rings. Few of these notions include essential elements, uniform elements, superfluous elements, etc. The study naturally found connections with the lattice properties using chains of submodules of a module. The notion of 'essential submodule' of a module over a ring is an analogy to the concept of 'dense subspace' in a topological space [2]. Unlike in topological spaces, in algebraic systems such as module over rings, there can be a situation where if a submodule is not essential in a given module, then it is possible to hold the essentiality with respect to (or relative to) an appropriate proper submodule. Nevertheless, the concept of a module over a ring is well interpreted in terms of the lattice structure of its submodules.
Grzeszczuk and Puczylowski [13] established the idea of the Goldie dimension from the module theory, to the modular lattices. They defined an essential element in a lattice with the least element 0. Later many developments were found in Calugareanu [9], wherein several ideas from modules over rings were generalized to the lattice theory. The notion of complement plays an important role in modules [5], to establish the dimension of a quotient submodule and the dimension of the sum of two submodules. Similarly, in a lattice with 0, the notion of pseudo-complement has been defined in [9], and some recent developments can be seen in [10]. The authors [17] have explored irreducible elements in almost semilattices. The s-complement undirected graph of lattice modules has been studied by Phadatare et.al [16], and we refer to Badawi [3] for the graph-theoretical properties linked to modules over commutative rings.
The study of graphs associated with algebraic structures is important to understand their structural aspects. The graphs obtained from various algebraic structures, namely, zero divisor graphs, annihilator graphs, and intersection graphs are the significant ones [11, 4, 3]. Amjadi [1], defined an essential ideal graph with respect to a commutative ring. However, in lattices, atoms

play a role to find some important connections between the lattice-theoretic properties and those of corresponding graph-theoretic properties. We consider a lattice $(L, \wedge, \vee)$ with the smallest element 0, and whenever necessary we assume 1 to be the greatest element in L.
For $x, y \in L$, and $x \leq y$, the interval between x and y is denoted by $[x, y] = \{a \in L \mid x \leq a \leq y\}$. If $a \neq 1$ in L, then a is called proper. In a bounded lattice an element a is called an atom (respectively, dual atom) if there is no $x \in L$ such that $0 < x < a$ (respectively, $a < x < 1$). In a sublattice $[m, n]$, an element a is called an atom (respectively, dual atom) if there is no $x \in L$ such that $m < x < a$ (respectively, $a < x < n$). The sum of all simple submodules of a module is called as its socle [5], and its lattice equivalent is the join of atoms in a lattice with 0.
The paper is organized into four sections. In section 2, we define θ-e-irreducibility in a lattice and show that a θ-complement is θ-e-irreducible but not the converse. We observe that in a bounded modular lattice, θ-e-irreducible elements are θ-essentially closed. In section 3, we define the notion θ-socle and prove several properties when a lattice is compactly generated. In section 4, we construct a generalized complement graph and relate the properties such as connectedness, diameter and cut vertices to atoms in a lattice.
For detailed literature in lattice theory, we refer to [8, 12].

2 θ-e-Irreducible

We start with some definitions and notions from [8, 9, 12]. Throughout, $\theta \in L$ is an arbitrary fixed element. An element a of a lattice L is proper if $a \neq 1$. A subset $\mathcal{C}$ of a poset is said to be upper directed, if every $S \subseteq \mathcal{C}$, where S is finite and has an upper bound in $\mathcal{C}$. A complete lattice L is called upper continuous if $a \wedge (\bigvee \mathcal{C}) = \bigvee\limits_{d \in \mathcal{C}} (a \wedge d)$ holds for every $a \in L$ and every upper directed subset $\mathcal{C} \subseteq L$. An element x of a complete lattice L is called compact if for every subset X of L and $x \leq \bigvee X$ there is a finite subset $F \subseteq X$ such that $x \leq \bigvee F$ and S-compact if for each upper directed subset $\mathcal{C} \subseteq L$ and $c \leq \bigvee \mathcal{C}$ there is an element $d_0 \in \mathcal{C}$ such that $c \leq d_0$. A complete lattice L is called compactly generated if each of its elements is a join of compact elements. If $y \in L$ is maximal with respect to the property $x \wedge y = 0$, then y is called a pseudo-complement of x in L, and if for every $x \in L$, there exists a pseudo-complement in L, then L is pseudo-complemented. In a lattice with 0 and 1, an element y is called a complement of x in L if $x \wedge y = 0$ and $x \vee y = 1$. $\theta \neq x \in L$ is θ-essential in y, if $x \wedge y \neq \theta$, for every $\theta \neq y \in L$. We denote it as $x \leq_\theta^e y$.

Definition 2.1. [9, 14] An element $x \in L$ is irreducible if for each $y, z \in L$, $y \wedge z = x$ implies $y = x$ or $z = x$.

We define θ-e-irreducible element analogue to the notion, e-irreducible submodule in a module over a ring.

Definition 2.2. An element $a \in L$ is said to be θ-e-irreducible, if $a = b \wedge c$, where $b, c \in L$ and $a \leq_\theta^e b$ implies $a = b$ or $a = c$.

Lemma 2.3. *Every irreducible element of L is θ-e-irreducible.*

Proof. Suppose a is irreducible. Let $a = b \wedge c$ and $a \leq_\theta^e b$. Then $a = b$ or $b = c$. □

Remark 2.4. The converse of the Lemma 2.3 need not be true, in general. Consider the lattice given in Figure 1.

Since $d \wedge f = b$, $b \neq f$ and $b \neq d$. Therefore, b is not irreducible. But b is θ-e-irreducible, since $d \wedge f = b$ and $b \nleq_\theta^e d$, $b \nleq_\theta^e f$, where $\theta = 0$.

Definition 2.5. If $c \in L$ is maximal with respect to $b \wedge c = \theta$, then we say that c is a θ-complement of b.

Definition 2.6. An element $x \in L$ is closed (or θ-essentially closed) in y, if x has no proper θ-essential extension in $(\theta, y]$, we denote it by $x \leq_\theta^{cl} y$.

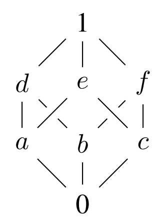

Figure 1.

Lemma 2.7. *In a upper continuous modular lattice L, $\theta < a \in L$ is a θ-complement (of an element) if and only if $a \leq_\theta^{cl} L$.*

Theorem 2.8. *A θ-complement element in L is θ-e-irreducible, but the converse need not be true.*

Proof. Let a be a θ-complement in L. Suppose $a = b \wedge c$, for $b, c \in L$ and $a \leq_\theta^e b$. We show that $a = b$ or $a = c$. Since a is θ-complement, by Lemma 2.7, we have a is θ-essentially closed. Now, since $a \leq_\theta^e b$, we have $a = b$. Therefore, a is e-irreducible. □

Converse of the Theorem 2.8 need not be true. Consider the following Example.

Example 2.9. Let L be a lattice given in Figure 2.

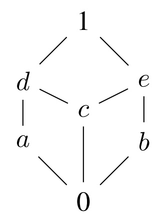

Figure 2.

Clearly, $c = d \wedge e$ and $c \not\leq_\theta^e d$. Therefore, c is θ-e-irreducible. But c is not a θ-complement element, because c is not maximal with respect to $c \wedge x = \theta$, for every $\theta \neq x \in L$.

Theorem 2.10. *Let L be a bounded modular lattice and $a \in L$. Then $a = 1$ or $a \leq_\theta^e L$, but a is θ-e-irreducible if and only if a is θ-essentially closed.*

Proof. If $a = 1$, then nothing to prove. Suppose $a \not\leq_\theta^e L$, but θ-e-irreducible. We show that $a \not\leq_\theta^e b$, for any $1 \neq b \in L$. Clearly $a \neq b$. Let $c \in L$ be such that $b \wedge c = \theta$. By modular law, $b \wedge (a \vee c) = a \vee (b \wedge c) = a \vee \theta = a$. Since a is θ-e-irreducible, $a \neq b$, and $a \leq_\theta^e b$, we get $a = a \vee c$, which implies $c \leq a < b$. So, $c = b \wedge c = \theta$. Therefore, $b \leq_\theta^e L$. Since $a \leq_\theta^e b$ and $b \leq_\theta^e L$, we get $a \leq_\theta^e L$, a contradiction. Therefore, a is θ-essentially closed. Conversely, suppose a is θ-essentially closed. Clearly $a = 1$. It remain to show that $a \not\leq_\theta^e L$, but θ-e-irreducible. Since $a \not\leq_\theta^e b$, for any $1 \neq b \in L$, we have $a \wedge c = \theta$, but $c \neq \theta$, where $c \in [\theta, b]$. Then $a \wedge c = \theta$, but $c \neq \theta$, where $c \in L$. Therefore, $a \not\leq_\theta^e L$. Also, since $a \not\leq_\theta^e b$, for any $1 \neq b \in L$, we have a is θ-e-irreducible. □

The conditions in the Theorem 2.10 are equivalent to a θ-complement if L is upper continuous ([9], Corollary 4.3).

3 θ-socle

Definition 3.1. Let $\theta \in L$. If θ is covered by $a \in L$, then a is said to be a θ-atom (means that, $\theta < a$ and $\theta < x \leq a$ implies $x = a$). The set of all θ-atoms in L is denoted by $A_\theta(L)$. Further, the join of all θ-atoms is called the θ-socle of L, we denote it by $Soc_\theta(L)$. If $Soc_\theta(L) = 1$, then L is called θ-semiatomic.

Note 3.1. If $a \in A_\theta(L)$, then for every $\theta \leq b \in L$, either $a \wedge b = a$ or $a \wedge b \leq \theta$.

Remark 3.2. Every θ-semiatomic need not be semiatomic (as defined in [8]).

Consider the following example.

Example 3.3. Consider the lattice given in Figure 3. Let $\theta = a$. Then, c, d, e are the θ-atoms, and so $Soc_\theta(L) = c \vee d \vee e = 1$. Hence, L is θ-semiatomic. But L is not semiatomic, as the atoms of L are a, b, and $Soc(L) = a \vee b = e \neq 1$.

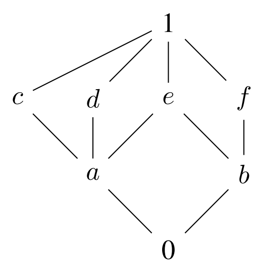

Figure 3.

Proposition 3.4. *In any lattice L,*

(i) $Soc_\theta[\theta, a] \leq a$;

(ii) $a \leq b$ *implies* $Soc_\theta[\theta, a] \leq Soc_\theta[\theta, b]$;

(iii) $Soc_\theta[\theta, (\bigwedge_{i \in I} a_i)] \leq \bigwedge_{i \in I} Soc_\theta[\theta, a_i]$;

(iv) $\bigvee_{i \in I} Soc_\theta[\theta, a_i] \leq Soc_\theta[\theta, (\bigvee_{i \in I} a_i)]$;

for all $\theta < a, b, a_i$ $(i \in I)$ *in* L.

Proof. (i) $Soc_\theta[\theta, a] = \bigvee_{x \in A_\theta[\theta, a]} x = \bigvee_{\theta < x \leq a} x \leq a$.

(ii) Suppose $a \leq b$. Then

$$Soc_\theta[\theta, a] = \bigvee_{x \in A_\theta[\theta, a]} x \leq \bigvee_{x \in A_\theta[\theta, b]} x = Soc_\theta[\theta, b].$$

(iii)

$$\begin{aligned} Soc_\theta[\theta, (\bigwedge_{i \in I} a_i)] &= \bigvee_{x \in A_\theta\left[\theta, \wedge a_i\right]} x \\ &\leq \bigvee_{x \in A_\theta[\theta, a_i]} x, \text{ for all } i \in I \\ &= \bigwedge_{i \in I} \Big(\bigvee_{x \in A_\theta[\theta, a_i]} x \Big) \\ &= \bigwedge_{i \in I} Soc_\theta[\theta, a_i]. \end{aligned}$$

(iv)

$$\begin{aligned}\bigvee_{i\in I} Soc_\theta[\theta, a_i] &= \bigvee_{i\in I} \Big(\bigvee_{x\in A_\theta[\theta,a_i]} x\Big)\\ &\leq \bigvee_{x\in A_\theta[\theta,a_i]} x, \text{ for some } i \in I\\ &= \bigvee_{x\in A_\theta[\theta, \vee a_i]} x\\ &= Soc_\theta[\theta, (\bigvee_{i\in I} a_i)].\end{aligned}$$

□

Lemma 3.5. *[9] L is upper continuous if and only if for every $a \in L$, $X \subseteq L$ and $\mathcal{X} = P_0(X)$, all the finite subsets of X,*

$$a \wedge \Big(\bigvee X\Big) = \bigvee_{F\in\mathcal{X}} (a \wedge (\vee F))$$

holds.

Definition 3.6. [18] Let $\theta \in L$. $S = \{a_i \mid i \in I\}$ be a finite subset of $L \setminus \{\theta\}$. S is said to be θ-$\vee$-independent if $a_i \wedge \big(\bigvee_{j\neq i} a_j\big) = \theta$, for every $i \in I$.

Lemma 3.7. *In an upper continuous lattice L, a subset is θ-$\vee$-independent if and only if each finite subset is θ-$\vee$-independent.*

Proof. Let $S = \{a_i\}_{i\in I}$ be θ-$\vee$-independent set. That is, $a_i \wedge \Big(\bigvee_{i\neq k\in I} a_k\Big) = \theta$. Let $J \subseteq I$.

Now,

$$a_i \wedge \Big(\bigvee_{i\neq j\in J} a_j\Big) \leq a_i \wedge \Big(\bigvee_{i\neq k\in I} a_k\Big) = \theta,$$

shows that J is θ-$\vee$-independent.
Conversely, let $\theta \neq \{a_i\}_{i\in I} \in L$ and i_0 be arbitrary (fixed) in I.
Using the equality

$$\bigvee_{j\neq i_0} a_j = \bigvee_{i_0\notin F\subseteq I} \Big(\bigvee_{i\in F} a_i\Big),$$

and by Lemma 3.5, we obtain

$$a_{i_0} \wedge \Big(\bigvee_{j\neq i_0} a_j\Big) = \bigvee_{i_0\notin F\subseteq I} \Big(a_{i_0} \wedge \bigvee_{i\in F} a_i\Big) = \theta.$$

□

Definition 3.8. If $y \in L$ is maximal with respect to the property $x \wedge y = \theta$, then y is called a θ-complement of x in L, and if for every $x \in L$, there exists a θ-complement in L, then L is θ-complemented.

Definition 3.9. An element a in L is called weak-θ-complement (abbr. ω-θ-complement) if there exists an element a' in L such that $a \wedge a' = \theta$ and $a \vee a' = 1$. Further, L is called ω-θ-complemented if every $l \in L$ has at least one ω-θ-complement. L is called relative ω-θ-complemented if for every $\theta \leq x \in L$ the quotient sublattice $[x, y]$ is ω-θ-complemented.

Example 3.10. Consider the lattice given in Figure 4:

$$L = \{0, a, b, c, d, e, f, g, h, 1\}.$$

Here, f is a ω-$(\theta = c)$-complement of g, but f is not a pseudo-complement of g, since $f \wedge g = c \neq 0$.

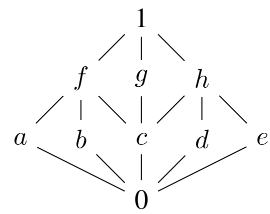

Figure 4.

Corollary 3.11. *A ω-θ-complemented lattice is relative ω-θ-complemented.*

Theorem 3.12. *Every $\theta < a \in L$ has a ω-θ-complement, when L is upper continuous and θ-semiatomic modular.*

Proof. Let L be a upper continuous θ-semiatomic modular lattice. Let $\theta < a \neq 1 \in L$. We show that a has a ω-θ-complement. Since L is θ-semiatomic, there exists a θ-atom s such that $a \wedge s = \theta$ (otherwise, by Note 3.1, we get $a \wedge s = s$ for every $s \in A_\theta(L)$, and so $a = 1$, a contradiction). Let $A_\theta(L) = \{s_i\}_{i \in I}$.
Consider

$$\mathcal{P} = \{J \subseteq I : \{s_i\}_{i \in I} \text{ is } \theta\text{-} \vee \text{-independent and } a \wedge \Big(\bigvee_{j \in J} s_j\Big) = \theta\}.$$

Then, $\mathcal{P} \neq \phi$, and $\mathcal{P}$ is partially ordered under set inclusion. Then by Lemma 3.7, the union of a chain $\{\{s_i\}_{i \in I_k}\}_{k \in K}$ of θ-$\vee$-independent subsets is also θ-$\vee$-independent. If $b_k = \bigvee_{i \in I_k} s_i$, then $a \wedge b_k = \theta$, $k \in K$. Also, since $\{b_k\}_{k \in K}$ is a chain, we get $a \wedge \Big(\bigvee_{k \in K} b_k\Big) = \theta$. Hence, by Zorn's lemma, there is a maximal θ-$\vee$-independent family $\{s_i\}_{i \in J}$ of θ-atoms, with respect to $a \wedge \Big(\bigvee_{i \in J} s_i\Big) = \theta$. Take $c = \bigvee_{i \in J} s_i$. Then $a \wedge c = \theta$. We now show that $a \vee c = 1$. Since the lattice is θ-semiatomic, it is enough to show that $x \leq a \vee c$, for every θ-atom x in L. Let $t \in A_\theta(L)$ such that $t \nleq a \vee c$. Then by Note 3.1, $t \wedge (a \vee c) = \theta$.
Now,

$$\begin{aligned} a \wedge (c \vee t) &\leq (a \vee c) \wedge (c \vee t) \\ &= [(a \vee c) \wedge t] \vee c, \ \text{since } c \leq a \vee c, \text{ and modular law} \\ &= \theta \vee c = c \end{aligned}$$

and $a \wedge (c \vee t) \leq c \wedge a = \theta$.
Now, $t \wedge c \leq t \wedge (a \vee c) = \theta$, and by Lemma 3.7, the subset $t \cup \{s_i\}_{i \in J} \in \mathcal{P}$, which is a contradiction to the maximality of J in $\mathcal{P}$. □

Theorem 3.13. *[9] In a modular lattice L, $[q, p \vee q]$ and $[p \wedge q, p]$ are isomorphic, for every p and q in L.*

Lemma 3.14. *In a ω-θ-complemented modular lattice L, $a \in A_\theta(L)$ if and only if every ω-θ-complement of a is maximal in L.*

Proof. Let $a \in L$ be a θ-atom. Since L is ω-θ-complemented, a has a ω-θ-complement in L, say a'. That is, $a \wedge a' = \theta$ and $a \vee a' = 1$. Now

$$\begin{aligned} [\theta, a] &= [a \wedge a', a] \\ &\cong [a', a \vee a'], \ \text{by Theorem 3.13}, \\ &= [a', 1]. \end{aligned}$$

Therefore, a' is a dual atom.
Converse follows dually. □

Lemma 3.15. *[9] Let $1 \neq a \in L$, where L is a compactly generated lattice. Then $[a, 1]$ has at least one maximal element $\neq 1$.*

Lemma 3.16. *If L is compactly generated and $a \in L$, then $[\theta, a]$ is also compactly generated.*

Proof. By ([9], Proposition 2.4), L is upper continuous, and so it suffices to show the compact elements in $[\theta, a]$ are same as the compact elements in $[\theta, a] \subseteq L$. If $c \in [\theta, a]$ is compact in L, then clearly it is also compact in $[\theta, a]$.
On the other hand, let $c \in [\theta, a]$ compact in $[\theta, a]$, and $c \leq \bigvee \mathcal{C}$, be an upper directed cover of $c \in L$.
Then

$$c = a \wedge c \leq a \wedge (\bigvee \mathcal{C}) = \bigvee_{x \in \mathcal{C}} (a \wedge x), \text{ where } a \wedge x \in [\theta, a],$$

which is a cover of c in $[\theta, a]$. Then there exists $x_1 \in \mathcal{C}$ such that $c \leq a \wedge x_1 \leq x_1$. □

Theorem 3.17. *Each compactly generated ω-θ-complemented modular lattice is θ-semiatomic.*

Proof. Let L be a compactly generated ω-θ-complemented modular lattice. To show L is θ-semiatomic, let $0 \neq \theta \neq a \in L$. By hypothesis, it is enough to prove for compact elements only. To show there exist a θ-atom in $[\theta, a]$. By Lemma 3.16 we have $[\theta, a]$ is compact, then by Lemma 3.15, there is a maximal element $m \neq a$, $m \in [\theta, a]$. Further, if $n \in L$ is a ω-θ-complement of m in L then by modularity, $a \wedge n$ is a (relative) θ-complement of m in $[\theta, a]$. That is, $(a \wedge n) \wedge m = \theta$ and $(a \wedge n) \vee m = a$. Also, by Lemma 3.14, it is a θ-atom in $[\theta, a]$. If $Soc_\theta(L) \neq 1$, let $p \neq \theta$ be a ω-θ-complement of $Soc_\theta(L)$ in L. In a similar way, we can show that $[\theta, p]$ contains one θ-atom, a contradiction to $Soc_\theta(L) \wedge p = \theta$. This shows that $Soc_\theta(L) = 1$, and so L is θ-semiatomic. □

Proposition 3.18. *[9] Each compactly generated lattice is upper continuous.*

Definition 3.19. L is called θ-inductive, if every quotient sublattice $[x, y]$ of L satisfies the condition that: for any chain $\{b_i\}_{i \in I}$ in L and for any $a \in [x, y]$ with $a \wedge b_i = \theta$, for all $i \in I$, imply $a \wedge (\bigvee_{i \in I} b_i) = \theta$.
If $\theta = 0$, θ-inductive coincides with the inductive defined in [9].
Clearly each upper continuous lattice is θ-inductive.

Lemma 3.20. *Let L be θ-inductive lattice. Then every $\theta < a \in L$ has a θ-complement in L.*

Corollary 3.21. *Every $\theta < a \in L$ has a θ-complement in a upper continuous lattice L.*

Proof. Follows from Lemma 3.20. □

Lemma 3.22. *In a compactly generated lattice, every $\theta < a \in L$ is θ-complemented.*

Proof. Follows from the Proposition 3.18 and the Corollary 3.21. □

Note 3.2. Let x, y be elements of L. If $x \vee y \leq^e_\theta L$, then $x \vee y \in [\theta, 1]$.

Lemma 3.23. *Let $1 \in L$, and $a, b \in L$. Then b is a θ-complement of a in L if and only if $a \wedge b = \theta$ and $a \vee b \leq^e_\theta [b, 1]$.*

Proof. Let b be a θ-complement of a in L. Clearly, $b \wedge a = \theta$, and for any $d \in L$, $b < d$ implies $d \wedge a \neq \theta$. In particular, $d \in [b, 1]$, $d \neq b$ implies $d \neq \theta$. Then by modular law and since $d \wedge a \leq d \nleq b$, we have $\theta \leq b < b \vee (d \wedge a) = (a \vee b) \wedge d$. Hence $(a \vee b) \wedge d \neq \theta$, shows that $a \vee b \leq^e_\theta [b, 1]$. Conversely, suppose that $a \wedge b = \theta$ and $a \vee b \leq^e_\theta [b, 1]$. Then, for every $d \in [b, 1]$, $d \neq b$, we have $b < (a \vee b) \wedge d = b \vee (a \wedge d)$. That is, $a \wedge b = \theta$, and for every $b < d$, we have $a \wedge d \nleq b$. This implies $a \wedge d \neq \theta$, showing that b is a θ-complement of a.

□

Theorem 3.24. *A compactly generated modular lattice is θ-semiatomic if and only if it has no proper θ-essential elements.*

Proof. If L is θ-semiatomic and $a \leq_\theta^e L$, then by Theorem 3.12, a has a ω-θ-complement in L, say b. That is $a \wedge b = \theta$ and $a \vee b = 1$. Since $a \leq_\theta^e L$, we get $b = \theta$. This implies $a \wedge \theta = \theta$. This shows that $\theta < a$. Now, $1 = a \vee b = a \vee \theta = a$.
Conversely, let L has no proper θ-essential element. Since L is compactly generated, by Lemma 3.22, we have L is θ-complemented. To show L is θ-semiatomic, by Theorem 3.17, it is enough to show that L is ω-θ-complemented. Let a be a θ-complement of b in L. Then by Lemma 3.23, $a \wedge b = \theta$ and $a \vee b \leq_\theta^e L$. Now by hypothesis, we get $a \vee b = 1$. Therefore, L is ω-θ-complemented. □

Corollary 3.25. *If L is modular compactly generated θ-semiatomic, then L is ω-θ-complemented.*

Proof. Follows from the Theorem 3.24. □

Proposition 3.26. *In an arbitrary lattice, $Soc_\theta(L) \leq \bigwedge_{m \in E_\theta(L)} m$, where $E_\theta(L)$ is the set of all θ-essential elements of L.*

Proof. Let $\theta \neq l$ be a θ-atom, and $\theta \neq m$, be an θ-essential element in L. Then $l \wedge m \neq \theta$. But $\theta < l \wedge m \leq l$ and l is a θ-atom, we have $l \wedge m = l$, implies $l \leq m$.
Therefore,

$$Soc_\theta(L) = \bigvee_{a_i \in A_\theta(L)} a_i \leq m.$$

Since m is arbitrary, we get

$$Soc_\theta(L) \leq \bigwedge_{m \in E_\theta(L)} m.$$

□

Proposition 3.27. *Each θ-atom in a upper continuous lattice is a compact element.*

Proof. Let L be an upper continuous lattice, and $a \in L$ be a θ-atom. Let $a \leq \bigvee \mathcal{C}$, where $\mathcal{C} \subseteq L$ be upper directed. If $a \nleq x$, for each $x \in \mathcal{C}$, then $a \wedge x \neq a$. Since a is a θ-atom, by Note 3.1 we get $a \wedge x \leq \theta$. Now, $a = a \wedge (\bigvee \mathcal{C}) = \bigvee_{x \in \mathcal{C}} (a \wedge x) \leq \bigvee \theta = \theta$, a contradiction to a is θ-atom. Therefore, $a \leq x$, for some $x \in \mathcal{C}$. □

Lemma 3.28. *Let L be a compactly generated ω-θ-complemented modular lattice. Then each element $\theta < p \in L$ is a join of θ-atoms.*

Proof. Clearly, L is upper continuous. Then by Proposition 3.27, each θ-atom is a compact element. Let $\theta < l$ be a compact element, which is not a θ-atom. We show that $[\theta, l]$ contains a dual atom. Let $K = \{k \in L : \theta \leq k < l\}$. Clearly, $\theta \in K$, and so $K \neq \emptyset$. By Lemma 3.15, for each $l \neq k \in K$, the sublattice $[k, l]$ has at least one maximal element $< l$. Then by Zorn's lemma, there exists a dual atom d such that $d < l$. Since d is maximal in $[k, l]$ and L is ω-θ-complemented, by Corollary 3.11, we have L is relative ω-θ-complemented, and so d has a ω-θ-complement in $[k, l]$, say t. Then by Lemma 3.14, t is a θ-atom. Since $t \leq l$, $[k, l]$ contains at least one θ-atom. Since L is compactly generated, we have each element is a compact element. Since l is arbitrary, each $\theta < l \in L$ has at least one θ-atom.
Let

$$m = \bigvee_{t \in A_\theta[\theta, x]} t,$$

where $\theta \leq x \in L$ is arbitrary.
If $m < x$ and let n be the ω-θ-complement of m in $[\theta, x]$, we get $m \wedge n \neq \theta$, a contradiction. Hence $m = x$. □

Theorem 3.29. *In a compactly generated modular lattice, $Soc_\theta(L) = \bigwedge_{e \in E_\theta(L)} e$.*

Proof. Let $k = \bigwedge_{a_i \in E_\theta(L)} a_i$. To show, $Soc_\theta(L) = k$. Clearly, by Lemma 3.26, $Soc_\theta(L) \le k$. To show $[\theta, k]$ is ω-θ-complemented, let $a \in [\theta, k]$. Then by Lemma 3.22, a is θ-complemented in L. Let c be the θ-complement of a in L. Then by Lemma 3.23, we have $a \vee c \le_\theta^e L$, and so $k \le a \vee c$. Now $a \wedge (c \wedge k) \le a \wedge c = \theta$, and since $a \le k$, by modular law, $a \vee (c \wedge k) = (a \vee c) \wedge k = k$. Therefore, $c \wedge k$ is a ω-θ-complement for a in $[\theta, k]$. Then by Lemma 3.28, each element in $[\theta, k]$ is a join of θ-atoms, including k itself. So, $k \le Soc_\theta(L)$. Therefore, $k = Soc_\theta(L)$. □

4 Generalized complement graph of a lattice

We consider a simple finite graph G, whose vertex set is $V(G)$ and the edge set is $E(G)$. We use ab to represent the edge between $a, b \in V(G)$, and we denote by $deg(v)$ the number of vertices associated with v. If a vertex is adjacent to all other vertices in G, then we refer it as a universal vertex. G is called disconnected if $ab \notin E(G)$, for some $a, b \in V(G)$. G is called a null graph if $V(G) = \emptyset$, and is called an empty graph if $E(G) = \emptyset$. The length of the shortest path between two vertices a, b in G, is denoted by $d(a, b)$, and $d(a, b) = \infty$, if such a path doesn't exist between a and b. Evidently, $d(a, a) = 0$. The diameter of a graph G, denoted by $diam(G)$, is equal to $\sup\{d(a, b) : a, b \in V(G)\}$. A vertex x of a connected graph G is a cut vertex of G if $G - \{x\}$ is disconnected. G is said to be k-regular ($k \in \mathbb{N}$), if every vertex is of degree k. $S \subseteq V(G)$ is said to be independent if $E(S) = \emptyset$, and the maximum size of an independent set is called as independence number $\alpha(G)$. A vertex whose neighborhood contains exactly one vertex is called pendant vertex.
For standard definitions and notations in graph theory, we refer to [6, 7].

Definition 4.1. An element $0 \neq x \neq 1$ of L is called semi-complement (abbr. s-complement) $x \wedge y = 0$, for some $0 \neq y \in L$.

Evidently, every pseudo-complement is a semi-complement. Now we construct a graph on semi-complement elements in a distributive lattice which we will refer it as generalized complement graph in the rest of the paper. We also study the properties such as connectedness, diameter and cut vertices relating to atoms in a lattice.

Definition 4.2. Let L be a distributive lattice. We define the generalized complement graph $gc(L)$, whose vertex set

$$V(gc(L)) = \{a : a \text{ is a } s\text{-complement element of } L\}$$

and the edge set

$$E(gc(L)) = \{ab : a \vee b \text{ is a } s\text{-complement element of } L\}.$$

Example 4.3. Consider the lattice given in Figure 3. Then graph $gc(L)$ corresponding to L is given in Figure 5:

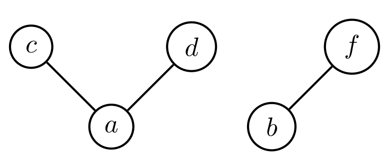

Figure 5.

Lemma 4.4. *[7, 8] Let L be a distributive lattice. Then every non-zero element a of L is either an atom or there exists an atom b of L such that $b < a$.*

Definition 4.5. An element $0 \neq a \neq 1$ of L is said to non s-complement if for any $0 \neq b \in L$, $a \wedge b \neq 0$.

Lemma 4.6. *(1) Let $a \in L$. Then a is a non s-complement if and only if $Soc(L) \le a$;*

(2) If x is an atom in L with $x \leq a \vee b$ for $0 \neq a, b \in L$, then $x \leq a$ or $x \leq b$.

Proof. (1) Suppose $a \in L$ is non s-complement. Then $a \wedge p \neq 0$, for every $0 \neq p \in L$. In particular, $a \wedge x_i \neq 0$, for every $x_i \in A(L)$. Now, $0 \neq a \wedge x_i \leq x_i$ implies $a \wedge x_i = x_i$, and so $x_i \leq a$. This is true for all $x_i \in A(L)$. Therefore, $Soc(L) \leq a$. Conversely, suppose that $Soc(L) \leq a$ and a is s-complement. Then there is an element $0 \neq x \in L$ with $a \wedge x = 0$. Then by Lemma 4.4, there exists an $a_i \in A(L)$ such that $a_i \leq x$. Now, $a \wedge a_i \leq a \wedge x = 0$. Hence, $a \wedge a_i = 0$ implies $a_i \nleq a$, a contradiction.

(2) Let $0 \neq x \in L$ be an atom and $0 \neq a, b \in L$ such that $x \leq a \vee b$. If $x \nleq a$ and $x \nleq b$, then $x \wedge a \neq x$ and $x \wedge b \neq x$. Since x is an atom, we have $x \wedge a = 0$ and $x \wedge b = 0$. Then there exists a diamond sublattice in L, a contradiction as L is distributive. Therefore, $x \leq a$ or $x \leq b$.

□

Lemma 4.7. *Let $A(L) = \{a_i\}_{i \in I}$ and $T \subset I$. Then, $\bigvee_{i \in T} a_i$ is s-complement.*

Proof. On a contrary, suppose $\bigvee_{i \in T} a_i$ is non s-complement. Then $\bigvee_{i \in T} a_i \wedge x \neq 0$, for all $0 \neq x \in L$. In particular, $\bigvee_{i \in T} a_i \wedge a_j \neq 1$, for every $a_j \in A(L)$. Since a_j is an atom, we have $\bigvee_{i \in T} a_i \leq a_j$, for each $j \in I \setminus T$. Then by Lemma 4.6 (2) we have $a_i \leq a_j$, for some $i \in I$, a contradiction as a_i is an atom. □

Theorem 4.8. *The following statements are equivalent.*

(1) $gc(L)$ has more than one component.

(2) $|A(L)| = 2$.

(3) $gc(L) = gc_1(L) \cup gc_2(L)$, where $gc_1(L)$ and $gc_2(L)$ are complete and disjoint subgraphs.

Proof. (1) ⇒ (2): Suppose $gc(L)$ is disconnected. Clearly, $|A(L)| \neq 1$, since $gc(L)$ has at least two components. Since $gc(L)$ is disconnected, without loss of generality we may assume that $gc(L)$ has two components, say C_1 and C_2. Let $a, b \in L$ such that $a \in V(C_1)$ and $b \in V(C_2)$. Since ab is not an edge, we have $a \vee b$ is non s-complement. Choose $x_1, x_2 \in A(L)$ such that $x_1 \leq a$ and $x_2 \leq b$. Then $x_1 \vee a = a$ is s-complement in L_1 and $x_2 \vee b = b$ is s-complement in L_2. That is, $x_1 a$ and $x_2 b$ are edges in C_1 and C_2 respectively.
Case (i): If $x_1 = x_2$, then we have $a - x_1 - b$ is a path in $gc(L)$, a contradiction.
Case (ii): If $x_1 \neq x_2$, then since $|A(L)| > 2$, by Lemma 4.7 we have $x_1 \vee x_2$ is a s-complement. Then $x_1 x_2 \in E(gc(L))$ in a path $a - x_1 - x_2 - b$, which is a contradiction as $gc(L)$ is disconnected. Therefore, $|A(L)| = 2$.

(2) ⇒ (3): Assume that $|A(L)| = 2$, and $A(L) = \{x_1, x_2\}$. Define

$$V(gc_i(L)) = \{t \in L : x_i \leq t \text{ and } t \text{ is } s\text{-complement}\},$$

for $i = 1, 2$. In order to show $gc(L) = gc_1(L) \cup gc_2(L)$, where $gc_1(L)$, $gc_2(L)$ are subgraphs of $gc(L)$, first we show that $gc_1(L)$ is complete subgraph, that is, for any $a, b \in V(gc_1(L))$, $ab \in E(gc_1(L))$. On a contrary, suppose that there exist $a, b \in V(gc_1(L))$ such that $ab \notin E(gc_1(L))$. That is, $a \vee b$ is non s-complement. Then by Lemma 4.6 (1), we have $x_1 \vee x_2 = Soc(L) \leq a \vee b$. Since $x_2 \leq x_1 \vee x_2 \leq a \vee b$, and x_2 is an atom, by Lemma 4.6 (2), we have $x_2 \leq a$ or $x_2 \leq b$. Also, since $a, b \in V(gc_1(L))$, we have $x_1 \leq a$ and $x_1 \leq b$. Thus, $x_1 \vee x_2 \leq a$ or $x_1 \vee x_2 \leq b$. Again, by Lemma 4.6 (1) we get a or b is non s-complement, a contradiction to $a, b \in V(gc_1(L))$. Therefore $ab \in E(gc_1(L))$, for every $a, b \in V(gc_1(L))$. The case for $gc_2(L)$ is similar. Now to prove that $gc_1(L)$ and $gc_2(L)$ are disjoint, let $y_1 \in V(gc_1(L))$ and $y_2 \in V(gc_2(L))$. Since $A(L) = \{x_1, x_2\}$ and y_1, y_2 are vertices, we have $Soc(L) = x_1 \vee x_2 \leq y_1 \vee y_2$. Therefore, by Lemma 4.6 (1), we have $y_1 \vee y_2$ is non s-complement. Hence, $y_1 y_2$ are not adjacent. Consequently, $gc_1(L)$ and $gc_2(L)$ are disjoint subgraphs.

(3) ⇒ (1): Obvious. □

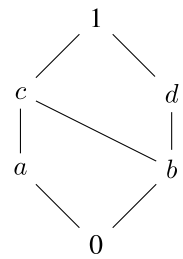

Figure 6.

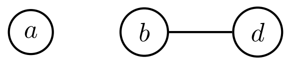

Figure 7.

Example 4.9. Consider the lattice given in Figure 6. Then graph $gc(L)$ corresponding to L is given in Figure 7: Here a and b are atoms, and so $gc(L)$ is disconnected. Also, $gc(L) = gc_1(L) \cup gc_2(L)$, where $gc_1(L)$ and $gc_2(L)$ are complete and disjoint subgraphs.

Theorem 4.10. *If $gc(L)$ is a connected graph, the diameter of $gc(L)$ is at most 2.*

Proof. Suppose that $gc(L)$ is a connected graph and $a, b \in V(gc(L))$. If $ab \in E(gc(L))$, then $d(a, b) < 2$. Suppose that $ab \notin E(gc(L))$. Since $a, b \in L$, by Lemma 4.4, there exist $x_1, x_2 \in A(L)$ such that $x_1 \leq a$ and $x_2 \leq b$. If $a \vee x_2$ is s-complement, there is a path $a - x_2 - b$, and so $d(a, b) = 2$. Similarly, if $b \vee x_1$ is s-complement, there is a path $b - x_1 - a$, and so $d(a, b) = 2$. Now, suppose both $a \vee x_2$ and $b \vee x_1$ are not s-complements. Then by Lemma 4.6 (1), we have $Soc(L) \leq a \vee x_2$ and $Soc(L) \leq b \vee x_1$. Since $gc(L)$ is connected, by Theorem 4.8, we have $|A(L)| > 2$. Therefore, we have $x_3 \in A(L)$ other than x_1 and x_2 with $x_3 \leq Soc(L) \leq a \vee x_2$ and $x_3 \leq Soc(L) \leq b \vee x_1$. Then by Lemma 4.6 (2), we have $x_3 \leq a$ and $x_3 \leq b$. Hence, $a - x_3 - b$ is a path. Therefore, $d(a, b) = 2$. □

Lemma 4.11. *Let $0 \neq a, b \neq 1 \in L$ be such that $a \leq b$. If b is an s-complement, then a is also s-complement.*

Proof. Straightforward. □

Theorem 4.12. *If $gc(L)$ is a connected graph, then $gc(L)$ has no cut vertex.*

Proof. Suppose that $gc(L)$ is a connected graph and $a \in V(gc(L))$ is a cut vertex of $gc(L)$. Then by definition, $gc(L) \setminus \{a\}$ is disconnected. This implies that for some $b, c \in V(gc(L))$, a lies in each path between b and c. Since $gc(L)$ is connected, by Theorem 4.10, $diam(gc(L)) \leq 2$. Therefore, we have a path $b - a - c$. If a is not an atom, by Lemma 4.4 there exists $x \in A(L)$ such that $x < a$ and so $x \vee b \leq a \vee b$. Since a and b are adjacent, we have $a \vee b$ is s-complement. Then by Lemma 4.11, $x \vee b$ is also s-complement. Similarly, $x \vee c \leq a \vee c$. Hence $x \vee c$ is s-complement. Therefore, $b - x - c$ is a path, a contradiction. Thus, a is an atom. Also, there exists $x \neq y_i \in A(L)$ with $y_i \not\leq b$. For if each $y_i \in A(L)$, $y_i \leq b$, then $Soc(L) \leq b \vee a$ and by Lemma 4.6 (1), we have $b \vee a$ is non s-complement, a contradiction. Similarly, $x \neq y_j \in A(L)$ with $y_i \not\leq c$. Since $b \vee c$ is non s-complement, by Lemma 4.6 (1), for each $y_k \in A(L)$, we have $y_k \leq Soc(L) \leq b \vee c$, and so by Lemma 4.7, $y_k \leq b$ or $y_k \leq c$. Thus, for distinct $y_i, y_j \in A(L)$ other than a, if $y_i \not\leq b$ and $y_j \not\leq c$, then $y_j \leq b$ and $y_i \leq c$. Hence $b - y_j - y_i - c$ is a path in $gc(L) \setminus \{a\}$, a contradiction. Therefore, $gc(L)$ has no cut vertex. □

Theorem 4.13. *If $|A(L)| \leq \infty$, then no vertex is a universal vertex in $gc(L)$.*

Proof. Suppose that $|A(L)| \leq \infty$ and there exists a vertex $a \in V(gc(L))$ which is universal. Then $x_i \in A(L)$ such that $x_i \leq a$, and by Lemma 4.7 for $x_j \in A(L)$, $t = \bigvee_{j \neq i} x_j$ is s-complement.

Since a is universal, $t \vee a$ is s-complement. Now, $x_i \leq a$ implies $Soc(L) = \bigvee_{j \neq i} x_j \vee x_i \leq t \vee a$. Hence by Lemma 4.6 (1), we have $t \vee a$ is non s-complement, a contradiction. Therefore, no vertex in $gc(L)$ is universal. □

Lemma 4.14. *If $gc(L)$ has a pendant vertex, then $|A(L)| = 2$.*

Proof. Suppose that $a \in V(gc(L))$ is a pendant vertex and $|A(L)| > 2$. By Lemma 4.7, for distinct $x_i, x_j \in A(L)$, $x_i \vee x_j$ is s-complement. This implies that $deg(x_i) \geq 2$, for each x_i, and so $a \notin A(L)$. Then by Lemma 4.4, there exists $x_1 \in A(L)$ such that $x_1 < a$. Hence $x_1 \vee a = a$ is s-complement, implies x_1 is the only element which is adjacent to a, as a pendant vertex. Thus, for $x_2 \in A(L)$, $a \vee x_2$ is non s-complement, and so by Lemma 4.6 (1), $Soc(L) \leq a \vee x_2$. This shows that $x_j \neq x_1, x_2$, $x_j \leq a \vee x_2$, and hence by Lemma 4.6 (2), we have $x_j \leq a$, a contradiction. □

Theorem 4.15. *$gc(L)$ is not a star graph.*

Proof. On a contrary, suppose that $gc(L)$ is a star graph. Then by definition, $gc(L)$ has a pendant vertex. Then by Lemma 4.14, $|A(L)| = 2$, and hence by Theorem 4.8, $gc(L)$ is disconnected, a contradiction. □

Corollary 4.16. *If $gc(L)$ is a k-regular graph, then $|A(L)| = 2$ and $|V(gc(L))| = 2k + 2$.*

Proof. Suppose that $gc(L)$ is a k-regular graph. If $|A(L)| \geq 3$, then $deg(x_1 \vee x_2) \leq deg(x_1)$, where $x_1, x_2 \in A(L)$. Note that $deg(x_1 \vee x_2) \neq deg(x_1)$, because by Lemma $\bigvee_{i \neq 2} x_i$ is adjacent to x_1, however, by Lemma 4.6 (1), $\bigvee_{i \neq 2} x_i$ is not adjacent to $x_1 \vee x_2$. Hence $deg(x_1 \vee x_2) < deg(x_1)$, and so $deg(x_1 \vee x_2) < k$, a contradiction to the given hypothesis. Therefore $|A(L)| \leq 2$. Since $gc(L)$ is not a null graph, $|A(L)| \neq 1$ and therefore $|A(L)| = 2$. Hence, by Theorem 4.8, $|V(gc(L))| = 2k + 2$. □

Theorem 4.17. $|A(L)| < \infty$. *Then $\alpha(gc(L)) = |A(L)|$.*

Proof. Suppose that $A(L) = \{x_1, x_2, \cdots, x_n\}$. Then by Lemma 4.6 (1), $\bigvee_{j \neq i = 1}^{n}$ is an independent set. This implies $\alpha(gc(L)) \geq n$. Suppose that $\alpha(gc(L)) = m > n$. Then we have maximal independent set $X = \{x_1, x_2, \cdots, x_m\}$. Since $m > n$, for $x_t \in A(L)$, there exists $x_i, x_j \in X$ such that $x_t \neq x_i$ and $x_t \nleq x_j$, where $1 \leq i, j \leq n$. Note that $x_i, x_j \in X$ and X is an independent set, therefore $x_i \vee x_j$ is non s-complement, and so by Lemma 4.6 (1), $x_t \leq Soc(L) \leq x_i \vee x_j$. Since $x_t \in A(L)$, we have by Lemma 4.6 (2), $x_t \leq x_i$ or $x_t \leq x_j$, a contradiction. Therefore, $m = n = \alpha(gc(L))$. □

References

[1] J. Amjadi, The essential ideal graph of a communicative ring, *Asian-Eur. J. Math.* **11**(4), 1850058 (9 pages) (2018).

[2] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules Graduate Texts in Mathematics*, (Springer-Verlag New York), (1992).

[3] A. Badawi, On the annihilator graph of a commutative ring. *Comm. Algebra*, **42**(1), 108–121 (2014).

[4] I. Beck, *Coloring of commutative rings*, *Journal of algebra*, **116**(1), 208–226 (1988). .

[5] S. Bhavanari, On modules with finite Goldie Dimension, *J. Ramanujan Math. Soc.* **5**(1), 61–75 (1990).

[6] S. Bhavanari and S.P. Kuncham, *Nearrings, fuzzy ideals, and graph theory*, (CRC press, 2013).

[7] S. Bhavanari and S.P. Kuncham, Discrete Mathematics and Graph theory, (Prentice Hall learning Ltd. Second edition, 2014).

[8] G. Birkoff, *Lattice theory*, volume 26, AMS, New York, (1967).

[9] G. Calugareanu, *Lattice concepts of module theory*, volume 22, Springer Science & Business Media (2003).

[10] I. Chajda and Länger, Filters and congruences in sectionally pseudocomplemented lattices and posets, *Soft Computing. 25*, 8827–8837 (2021).

[11] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.* **309**(17), 5381–5392 (2009).

[12] G. Gratzer, *Lattice theory: foundation*, Springer Science & Business Media (2011).

[13] P. Grzeszczuk and E. R. Puczyłowski, On Goldie and dual Goldie dimensions, *J. Pure Appl. Algebra* **31**(1-3), 47–54 (1984).

[14] S. Nimbhokar and V. Deshmukh, The essential element graph of a lattice, *Asian-Eur. J. Math.* **13**(1), 2050023 (9 pages) (2020).

[15] P. Pallavi, S. P. Kuncham, G. R. B. Vadiraja and P. K. Harikrishnan, Computation of prime hyperideals in meet hyperlattices, *Bull. Comput. Appl. Math.* **10**(1), (25 pages) (2022).

[16] N. Phadatare, K. Vilas and B. Sachin, Semi-complement graph of lattice modules, *Soft Computing* **23**, 3973–3978 (2019).

[17] G. N. Rao and T. G. Beyene, Relative Annihilators and Filters in Almost Semilattice, *Southeast Asian Bull. Math.* **43**(4), 553–576 (2019).

[18] S. Tapatee, B. S. Kedukodi, K. P. Shum, P. K. Harikrishnan and S. P. Kuncham, On essential elements in a lattice and Goldie analogue theorem, *Asian-Eur. J. Math.* (2021). https://doi.org/10.1142/S1793557122500917

[19] S. Tapatee, B. Davvaz, P. K. Harikrishnan, B. S. Kedukodi and S. P. Kuncham, Relative Essential Ideals in N-groups, *Tamkang J. Math.* **54**, (2021). https://doi.org/10.5556/j.tkjm.54.2023.4136

[20] S. Tapatee, P. K. Harikrishnan, B. S. Kedukodi and S. P. Kuncham, Graph with respect to superfluous elements in a lattice, *Miskolc Math. Notes* (to appear) (2022).

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