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ANALYSIS OF A RANDOM INCOME RISK MODEL WITH LATENT TAX PAYMENTS

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Abstract This paper introduces the tax strategy in a random income risk model. The inter-arrival times of independent claims and premiums are assumed to be Markovian arrival processes, to which the independent amount sizes are assumed to follow Phase-type distributions. We analyse the risk process using the corresponding Markovian fluid queue process. Under the latent tax strategy, we deduce the differential equations from the integral equations satisfied by survival probability, the Laplace transform of ruin time and the moments of discounted tax payments up to ruin. Further, for the corresponding single-phase cash-flow model, we establish the tax identity. Expressions for Laplace transform of ruin time and moments of discounted tax payments up to ruin are also derived. Further, we investigate the existence of an optimal surplus level for starting tax collection. Finally, the findings are illustrated numerically.

1 Introduction

This paper deals with an insurance risk process featuring random incomes. The classical constant premium risk model found inefficient when the insurance portfolio will have a higher volatility due to the premiums. This fluctuation is phenomenal in insurance businesses under a beginner stage compared to that of established ones and in developing countries due to the count of insured is squat in these situations and territories, respectively. More precisely, the assumption of random income portrays risk models with more realistic cash inflows over classical models with a constant premium rate.

Boucherie et al. [1] proposed the risk model with stochastic income in the classical risk model as mathematical accessibility of $M/G/1$ queue with work removal of negative customers. To model the uncertainty in premium arrival time and amount sizes, Boikov [2] and Melnikov [3] replaced the time-homogeneous premium component in the classical risk process with a compound Poisson process to analyse the survival probabilities. Later, Temnov [4] estimate the ruin probabilities of this Boikov-Melnikov risk process. The analysis under random income strategy was extended to the Gerber-Shiu function (See Gerber and Shiu [5]) by Bao and Ye [6] in the delayed renewal model, and Labbé and Sendova [7] in the Boikov-Melnikov process. Further, Hao and Yang [8] studied the Gerber-Shiu function in a compound Poisson risk model featuring random incomes and delayed claims. In the barrier strategy, Dong et al. [9] analysed the Gerber-Shiu function, and Zou et al. [10] dealt with an optimal dividend problem considering inter-dependent claims. Recently, Su et al. [11] used the Laguerre numerical scheme to estimate Gerber-Shiu function in the random income risk model.

An insurance risk process is a realisation of an insurer's surplus. The pioneer model of this process, detailed in Lundberg [12], is modified to incorporate several strategic industrial, economic and regulatory features. In 1957, De Finetti [13] propose a risk model featuring the instantaneous reflection of surplus if it exceeds a horizontal barrier. This strategy is considered as the pioneer dividend strategy, in which the reflected amount that exceeds the barrier is paid off as dividends to shareholders. Another kind of dividend payment is done via the threshold strategy, in which a proportion of excess amount is given as dividend so that the process refracts (see Albrecher and Hartinger [14] and references therein) at multiple surplus levels establishing a multi-step premium rate. Along with the reflection and refraction processes, an entirely different class of process exists, reflecting partially under increasing barrier steps associated with the profits of an insurer at new record highs. This kind of partial reflection corresponds to the tax payments under loss-carry forward (latent) strategy in which the tax is a function of surplus, without applying tax, which is collected only if the insurer is in a profitable condition. Our work explicitly focuses on the latent tax scheme in a random income model.

Many authors have considered the risk models with tax payments since its inception by Albrecher and Hipp [15]. In this pioneer work, the latent tax strategy is assumed in the compound Poisson risk model, and a simple but robust relation on survival probabilities between tax and tax-free processes, so-called the tax identity, is established. Later, the tax identity was derived for the Levy risk process by Albrecher et al. [16] and for the dual risk model by Albrecher et al. [17]. However, Albrecher et al. [18] confirmed that the tax identity cannot hold good generally for Markov additive process but can exist definitely in the Sparre Anderson process with Phase-type renewals and claims.

Under the assumption of surplus-dependent (natural) tax rates, Albrecher et al. [19] verified the tax identity in the Cramer-Lundberg model. In an extension study, Cheung and Landriault [20] analysed a reward-based generalisation of Geber-Shiu function in a compound Poisson process with surplus-dependent premiums, and Cui and Nguyen [21] analysed the Laplace transform of time of bankruptcy in the Omega-diffusion process with surplus-dependent capital injections. Remarkably, Al Ghanim et al. [22] investigated and proved the equivalence between natural and latent tax systems. Distinguished from the works mentioned above, Schmidli [23] extended the tax model with capital injection proposed by Albrecher and Ivanovs [24] to a taxable dividend and non-taxable capital injection strategy and shown that a double barrier dividend strategy is optimal in the scenario. The aforementioned papers dealt with tax strategies for the standard risk models with premium rate homogeneous to time. However, the homogeneous premium capture less volatility when the insurer is in a beginner stage as well as for a swat insured count. The proposed model triggers the analysis of tax strategies with models suitable to more volatile cash inflows. Henceforth, this paper analyse a random income risk model under the latent tax strategy.

For the proposed model, the risk process is characterised by the Markovian Arrival Processes (MAP) to govern the inter-arrival times of premiums and claims. The MAP risk model is quite generally handled in the literature since it can be used to model multiple-phase transactions along with phase correlations. In the literature, it is quite acceptable to fuse the Phase-type (PH) claim sizes with the MAP inter-arrival times. The MAP/PH fusion allows furnishing correlations between claim amounts and arrival times. The classical analyses of MAP/PH risk process on the Gerber-Shiu function can be seen in Ahn and Badescu [25]. The risk models under MAP structure were further studied by Cheung and Landriault [26] under perturbation and phase-dependent barrier strategy, by Zhang et al. [27] under absolute ruin with debit interest, and Zhang et al. [28] in a delayed dividend strategy.

We, in the upcoming sections, try to answer the question of the existence of tax identity in a random income risk model. Further, the optimal surplus level to start tax collection, subject to the maximisation of the expected discounted tax payments before ruin, is investigated. Under strong Markov laws and utilising renewal theory, the corresponding governing (renewal) equations turned out to be (in)homogeneous Fredholm or Volterra equations for which expressions of solutions in feasible forms are obtained.

We organise the rest of the paper as follows. In Section 2, we introduce the random income tax risk process with the latent tax strategy under MAP/PH compound premiums and incomes. The tax identity in the random income scenario is brought out in Section 3 when the premium arrivals and sizes are exponentially distributed. The Laplace transform of the ruin time is derived in Section 4 while the q^{th} moment of discounted tax payments is deduced in Section 5. The optimal surplus level for starting tax collection is investigated in Section 6. The expressions derived are numerically illustrated in Section 7. Throughout this paper, we denote parameters, vectors and matrices in bold scripts while that for single-phase variables and functions in unbold.

2 Latent tax strategy with MAP premium arrivals

For the initial capital $x \geqslant 0$, a random income risk process is given by

$$R(t) = x + S_p(t) - S_c(t), \tag{2.1}$$

where $S_p(t)$ and $S_c(t)$ are independent aggregate pure jump processes for premiums and claims respectively. The inter-premium times, T^p follow an $n-$phased MAP with representation $\text{MAP}_n(\boldsymbol{\alpha}, \mathbf{E}_0, \mathbf{E}_1)$. Let $\{N_p(t)\}_{t\geqslant 0}$ be the premium counting process and $\{J_p(t)\}_{t\geqslant 0}$, the underlying Continuous Time Markov Chain (CTMC) with state space $\mathcal{E}_p = \{1, 2, \ldots, n\}$. The premium sizes, W follow an m_1-phased i.i.d. PH distribution with representation $\text{PH}_{m_1}(\boldsymbol{\mu}, \mathbf{G})$. The inter-claim times, T^c of risk process (2.1) follow a $m-$phased MAP with representation $\text{MAP}_m(\boldsymbol{\beta}, \mathbf{F}_0, \mathbf{F}_1)$. In the MAP claim arrivals, let $\{N_c(t)\}_{t\geqslant 0}$ is the claim counting process and $\{J_c(t)\}_{t\geqslant 0}$ is the underlying CTMC with state space $\mathcal{E}_c = \{1, 2, \ldots, m\}$. The claim sizes, Z follow an n_1-phased i.i.d. PH distribution denoted by $\text{PH}_{n_1}(\boldsymbol{\nu}, \mathbf{H}))$.

The random income risk process (2.1) can be now alternatively written as

$$R(t) = x + \sum_{j=1}^{N_p(t)} W_j - \sum_{k=1}^{N_c(t)} Z_k, \tag{2.2}$$

which is renewed either by a premium or by a claim arrival. Let $T = \min(T^p, T^c)$ be the sequence of renewals for which we define $\{U_i = T_0+T_1+T_2+\cdots+T_i\}_{i\in\mathbb{N}_0}$, with the understanding that $U_0 = T_0 = 0$. Denoting $\oplus$ as Kronecker sum and $\otimes$ as Kronecker product, T will then follow a $nm-$phased MAP with representation $\text{MAP}_{nm}([\boldsymbol{\alpha} \otimes \boldsymbol{\beta}], \mathbf{D}_0, \mathbf{D}_1)$ where $\mathbf{D}_0 = \mathbf{E}_0 \oplus \mathbf{F}_0$ and $\mathbf{D}_1 = \mathbf{E}_1 \oplus \mathbf{F}_1$. The renewal point process, T will be then governed by the bi-variate Markov process $\{N(t), J(t)\}_{t\geqslant 0}$ having state space, $\mathbb{N} \times \mathcal{E}$ where $\mathcal{E} = \{1, 2, \ldots, nm\}$. Then, $\{N(t)\}_{t\geqslant 0}$ denotes the number of renewals due to a premium or claim arrival and$\{J(t)\}_{t\geqslant 0}$ is the state of underlying CTMC of renewals.

In this paper, we furnish correlated amount sizes with inter-arrival times of independent premiums and claims. We fuse (see Example 2.4 of Badecu et al. [29]) the rate matrices of PH amount sizes to that of MAP inter-renewal times to analyse the risk process (2.1) as a Markovian-fluid Queue Process (M-FQP). Thus, by this fusion, the process (2.2) will be governed by an irreducible CTMC with Transition Rate Matrix (TRM)

$$\begin{bmatrix} \mathbf{T}_{11_{nm\times nm}} & \mathbf{T}_{12_{nm\times nmm_1}} & \mathbf{T}_{13_{nm\times nmn_1}} \\ \mathbf{T}_{21_{nmm_1\times nm}} & \mathbf{T}_{22_{nmm_1\times nmm_1}} & \mathbf{T}_{23_{nmm_1\times nmn_1}} \\ \mathbf{T}_{31_{nmn_1\times nm}} & \mathbf{T}_{32_{nmn_1\times nmm_1}} & \mathbf{T}_{33_{nmn_1\times nmn_1}} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_0 & \boldsymbol{\mu} \otimes [\mathbf{E}_1 \otimes \boldsymbol{I}_m] & \boldsymbol{\nu} \otimes [\mathbf{F}_1 \otimes \boldsymbol{I}_n] \\ \mathbf{g}^\top \otimes \boldsymbol{I}_{nm} & \mathbf{G} \otimes \boldsymbol{I}_{nm} & \mathbf{0}_{nmm_1\times nmn_1} \\ \mathbf{h}^\top \otimes \boldsymbol{I}_{nm} & \mathbf{0}_{nmn_1\times nmm_1} & \mathbf{H} \otimes \boldsymbol{I}_{nm} \end{bmatrix}, \tag{2.3}$$

where $\boldsymbol{I}_.$ and $\mathbf{0}_.$ denote the identity matrix and matrix of zeroes of corresponding order respectively, $\mathbf{g}^\top = -\mathbf{G}\mathbf{e}_{m_1}^\top$ and $\mathbf{h}^\top = -\mathbf{H}\mathbf{e}_{n_1}^\top$, in which $\mathbf{e}_.^\top$ denotes the column vector of ones with the order suffixed. During the sojourn times, the elements of sub-TRM $\mathbf{T}_{11}$ represents the transition rates of switching phases for which fluid level has void acceleration(deceleration). On the other hand, the fluid process is characterised by positive jumps in the states governed by the sub-TRM $\mathbf{T}_{22}$ and negative jumps in the states governed by the sub-TRM $\mathbf{T}_{33}$.

For the single-phased cash-flow model (SPCM) of risk process (2.2), we assume that $T^p \sim \exp(\boldsymbol{\lambda}_p)$, $W \sim \exp(\boldsymbol{\kappa}_p)$, $T^c \sim \exp(\boldsymbol{\lambda}_c)$ and $Z \sim \exp(\boldsymbol{\kappa}_c)$ (all mutually independent). Corresponding, the TRM reduces to

$$\begin{bmatrix} D_0 & E_1 & F_1 \\ g & G & 0 \\ h & 0 & H \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{\lambda}_p + \boldsymbol{\lambda}_c) & \boldsymbol{\lambda}_p & \boldsymbol{\lambda}_c \\ \boldsymbol{\kappa}_p & -\boldsymbol{\kappa}_p & 0 \\ \boldsymbol{\kappa}_c & 0 & -\boldsymbol{\kappa}_c \end{bmatrix}. \tag{2.4}$$

We amend the risk process (2.2) by considering the latent tax strategy as in Figure 1. Whenever the surplus is at the running maximum and in state $a \in \mathcal{E}_p$, the tax is paid at rate $\gamma_a \in [0, 1)$. Let $\sigma_0 = 0$ and define

$$\sigma_d = \inf_{\substack{l\,=\,u\,+\,v \\ u,\,v\,\in\,\mathbb{N}}} \left\{ l > \sigma_{d-1} : \sum_{j=N_p(U_{\sigma_{d-1}})}^{u} W_j - \sum_{k=N_c(U_{\sigma_{d-1}})}^{v} Z_k > 0 \right\} \tag{2.5}$$

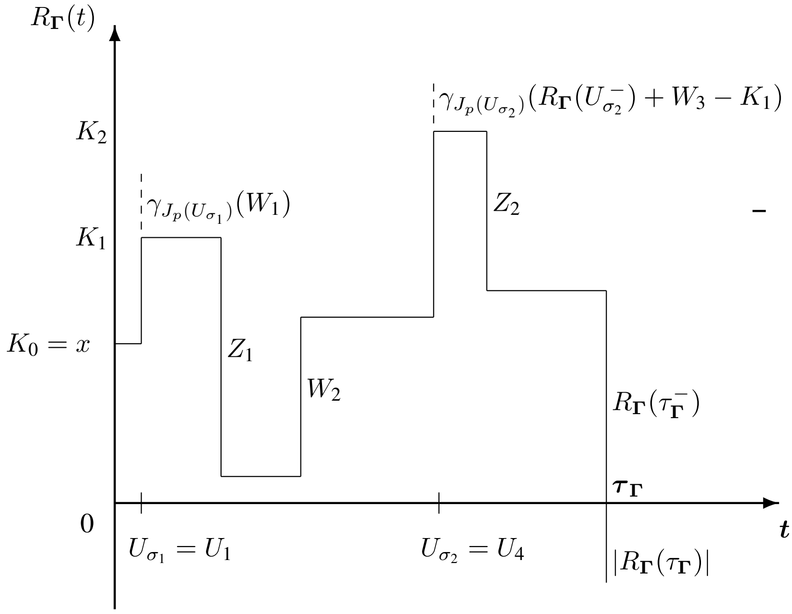


Figure 1: A realisation of the surplus Process $R_{\mathbf{\Gamma}}(t)$

to be the number of renewals up to the d^{th} record high time point. We associate the same number of phases in the premium arrivals to tax so that the tax matrix is taken as $\mathbf{\Gamma} = \text{diag}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_n)$. For $K_0 = x$, let

$$K_d = K_{d-1} + (1 - \boldsymbol{\gamma}_{J_p(U_{\sigma_d})}) \left(\sum_{j=N_p(U_{\sigma_{d-1}})}^{N_p(U_{\sigma_d})} W_j - \sum_{k=N_c(U_{\sigma_{d-1}})}^{N_c(U_{\sigma_d})} Z_k \right) \tag{2.6}$$

be d^{th} record high surplus. The resulting surplus process in the random income model with tax is given by

$$R_{\mathbf{\Gamma}}(t) = K_{\mathbf{\Xi}(t)} - \left(\sum_{k=N_c(U_{\sigma_{d-1}})}^{N_c(t)} Z_k - \sum_{j=N_p(U_{\sigma_{d-1}})}^{N_p(t)} W_j \right), \tag{2.7}$$

where $\mathbf{\Xi}(t) = \sup\{d \in \mathbb{N} : U_{\sigma_d} \leqslant t\}$. For practical considerations, we assume the security loading factor

$$\theta = \frac{\text{AIF}}{\text{AOF}} - 1 \tag{2.8}$$

in the system to be positive by taking the average cash-inflow (AIF) as

$$AIF = -\boldsymbol{\pi}_p \left[\boldsymbol{I}_n - \mathbf{\Gamma} \right] \left[\boldsymbol{\mu} \otimes \mathbf{E}_1 \right] \left[\mathbf{G} \otimes \boldsymbol{I}_n \right]^{-1} \mathbf{e}_{nm_1}^{\top}$$

and the average cash-outflow (AOF) given by

$$AOF = -\boldsymbol{\pi}_c \left[\boldsymbol{\nu} \otimes \mathbf{F}_1 \right] \left[\mathbf{H} \otimes \boldsymbol{I}_m \right]^{-1} \mathbf{e}_{mn_1}^{\top},$$

where $\boldsymbol{\pi}_p$ and $\boldsymbol{\pi}_c$ are the stationary probability row vectors of the CTMC, $\{J_p(t)\}_{t \geqslant 0}$ and $\{J_c(t)\}_{t \geqslant 0}$ respectively.

Let the ultimate ruin time for the tax risk process (2.7) be $\tau_{\mathbf{\Gamma}} = \inf\{t \geqslant 0 : R_{\mathbf{\Gamma}}(t) < 0\}$ ($\tau_{\mathbf{\Gamma}} = \infty$ if the set is empty). Then for the discounting factor $\boldsymbol{\delta} > 0$, let the Laplace transform of ruin time be

$$\psi_{\mathbf{\Gamma}, \boldsymbol{\delta}}(x) = \left[\boldsymbol{\alpha} \otimes \boldsymbol{\beta} \right] \mathbf{\Psi}_{\mathbf{\Gamma}, \boldsymbol{\delta}}(x) \mathbf{e}_{nm}^{\top}, \tag{2.9}$$

in which for the $(a,b)^{th}$ transition

$$[\mathbf{\Psi}_{\mathbf{\Gamma},\boldsymbol{\delta}}(x)]_{(a,b)} = \mathbb{E}\left[\mathrm{e}^{-\boldsymbol{\delta}\tau_{\mathbf{\Gamma}}}\mathbb{I}\left(\tau_{\mathbf{\Gamma}}<\infty\right)|R_{\mathbf{\Gamma}}(0)=x, J(0)=a, J(\tau_{\mathbf{\Gamma}})=b\right], \quad \text{for} \quad a,b\in\mathcal{E}, \tag{2.10}$$

where $\mathbb{I}(.)$ denotes the indicator function. The ruin probability can be then determined from Eq. (2.10) by $\psi_{\mathbf{\Gamma},0}(x) = [\boldsymbol{\alpha}\otimes\boldsymbol{\beta}]\,\mathbf{\Psi}_{\mathbf{\Gamma},0}(x)\mathbf{e}_{nm}^{\top}$.

3 Tax identity in the random income risk model

In this section, we investigate the impact of latent tax strategy on survival probability, $\phi_{\mathbf{\Gamma},0}(x) = 1-\psi_{\mathbf{\Gamma},0}(x)$, for the tax risk process (2.7). We try to establish the tax identity which determines the association of survival probability of tax risk processes to that of corresponding non-tax risk process. For the transition from a^{th} state to b^{th} state, the premium arrives and the surplus shoots with a $q^{th}(q\in\mathbb{N})$-order discounted ($\boldsymbol{\delta}>0$) probability measure, $[\mathbf{C}_{q\boldsymbol{\delta}}]_{a,b}$ where $\mathbf{C}_{q\boldsymbol{\delta}} = [q\boldsymbol{\delta}\boldsymbol{I}_{nm}-\mathbf{D}_0]^{-1}\mathbf{D}_1[-\mathbf{D}_0]^{-1}[\mathbf{E}_1\otimes\boldsymbol{I}_m]$. Using Eq. (2.4), $\mathbf{C}_{q\boldsymbol{\delta}}$ reduces to $C_{q\boldsymbol{\delta}} = \boldsymbol{\lambda}_p/(\boldsymbol{\lambda}_p+\boldsymbol{\lambda}_c+q\boldsymbol{\delta})$ for the SPCM. On the other hand, if a claim arrives at T_1 and the process survives, it may (up-) cross level x in order to avoid ruin. Let

$$\chi_x = \inf\{t>0 : R_{\mathbf{\Gamma}}(t)\geqslant x|\, T_1 = T^c\} \tag{3.1}$$

be the time of first shooting of $R_{\mathbf{\Gamma}}(t)$ above initial surplus x, after a non-ruin claim arrival at the first renewal time T_1. Then, χ_x will be the time at which the surplus process (2.7) reaches the first record high. Define the lowest surplus up to the crossing time χ_x by $Q_x = \inf\{R_{\mathbf{\Gamma}}(t) : 0\leqslant t<\chi_x\}$. Again with a transition from a^{th} state to b^{th} state, let the q^{th}-moment of Laplace transform of χ_x for which the risk process survives up to χ_x be

$$[\mathbf{V}_{q\boldsymbol{\delta}}(x)]_{a,b} = \mathbb{E}\left[\mathrm{e}^{-q\boldsymbol{\delta}\chi_x}\mathbb{I}(Q_x>0)|R_{\mathbf{\Gamma}}(0)=x, J(0)=a, J(\chi_x)=b\right] \quad \text{for} \quad a,b\in\mathcal{E}. \tag{3.2}$$

For convenience of the analysis, we denote $\mathbf{\Lambda}_{q\boldsymbol{\delta}}(x) = \mathbf{C}_{q\boldsymbol{\delta}}+\mathbf{V}_{q\boldsymbol{\delta}}(x)$. Note that the quantities χ_x, Q and $\mathbf{V}_{q\boldsymbol{\delta}}(x)$ are independent of the tax. Further, for $\boldsymbol{\delta}=0$, $\mathbf{V}_0(x)$ is equivalent to the probability that the dual of the risk process (2.2) starting at $x=0$ (with negative safety loading) priorly got ruined without reaching the level x, undergoing a state transition from a to b.

Lemma 3.1. *Assuming positive cash flow in the system, we have* $\lim_{x\to\infty}\psi_{\mathbf{\Gamma},0}(x)=0$.

Proof. Under the postive cash flow, the risk process (2.2) up-crosses the initial capital x countably often. Then, it yields $\sup_{n\in\mathbb{N}}\{U_{\sigma_{n+1}}-U_{\sigma_n}\}<\infty$ a.s. By process (2.7), the ruin probability $\psi_{\Gamma,0}(x)$ is bounded as

$$\psi_{\mathbf{\Gamma},0}(x)\leqslant \Pr\left[\inf_{d\in\mathbb{N}}\left\{K_d-\left(\sum_{k=N_c(U_{\sigma_d})}^{N_c(U_{\sigma_{d+1}})}Z_k-\sum_{j=N_p(U_{\sigma_d})}^{N_p(U_{\sigma_{d+1}})}W_j\right)\right\}<0\right]. \tag{3.3}$$

Also, from process (2.6), it is clear that $\inf_{d\in\mathbb{N}}\{K_d\}=x$ from which

$$\psi_{\mathbf{\Gamma},0}(x)\leqslant \Pr\left[\sup_{d\in\mathbb{N}}\left(\sum_{k=N_c(U_{\sigma_n})}^{N_c(U_{\sigma_{n+1}})}Z_k-\sum_{j=N_p(U_{\sigma_n})}^{N_p(U_{\sigma_{n+1}})}W_j\right)>x\right]. \tag{3.4}$$

Finally,

$$\lim_{x\to\infty}\psi_{\mathbf{\Gamma},0}(x)\leqslant \lim_{x\to\infty}\Pr\left[\sup_{d\in\mathbb{N}}\left(\sum_{k=N_c(U_{\sigma_n})}^{N_c(U_{\sigma_{n+1}})}Z_k-\sum_{j=N_p(U_{\sigma_n})}^{N_c(U_{\sigma_{n+1}})}W_j\right)>x\right]=0. \tag{3.5}$$

□

Proposition 3.2. *For the risk process* (2.7)*, the nm-dimensional square matrix $\Phi_{\Gamma,0}(x)$ satisfies the first order homogeneous differential equation (DEq.):*

$$\Phi'_{\Gamma,0}(x) = [\mu_\alpha \otimes \Lambda'_0(x)]\, [\mu_\alpha \otimes \Lambda_0(x)]^{-1}\, \Phi_{\Gamma,0}(x) - [\mu_\alpha \otimes \Lambda_0(x)]\, \Big\{[\mathbf{g}_\Gamma \otimes I_{nm}] + [\mathbf{G}_\Gamma \otimes I_{nm}]\, [\mu_\alpha \otimes \Lambda_0(x)]^{-1}\Big\}\, \Phi_{\Gamma,0}(x), \tag{3.6}$$

where $\mu_\alpha = \alpha \otimes \mu$, $\Lambda_0(x) = \mathbf{C}_0 + \mathbf{V}_0(x)$, $\mathbf{G}_\Gamma = (I_n - \Gamma)^{-1} \otimes \mathbf{G}$ and $\mathbf{g}_\Gamma = \left[(I_n - \Gamma)^{-1} \otimes \mathbf{g}^\top\right] \mathbf{e}_n^\top$. Further, $[\mu_\alpha \otimes \Lambda_0(x)]^{-1}$ is an appropriate generalised inverse of $[\mu_\alpha \otimes \Lambda_0(x)]$.

Proof. Due to the premium arrival, a new record high of the process (2.7) may occur in two different manners: $a)$ if the first renewal happens due to a premium arrival and $b)$ if the first renewal happens due to a claim arrival and the surplus process survives to up-cross the level x avoiding the ruin. Then, the (after-tax) excess of the surplus level over x follows the PH distribution, $\text{PH}_{nm_1}(\mu_\alpha, \mathbf{G}_\Gamma)$. As a result, the non-discounted survival probability $\phi_{\Gamma,0}(u)$ satisfies the Fredholm integral Eq.,

$$\begin{aligned}\Phi_{\Gamma,0}(x) &= [\mu_\alpha \otimes \Lambda_0(x)] \int_{w=0}^{\infty} \left[\left(e^{\mathbf{G}_\Gamma w}\mathbf{g}_\Gamma\right) \otimes I_{nm}\right] \Phi_{\Gamma,0}(x+w)\mathrm{d}w \\ &= [\mu_\alpha \otimes \Lambda_0(x)] \int_{w=x}^{\infty} \left[\left(e^{\mathbf{G}_\Gamma (w-x)}\mathbf{g}_\Gamma\right) \otimes I_{nm}\right] \Phi_{\Gamma,0}(w)\mathrm{d}w.\end{aligned} \tag{3.7}$$

Differentiating Eq. (3.7) w.r.to x, we can obtain the DEq. (3.6). □

By proposition 3.2, we have the DEq. satisfying the non-discounted survival probability of the risk process (2.7). For convenience of analysis, denote nm-dimensional matrix function

$$\Omega_{\Gamma,q\delta}(x) = \exp\left[\int_{w=0}^{x} \Lambda_{q\delta}(w) \left[(\mathbf{g}_\Gamma \otimes I_m) + (\mathbf{G}_\Gamma \otimes I_m)\, \Lambda_{q\delta}^{-1}(w)\right] \mathrm{d}w\right] \tag{3.8}$$

and

$$\Delta_{\Gamma,q\delta}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln\left[\Omega_{\Gamma,q\delta}^{-1}(x)\Lambda_{q\delta}(x)\right]. \tag{3.9}$$

We proceed the analysis by reducing the expression (3.6) to the case of single-phase (cash-inflow) premium size. Correspondingly, we particularise the process (2.7) with single-phase tax, $\gamma \in [0,1)$ so that $\mathbf{G}_\Gamma$ reduces to $G_\gamma = [-\kappa_p/(1-\gamma)]$. Then using Eqs. (2.4) and (3.9), Eq. (3.6) can be written as

$$\Phi'_{\gamma,0}(x) - \Delta_{\gamma,0}(x)\Phi_{\gamma,0}(x) = \mathbf{0}. \tag{3.10}$$

Theorem 3.3. *Assuming that the m-dimensional square matrix $\Phi_{\gamma,0}(x)$ is invertible, the solution of Eq.* (3.10) *is*

$$\Phi_{\gamma,0}(x) = \Lambda_0^{-1}(\infty)\Phi_{\gamma,0}(\infty)\Lambda_0(x)\Omega_{\gamma,0}^{-1}(x)\Omega_{\gamma,0}(\infty), \tag{3.11}$$

where $\Lambda_0(\infty) = -[\mathbf{D}_0]^{-1}\mathbf{D}_1$.

Further, for the SPCM, we can have

$$\Omega_{\gamma,q\delta}(x) = \exp\left[\int_{w=0}^{x} -\frac{\kappa_p}{1-\gamma}\left(1 - \Lambda_{q\delta}(w)\right)\mathrm{d}w\right] \tag{3.12}$$

and hence we can state the theorem below.

Theorem 3.4. *For tax rate $\gamma < 1$, the non-discounted survival probability $\phi_{\gamma,0}(x)$ i.e., the tax identity for the SPCM of risk process* (2.7) *is given by*

$$\phi_{\gamma,0}^{1-\gamma}(x) = \frac{\phi_{0,0}(x)}{\Lambda_0^{\gamma}(x)}. \tag{3.13}$$

Proof. For the SPCM of process (2.7) which is governed by TRM (2.4), we can have $\phi_{\boldsymbol{\gamma},0}(\infty) = 1$ and $V_0(\infty) = \boldsymbol{\lambda}_c / (\boldsymbol{\lambda}_p + \boldsymbol{\lambda}_c)$ by Lemma 3.1 and it follows

$$\phi_{\boldsymbol{\gamma},0}(x) = \Lambda_0(x) \frac{\Omega_{\boldsymbol{\gamma},0}(\infty)}{\Omega_{\boldsymbol{\gamma},0}(x)}. \tag{3.14}$$

Since Eq. (3.14) also holds for the corresponding non-tax process, we arrive at the tax identity (3.13). □

Since $\Lambda_0(x) = C_0(x) + V_0(x)$, the right-hand side of tax identity (3.13) would be completely known by determining the expression for $V_0(x)$. To derive $V_0(x)$, consider a negatively loaded SPCM of risk process (2.2) under the TRM (2.4). Let $\xi(x)$ be the probability that ruin happens for this negatively loaded process. By applying the Markovian property on exponential jumps for a process with barrier strategy, we have (See Dickson [30])

$$V_0(x) = \frac{\xi(0) - \xi(x)}{1 - \xi(x)}, \tag{3.15}$$

where (see Dong et al. [9])

$$\xi(x) = \frac{\boldsymbol{\kappa}_c - L}{\boldsymbol{\kappa}_c} e^{-Lx} = \frac{\boldsymbol{\lambda}_p(\boldsymbol{\kappa}_p + L)}{(\boldsymbol{\lambda}_c + \boldsymbol{\lambda}_p)(\boldsymbol{\kappa}_p + L) - \boldsymbol{\kappa}_p \boldsymbol{\lambda}_c} e^{-Lx}, \tag{3.16}$$

in which L (for the dual process) satisfies the Lundberg Eq.

$$\boldsymbol{\lambda}_c + \boldsymbol{\lambda}_p = \frac{\boldsymbol{\kappa}_p \boldsymbol{\lambda}_c}{\boldsymbol{\kappa}_c - L} + \frac{\boldsymbol{\kappa}_p \boldsymbol{\lambda}_p}{L + \boldsymbol{\kappa}_p}. \tag{3.17}$$

Eq. (3.13) determines the ruin probability with tax as a function of the ruin probability without tax. Further, applying probabilistic arguments on the passage times, a relation between $\Lambda_{q\boldsymbol{\delta}}(x)$ and the q^{th} moment of discounted dividends $P_{q\boldsymbol{\delta}}(x;\varsigma)$ paid before ruin in the SPCM of risk process (2.2) under a dividend barrier strategy at level $\varsigma = x$ is given by

$$\Lambda_{q\boldsymbol{\delta}}(x) = \left(\frac{1}{\boldsymbol{\kappa}_p} + P_{q\boldsymbol{\delta}}(x,x) \right)^{-1} P_{q\boldsymbol{\delta}}(x,x) \quad \text{for} \quad x \geqslant 0. \tag{3.18}$$

4 The Laplace transform of ruin time

In this section, we analyse the Laplace transform of ruin time $\tau_{\boldsymbol{\Gamma}}$ for the risk process (2.7). Let us consider what happens after a claim arrives at the first renewal time T_1. After the time $T_1 (= T^c \geqslant 0)$, the process (2.7) can either up-cross the level x to reach a new record high at time χ_x so as to avoid ruin or ruin happens before the time χ_x. Denote this ruin time by

$$\zeta_x = \inf\{t > 0 : R_{\boldsymbol{\Gamma}}(t) < 0 | T_1 = T^c, 0 \leqslant t < \chi_x\} \tag{4.1}$$

be the time of ruin, occurs without up-crossing the level x, after a non-ruin claim arrival at the first renewal time T_1. Define the highest surplus up to the crossing time ζ_x by $O = \sup\{R_{\boldsymbol{\Gamma}}(t) : 0 \leqslant t < \zeta_x\}$. Define the Laplace transform of the ruin time ζ_x along with a transition from state a to state b by

$$[\mathbf{A}_{q\boldsymbol{\delta}}(x)]_{a,b} = \mathbb{E}\left[\mathrm{e}^{-q\delta\zeta_x} \mathbb{I}(O \leqslant x) | R_{\boldsymbol{\Gamma}}(0) = x, J(0) = a, J(\zeta_x) = b \right], \quad \text{for} \quad a, b \in \mathcal{E}. \tag{4.2}$$

Sample path of the risk process (2.2) that realises $\mathbf{A}_{\boldsymbol{\delta}}(x)$ is equivalent to the path of dual (negatively loaded) of same process that attains the level x from initial surplus level 0, with out a ruin . For a positively loaded corresponding risk process, let $\mathbf{B}_{\boldsymbol{\delta}}(x,\varsigma)$ be the Laplace transform of first passage time from level x to any level above ς avoiding ruin en route. Furthermore, an expression for $\mathbf{B}_{\boldsymbol{\delta}}(x,\varsigma)$ can be obtained in terms of $\boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)$. Indeed, by conditioning on the ascending ladder height, one obtains

$$\mathbf{B}_{\boldsymbol{\delta}}(x,\varsigma) = [\boldsymbol{\mu} \otimes \boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)] \left\{ \int_{w=x}^{\varsigma} \left[\left(e^{\mathbf{G}(w-x)} \mathbf{g} \right) \otimes \boldsymbol{I}_m \right] \mathbf{B}_{\boldsymbol{\delta}}(w,\varsigma) \mathrm{d}w + \left[\left(e^{\mathbf{G}(\varsigma - x)} \mathbf{e}_{m_1}^{\top} \right) \otimes \boldsymbol{I}_m \right] \right\}. \tag{4.3}$$

Differentiating Eq. (4.3) w.r.t. x, we obtain

$$\mathbf{B}'_{\boldsymbol{\delta}}(x,\varsigma) = \left\{ \left[\boldsymbol{\mu} \otimes \boldsymbol{\Lambda}'_{\boldsymbol{\delta}}(x)\right] \left[\boldsymbol{\mu} \otimes \boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)\right]^{-1} - \frac{\mathrm{d}}{\mathrm{d}x} \ln \boldsymbol{\Omega}_{0,\boldsymbol{\delta}}(x) \right\} \mathbf{B}_{\boldsymbol{\delta}}(x,\varsigma). \tag{4.4}$$

For the SPCM of risk process (2.2) which is governed by TRM (2.4), we have the following from Eq. (4.4):

$$B'_{\boldsymbol{\delta}}(x,\varsigma) - \Delta_{0,0}(x) B_{\boldsymbol{\delta}}(x,\varsigma) = 0, \tag{4.5}$$

from which

$$B_{\boldsymbol{\delta}}(x,\varsigma) = \Lambda_{\boldsymbol{\delta}}(x) \frac{\Omega_{0,\boldsymbol{\delta}}(\varsigma)}{\Omega_{0,\boldsymbol{\delta}}(x)}. \tag{4.6}$$

And hence from the single-phased dual process (with negative loading) of risk process (2.2), it yields that $\mathbf{A}_{\boldsymbol{\delta}}(x)$ reduces to $A_{\boldsymbol{\delta}}(x) = B_{\boldsymbol{\delta}}(0,x)$.

Proposition 4.1. *For the risk process (2.7), the nm-dimensional square matrix $\boldsymbol{\Psi}_{\boldsymbol{\Gamma},\delta}(x)$ satisfies the first order inhomogeneous DEq.:*

$$\begin{aligned} \boldsymbol{\Psi}'_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(x) =& \boldsymbol{\Lambda}'_{\boldsymbol{\delta}}(x) + \Big\{ \left[\boldsymbol{\mu}_{\boldsymbol{\alpha}} \otimes \boldsymbol{\Lambda}'_{\boldsymbol{\delta}}(x)\right] \left[\boldsymbol{\mu}_{\boldsymbol{\alpha}} \otimes \boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)\right]^{-1} - \left[\boldsymbol{\mu}_{\boldsymbol{\alpha}} \otimes \boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)\right] \{ \left[\mathbf{g}_{\boldsymbol{\Gamma}} \otimes \boldsymbol{I}_{nm}\right] \\ &+ \left[\mathbf{G}_{\boldsymbol{\Gamma}} \otimes \boldsymbol{I}_{nm}\right] \left[\boldsymbol{\mu}_{\boldsymbol{\alpha}} \otimes \boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)\right]^{-1} \} \Big\} \left[\boldsymbol{\Psi}_{\boldsymbol{\Gamma},\delta}(x) - \mathbf{A}_{\boldsymbol{\delta}}(x)\right]. \end{aligned} \tag{4.7}$$

Proof. Considering the after-tax excess surplus level over x at times $T_1 = T^p$ and χ_x, and the claim amount size at ξ_x, the mn-dimensional matrix $\boldsymbol{\Psi}_{\boldsymbol{\Gamma},\delta}(x)$ satisfies the Fredholm integral Eq.:

$$\begin{aligned} \boldsymbol{\Psi}_{\boldsymbol{\Gamma},\delta}(x) &= \left[\boldsymbol{\mu}_{\boldsymbol{\alpha}} \otimes \boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)\right] \int_{w=0}^{\infty} \left[\left(e^{\mathbf{G}_{\boldsymbol{\Gamma}} w} \mathbf{g}_{\boldsymbol{\Gamma}}\right) \otimes \boldsymbol{I}_{nm} \right] \boldsymbol{\Psi}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(x+w) \mathrm{d}w + \mathbf{A}_{\boldsymbol{\delta}}(x) \\ &= \left[\boldsymbol{\mu}_{\boldsymbol{\alpha}} \otimes \boldsymbol{\Lambda}_{\boldsymbol{\delta}}(x)\right] \int_{w=x}^{\infty} \left[\left(e^{\mathbf{G}_{\boldsymbol{\Gamma}}(w-x)} \mathbf{g}_{\boldsymbol{\Gamma}}\right) \otimes \boldsymbol{I}_{nm} \right] \boldsymbol{\Psi}_{\boldsymbol{\Gamma},\delta}(w) \mathrm{d}w + \mathbf{A}_{\boldsymbol{\delta}}(x). \end{aligned} \tag{4.8}$$

Differentiating Eq. (4.8) w.r.t. x, we can deduce Eq. (4.7). □

Theorem 4.2. *For the SPCM of risk process (2.7), the solution for $\psi_{\gamma,\boldsymbol{\delta}}(y)$ is given by*

$$\psi_{\gamma,\boldsymbol{\delta}}(y) = A_{\boldsymbol{\delta}}(y) + g_{\gamma} \frac{\Lambda_{\boldsymbol{\delta}}(y)}{\Omega_{\gamma,\delta}(y)} \int_{y}^{\infty} \Omega_{\gamma,\delta}(x) A_{\boldsymbol{\delta}}(x) \mathrm{d}x. \tag{4.9}$$

Proof. For the SPCM of risk process (2.7), Eq. (4.7) can be rewritten as

$$\psi'_{\boldsymbol{\gamma},\boldsymbol{\delta}}(x) - \Delta_{\boldsymbol{\gamma},\boldsymbol{\delta}}(x) \psi_{\boldsymbol{\gamma},\boldsymbol{\delta}}(x) = A'_{\boldsymbol{\delta}}(x) - \Delta_{\boldsymbol{\gamma},\boldsymbol{\delta}}(x) A_{\boldsymbol{\delta}}(x) - g_{\boldsymbol{\gamma}} \Lambda_{\boldsymbol{\delta}}(x) A_{\boldsymbol{\delta}}(x). \tag{4.10}$$

Applying the multiplicative factor $\Omega_{\gamma,\delta}(x)/\Lambda_{\boldsymbol{\delta}}(x)$ on the inhomogeneous DEq.. (4.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\Omega_{\boldsymbol{\gamma},\delta}(x) \psi_{\boldsymbol{\gamma},\boldsymbol{\delta}}(x)}{\Lambda_{\boldsymbol{\delta}}(x)} \right] = \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\Omega_{\boldsymbol{\gamma},\delta}(x) A_{\boldsymbol{\delta}}(x)}{\Lambda_{\boldsymbol{\delta}}(x)} \right] - g_{\boldsymbol{\gamma}} \boldsymbol{\Omega}_{\boldsymbol{\gamma},\delta}(x) A_{\boldsymbol{\delta}}(x). \tag{4.11}$$

Integrate Eq. (4.11) with respect to x from $y(>0)$ to ∞ together with

$$\lim_{x\to\infty} \psi_{\boldsymbol{\gamma},\boldsymbol{\delta}}(x) = \lim_{x\to\infty} A_{\boldsymbol{\delta}}(x) = 0$$

and $\lim_{x\to\infty} V_{\delta}(x) > 0$ due to Lemma 3.1, one can conclude the statement. □

5 The discounted tax payments

This section focuses on the solution analysis for moments of discounted tax payments until ruin. Let $[\boldsymbol{X}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(x)]_{a,b\in\mathcal{E}}$ denote the discounted tax payments before ruin in the surplus process (2.7) undergoing $(a,b)^{th}$ transition. Then, the q^{th} moment of $\boldsymbol{X}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(x)$ is given by $\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)=\mathbb{E}\left[\boldsymbol{X}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(x)\right]^q$. By Proposition 5.1, we can have the DEq. satisfied by $\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)$.

Proposition 5.1. *For the risk process (2.7), the nm-dimensional square matrix $\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)$ satisfies the first order inhomogeneous DEq.:*

$$\begin{aligned}\mathbf{M}'_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)&=\left[\boldsymbol{\mu}_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Lambda}'_{q\boldsymbol{\delta}}(x)\right]\left[\boldsymbol{\mu}_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Lambda}_{q\boldsymbol{\delta}}(x)\right]^{-1}\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)-\left[\boldsymbol{\mu}_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Lambda}_{q\boldsymbol{\delta}}(x)\right]\left[(\mathbf{g}_{\boldsymbol{\Gamma}}\otimes\boldsymbol{I}_{nm})+(\mathbf{G}_{\boldsymbol{\Gamma}}\otimes\boldsymbol{I}_{nm})\right.\\&\quad\left.\times\left[\boldsymbol{\mu}_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Lambda}_{q\boldsymbol{\delta}}(x)\right]^{-1}\right]\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)-q\left\{\left[(\boldsymbol{I}_n-\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\right]\otimes\boldsymbol{I}_m\right\}\int_{w=x}^{\infty}\left[\left(e^{\mathbf{G}_{\boldsymbol{\Gamma}}(w-x)}\mathbf{g}_{\boldsymbol{\Gamma}}\right)\otimes\boldsymbol{I}_{nm}\right]\\&\quad\times\mathbb{E}\left[\boldsymbol{X}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(w)+\left\{\left[(\boldsymbol{I}_n-\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\right]\otimes\boldsymbol{I}_m\right\}(w-x)\right]^{q-1}\mathrm{d}w.\end{aligned}\tag{5.1}$$

Proof. By conditioning on the first upper exit time χ_x of the risk process (2.7), one finds

$$\begin{aligned}\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)&=\left[\boldsymbol{\mu}_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Lambda}_{q\boldsymbol{\delta}}(x)\right]\int_{w=0}^{\infty}\left[\left(e^{\mathbf{G}_{\boldsymbol{\Gamma}}w}\mathbf{g}_{\boldsymbol{\Gamma}}\right)\otimes\boldsymbol{I}_{nm}\right]\\&\qquad\times\mathbb{E}\left[\boldsymbol{X}_{\boldsymbol{\Gamma},\delta}(x+w)+\left\{\left[(\boldsymbol{I}_n-\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\right]\otimes\boldsymbol{I}_m\right\}w\right]^q\mathrm{d}w\\&=\left[\boldsymbol{\mu}_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Lambda}_{q\boldsymbol{\delta}}(x)\right]\int_{w=x}^{\infty}\left[\left(e^{\mathbf{G}_{\boldsymbol{\Gamma}}(w-x)}\mathbf{g}_{\boldsymbol{\Gamma}}\right)\otimes\boldsymbol{I}_{nm}\right]\\&\qquad\times\mathbb{E}\left[\boldsymbol{X}_{\boldsymbol{\Gamma},\delta}(w)+\left\{\left[(\boldsymbol{I}_n-\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\right]\otimes\boldsymbol{I}_m\right\}(w-x)\right]^q\mathrm{d}w.\end{aligned}\tag{5.2}$$

Differentiating Eq. (5.2) w.r.t. to x, the Eq. (5.1) holds. □

By Proposition 5.1, we have the recurrence relation of $\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)$ in the order of q. The solution of $\mathbf{M}_{\boldsymbol{\Gamma},q\boldsymbol{\delta}}(x)$ for the SPCM is derived in Theorem 5.2.

Theorem 5.2. *For the SPCM, the solution for $M_{\gamma,q\boldsymbol{\delta}}(y)$ is given by*

$$M_{\gamma,q\boldsymbol{\delta}}(y)=\frac{q\gamma\Lambda_{q\boldsymbol{\delta}}(y)}{(1-\gamma)\Omega_{\gamma,q\boldsymbol{\delta}}(y)}\int_{x=y}^{\infty}\frac{\Omega_{\gamma,q\boldsymbol{\delta}}(x)M_{\boldsymbol{\Gamma},(q-1)\boldsymbol{\delta}}(x)}{\Lambda_{(q-1)\boldsymbol{\delta}}(x)}\mathrm{d}x\tag{5.3}$$

Proof. Taking account of the SPCM of risk process (2.7) which is governed by TRM (2.4) and replacing q by $q-1$ in Eq. (5.2), the Eq. (5.1) can be simplified to

$$M'_{\gamma,q\boldsymbol{\delta}}(x)-\Delta_{\gamma,q\boldsymbol{\delta}}(x)M_{\gamma,q\boldsymbol{\delta}}(x)=-\frac{q\gamma\Lambda_{q\boldsymbol{\delta}}(x)M_{\gamma,(q-1)\boldsymbol{\delta}}(x)}{(1-\gamma)\Lambda_{(q-1)\boldsymbol{\delta}}(x)}.\tag{5.4}$$

Multiplying both sides of Eq. (5.4) by the multiplicative factor $\Omega_{\gamma,q\boldsymbol{\delta}}(x)/\Lambda_{q\boldsymbol{\delta}}(x)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\Omega_{\gamma,q\boldsymbol{\delta}}(x)M_{\gamma,q\boldsymbol{\delta}}(x)}{\Lambda_{q\boldsymbol{\delta}}(x)}\right]=-\frac{q\gamma M_{\gamma,(q-1)\boldsymbol{\delta}}(x)}{(1-\gamma)\Omega_{\gamma,q\boldsymbol{\delta}}(x)\Lambda_{(q-1)\boldsymbol{\delta}}(x)}.\tag{5.5}$$

Integrating Eq. (5.5) from y to ∞ then yields

$$M_{\gamma,q\boldsymbol{\delta}}(y)=\frac{\Lambda_{q\boldsymbol{\delta}}(y)\Omega_{\gamma,q\boldsymbol{\delta}}(\infty)M_{\gamma,q\boldsymbol{\delta}}(\infty)}{\Omega_{\gamma,q\boldsymbol{\delta}}(y)\Lambda_{q\boldsymbol{\delta}}(\infty)}+\frac{q\gamma}{1-\gamma}\int_{x=y}^{\infty}\frac{\Omega_{\gamma,q\boldsymbol{\delta}}(x)M_{\gamma,(q-1)\boldsymbol{\delta}}(x)}{\Lambda_{(q-1)\boldsymbol{\delta}}(x)}\mathrm{d}x,\tag{5.6}$$

where $M_{\gamma,q\boldsymbol{\delta}}(\infty):=\lim_{x\to\infty}M_{\gamma,q\boldsymbol{\delta}}(x)$ and $\Lambda_{q\boldsymbol{\delta}}(\infty):=\lim_{x\to\infty}\Lambda_{q\boldsymbol{\delta}}(x)$. Under the positive cash flows and $\delta\geqslant 0$, we have $\lim_{x\to\infty}\Lambda_{q\boldsymbol{\delta}}(x)>0$, $\lim_{x\to\infty}M_{\gamma,q\boldsymbol{\delta}}(x)$ is finite a.s. and

$$\lim_{x->\infty}\frac{\Omega_{\gamma,q\boldsymbol{\delta}}(x)}{\Omega_{\gamma,q\boldsymbol{\delta}}(y)}\leqslant\lim_{x->\infty}\exp\left[\left\{\Lambda_{q\boldsymbol{\delta}}(x)g_\gamma+G_\gamma\right\}(y-x)\right]=0,\tag{5.7}$$

by which Eq. (5.6) becomes Eq. (5.3).

□

Corollary 5.3. *For $q = 1$, the expected discounted tax payments until ruin is given by*

$$M_{\gamma,\boldsymbol{\delta}}(y) = \frac{\gamma \Lambda_{\boldsymbol{\delta}}(y)}{(1-\gamma)\Omega_{\gamma,\boldsymbol{\delta}}(y)} \int_{x=y}^{\infty} \Omega_{\gamma,\boldsymbol{\delta}}(x) \mathrm{d}x, \tag{5.8}$$

since $M_{\gamma,0}(x) = \Lambda_0(x)$ by Eq. (5.2).

6 Delayed start of tax payments

In this section, we investigate the existence of optimal surplus level to start tax collection. The optimal surplus is determined under the objective to maximise the expected discounted tax paid out until ruin. Further, by delaying the tax collection will decrease the ruin probability of risk process. We consider the same tax system with tax payments allowed to start only at a threshold level $\varsigma > x$, for which let $[\mathbf{Y}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(x,\varsigma)]_{a,b}$ be the element denoting $(a,b)^{th}$ transition of the resulting expected discounted tax payments. The immediate probabilistic argument is that

$$\mathbf{Y}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(x,\varsigma) = \mathbf{B}_{\boldsymbol{\delta}}(x,\varsigma) \left\{ \int_{w=0}^{\infty} w \left[\boldsymbol{\mu}_{\boldsymbol{\alpha}} \otimes \left\{ \left[(\boldsymbol{I}_n - \boldsymbol{\Gamma})^{-1} \boldsymbol{\Gamma} \right] \otimes \boldsymbol{I}_m \right\} \right] \left[\left(e^{\mathbf{G}_{\boldsymbol{\Gamma}} w} \mathbf{g}_{\boldsymbol{\Gamma}} \right) \otimes \boldsymbol{I}_{nm} \right] \mathrm{d}w \right.$$
$$\left. + \int_{w=0}^{\infty} \left[\left(\boldsymbol{\mu}_{\boldsymbol{\alpha}} e^{\mathbf{G}_{\boldsymbol{\Gamma}} w} \mathbf{g}_{\boldsymbol{\Gamma}} \right) \otimes \boldsymbol{I}_{nm} \right] \mathbf{M}_{\boldsymbol{\Gamma},\boldsymbol{\delta}}(b+w) \mathrm{d}w \right\}. \tag{6.1}$$

Combining (5.2) at $q = 1$ and (6.1), we can have

$$Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma) = B_{\boldsymbol{\delta}}(x,\varsigma) \frac{M_{\gamma,\boldsymbol{\delta}}(\varsigma)}{\Lambda_{q\boldsymbol{\delta}}(\varsigma)} = \frac{\Lambda_{\boldsymbol{\delta}}(x)\Omega_{0,\boldsymbol{\delta}}(\varsigma) M_{\gamma,\boldsymbol{\delta}}(\varsigma)}{\Omega_{0,\boldsymbol{\delta}}(x)\Lambda_{\boldsymbol{\delta}}(\varsigma)}. \tag{6.2}$$

On substituting Eq. (4.6) in Eq. (6.2),

$$Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma) = \frac{\Lambda_{\boldsymbol{\delta}}(x)\Omega_{0,\boldsymbol{\delta}}(\varsigma) M_{\gamma,\boldsymbol{\delta}}(\varsigma)}{\Omega_{0,\boldsymbol{\delta}}(x)\Lambda_{\boldsymbol{\delta}}(\varsigma)}. \tag{6.3}$$

Theorem 6.1. *If there is an optimal level $\varsigma^* > x \geqslant 0$ to start taxation at rate $0 < \gamma < 1$ in the SPCM of risk process (2.7), it has to fulfil the condition*

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma} \frac{M_{\gamma,\boldsymbol{\delta}}(\varsigma)}{\Lambda_{\boldsymbol{\delta}}(\varsigma)} \bigg|_{\varsigma=\varsigma^*} = 1 \tag{6.4}$$

together with

$$\Lambda'_{\boldsymbol{\delta}}(\varsigma^*) > \boldsymbol{\kappa}_p \left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right)^2, \tag{6.5}$$

and the optimal expected discounted tax payment is then given by

$$Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma^\star) = \frac{B_{\gamma,\boldsymbol{\delta}}(x,\varsigma^*)}{\boldsymbol{\kappa}_p \left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right)}, \quad \text{for} \quad x < \varsigma^*. \tag{6.6}$$

A sufficient condition for the existence of such an optimal positive level $\varsigma^* > 0$ is

$$\lim_{x \to 0} \frac{M_{\gamma,\boldsymbol{\delta}}(x)}{\Lambda_{\boldsymbol{\delta}}(x)} > 1/\boldsymbol{\kappa}_p. \tag{6.7}$$

On the other hand, if such a $\varsigma^* > 0$ does not exist, then the optimal level to start taxation is $\varsigma^* = 0$ (i.e. start immediately), so that in this case $Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma^\star) = Y_{\gamma,\boldsymbol{\delta}}(x,0)$.

Proof. To identify the optimal surplus level ς^* for the authority to start tax collection, we shall look for the solution of

$$\frac{\partial}{\partial\varsigma} Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma) = 0. \tag{6.8}$$

Combining Eqs. (6.3) and (6.8) yields

$$\left[\frac{M'_{\gamma,\boldsymbol{\delta}}(\varsigma^*)}{M_{\gamma,\boldsymbol{\delta}}(\varsigma^*)} - \left(\frac{\Lambda'_{\boldsymbol{\delta}}(\varsigma^*)}{\Lambda_{\boldsymbol{\delta}}(\varsigma^*)} + \boldsymbol{\kappa}_p \left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right) \right) \right] Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma^\star) = 0. \tag{6.9}$$

Since $Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma) > 0$ for $x, \varsigma > 0$, we get

$$\frac{M'_{\gamma,\boldsymbol{\delta}}(\varsigma^*)}{M_{\gamma,\boldsymbol{\delta}}(\varsigma^*)} = \frac{\Lambda'_{\boldsymbol{\delta}}(\varsigma^*)}{\Lambda_{\boldsymbol{\delta}}(\varsigma^*)} + \boldsymbol{\kappa}_p\left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right). \tag{6.10}$$

On the other hand, we know from Eq. (5.5) at $q = 1$ that

$$\frac{M'_{\gamma,\boldsymbol{\delta}}(x)}{M_{\gamma,\boldsymbol{\delta}}(x)} = \frac{\Lambda'_{\boldsymbol{\delta}}(x)}{\Lambda_{\boldsymbol{\delta}}(x)} + \frac{\boldsymbol{\kappa}_p\left(1 - \Lambda_{\boldsymbol{\delta}}(x)\right)}{1-\gamma} - \frac{\gamma\Lambda_{\boldsymbol{\delta}}(x)}{(1-\gamma)M_{\gamma,\boldsymbol{\delta}}(x)} \quad \text{for} \quad x \geqslant 0. \tag{6.11}$$

Hence, Eq. (6.10) can be written as

$$\frac{M_{\gamma,\boldsymbol{\delta}}(\varsigma^*)}{\Lambda_{\boldsymbol{\delta}}(\varsigma^*)} = \frac{1}{\boldsymbol{\kappa}_p\left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right)} \tag{6.12}$$

or alternatively,

$$M_{\gamma,\boldsymbol{\delta}}(\varsigma^*) = P_{\boldsymbol{\delta}}(\varsigma^*, \varsigma^*). \tag{6.13}$$

Replacing Eq. (6.12) in Eq. (6.10) eventually leads to Eq. (6.4).

In order to ensure that ς^* is indeed a maximum, we have to prove that

$$\left.\frac{\partial^2}{\partial\varsigma^2} Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma)\right|_{\varsigma=\varsigma^*} < 0. \tag{6.14}$$

From Eq. (6.3), we have

$$\left.Y''_{\gamma,\boldsymbol{\delta}}(x,\varsigma)\right|_{\varsigma=\varsigma^*} = \left(\left(\frac{M'_{\gamma,\boldsymbol{\delta}}(\varsigma)}{M_{\gamma,\boldsymbol{\delta}}(\varsigma)}\right)' - \left(\frac{\Lambda'_{\boldsymbol{\delta}}(\varsigma)}{\Lambda_{\boldsymbol{\delta}}(\varsigma)}\right)' + \boldsymbol{\kappa}_p\Lambda'_{\boldsymbol{\delta}}(\varsigma)\right) \left.Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma)\right|_{\varsigma=\varsigma^*}. \tag{6.15}$$

Differentiating Eq. (6.11) w.r.t. ς, we also get

$$\left(\frac{M'_{\gamma,\boldsymbol{\delta}}(\varsigma)}{M_{\gamma,\boldsymbol{\delta}}(\varsigma)}\right)' = \left(\frac{\Lambda'_{\boldsymbol{\delta}}(\varsigma)}{\Lambda_{\boldsymbol{\delta}}(\varsigma)}\right)' - \frac{\boldsymbol{\kappa}_p}{1-\gamma}\Lambda'_{\boldsymbol{\delta}}(\varsigma) - \frac{\gamma}{1-\gamma}\left(\frac{\Lambda_{\boldsymbol{\delta}}(\varsigma)}{M_{\gamma,\boldsymbol{\delta}}(\varsigma)}\right)' \tag{6.16}$$

and combining the last two Eqs., one arrives at

$$\begin{aligned} Y''_{\gamma,\boldsymbol{\delta}}(x,\varsigma^*) &= -\frac{\gamma}{1-\gamma}\left(\boldsymbol{\kappa}_p\Lambda'_{\boldsymbol{\delta}}(\varsigma^*) + \left(\frac{\Lambda_{\boldsymbol{\delta}}(\varsigma^*)}{M_{\gamma,\boldsymbol{\delta}}(\varsigma^*)}\right)'\right) Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma^*) \\ &= -\frac{\gamma}{1-\gamma}\left(\boldsymbol{\kappa}_p\Lambda'_{\boldsymbol{\delta}}(\varsigma^*) + \left(\frac{\Lambda_{\boldsymbol{\delta}}(\varsigma^*)}{M_{\gamma,\boldsymbol{\delta}}(\varsigma^*)}\right)^2\right) Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma^*) \end{aligned} \tag{6.17}$$

or, by virtue of Eq. (6.11)

$$Y''_{\gamma,\boldsymbol{\delta}}(x,\varsigma^*) = -\frac{\gamma}{1-\gamma}\left(\boldsymbol{\kappa}_p\Lambda'_{\boldsymbol{\delta}}(\varsigma^*) - \boldsymbol{\kappa}_p^2\left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right)^2\right) Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma^*). \tag{6.18}$$

Hence $\Lambda'_{\boldsymbol{\delta}}(\varsigma^*) - \boldsymbol{\kappa}_p\left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right)^2 > 0$ guarantees $Y''_{\gamma,\boldsymbol{\delta}}(x,\varsigma^*) < 0$, identifying ς^* as a (local) maximum. Note that Eq. (6.5) also translates into

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varsigma}\left(\frac{1}{\boldsymbol{\kappa}_p\left(1 - \Lambda_{\boldsymbol{\delta}}(\varsigma^*)\right)}\right)\right|_{\varsigma=\varsigma^*} > 1, \tag{6.19}$$

which means that the derivative of the right-hand side exceeds the one of the left-hand side of Eq. (6.11) in the intersection point ς^*. From Eq. (5.8),

$$\begin{aligned} \lim_{x\to\infty} \frac{M_{\gamma,\boldsymbol{\delta}}(x)}{\Lambda_{\boldsymbol{\delta}}(x)} &= \frac{\gamma}{1-\gamma}\int_{w=0}^{\infty} \mathrm{e}^{-\frac{\boldsymbol{\kappa}_p}{1-\gamma}\int_0^x (1-\Lambda_{\boldsymbol{\delta}}(\infty))\mathrm{d}w}\mathrm{d}x \\ &= \frac{\gamma}{\boldsymbol{\kappa}_p\left(1 - \Lambda_{\boldsymbol{\delta}}(\infty)\right)}. \end{aligned} \tag{6.20}$$

Note that Eq. (6.20) can also be obtained directly by probabilistic reasoning (in the absence of ruin):

$$M_{\gamma,\boldsymbol{\delta}}(\infty) = \Lambda_{\boldsymbol{\delta}}(\infty)\left(\frac{\gamma}{\boldsymbol{\kappa}_p} + M_{\gamma,\boldsymbol{\delta}}(\infty)\right). \tag{6.21}$$

Altogether it is then clear that

$$\lim_{x\to\infty}\frac{M_{\gamma,\boldsymbol{\delta}}(x)}{\Lambda_{\boldsymbol{\delta}}(x)} < \lim_{x\to\infty}\frac{1}{\boldsymbol{\kappa}_p\left(1-\Lambda_{\boldsymbol{\delta}}(x)\right)}. \tag{6.22}$$

Hence, for $\lim_{x\to 0} M_{\gamma,\boldsymbol{\delta}}(x)/\Lambda_{\boldsymbol{\delta}}(x) > 1/\boldsymbol{\kappa}_p$, the continuity of the functions $M_{\gamma,\boldsymbol{\delta}}(x)/\Lambda_{\boldsymbol{\delta}}(x)$ and $1/\boldsymbol{\kappa}_p\left(1-\Lambda_{\boldsymbol{\delta}}(x)\right)$ guarantees the existence of an optimal $\varsigma^* > 0$ (in case there should be several positive solutions of Eq. (6.4) with Eq. (6.5), one would have to pick the one leading to the largest value of $Y_{\gamma,\boldsymbol{\delta}}(x,\varsigma)$.

Finally, in the absence of a positive local maximum, the fact that $Y_{\gamma,\boldsymbol{\delta}}(x,\infty) = 0$ then establishes $\varsigma^* = 0$ as the optimal taxation level. □

7 Numerical Examples

In this section, we quantify the expressions of ruin probabilities, Laplace transform of ruin time, expected discounted tax paid until ruin in the SPCM for some set of parameters. The values of corresponding expressions are obtained, and the related graphs are plotted using MATLAB. We provide illustrations using two different examples to point out the existence of the optimal surplus level to start tax collection. In the first example, we consider parameters that do not satisfy the condition (6.7) for the existence of optimal surplus level to start tax collection. While in the second one, we consider the parameters in such a way to satisfy the sufficient condition (6.7).

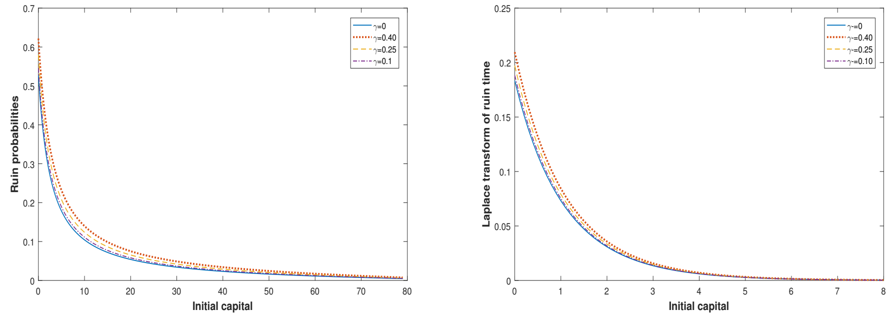
Example 7.1. We consider the following parameters for the SPCM: $\boldsymbol{\lambda}_p = 1$, $\boldsymbol{\lambda}_c = 1$, $\boldsymbol{\kappa}_p = 1$ and $\boldsymbol{\kappa}_c = 2$. The cases of tax rates $\boldsymbol{\gamma} = 0.1, 0.25$ and 0.4 are taken along with the non-tax case for which the security loading factor (by Eq. (2.8)) is 0.2, 0.5 ,0.8 and 1 respectively.

Remark 7.2. From Figure2a, the plot indicates that the ruin probabilities are proportional to the tax rates (also see Table 1). For $\boldsymbol{\delta} = 0.6$, the corresponding Laplace transform values of ruin time are plotted and tabled in Figure 2b and Table 2 respectively, which also behaves the same.

Remark 7.3. We further obtain the values of expected discounted tax paid until ruin $M_{\gamma,\delta}(u)$. By Figure 3a and Table 3, we observe that the tax payments converge to a constant when the insurer has unbounded capital. Finally, we determine the values of expected discounted tax paid until ruin $Y_{\gamma,\delta}(0,\varsigma)$ for the same $\boldsymbol{\delta}$ against the threshold surplus level to start tax collection. By Figure 3b and Table 4, we can conclude that the curve will decay to zero as the increment in threshold to start tax collection. In other words, the possibility to start tax collection will decrease since the surplus probably may not attain the threshold when it is large. Hence, the optimal threshold that maximises the expected discounted tax does not exist, which is natural since the choice of parameters does not satisfy the sufficient condition (6.7).

Example 7.4. In this example, we try to make an analogue of the example illustrated by Albrecher and Hipp [15] for the random income risk process. The values of expected discounted tax paid until ruin are obtained for the set of parameters satisfying the condition (6.7). We consider the following parameters: $\boldsymbol{\lambda}_p = 2$, $\boldsymbol{\lambda}_c = 1$, $\boldsymbol{\kappa}_p = 1$ and $\boldsymbol{\kappa}_c = 1$. The tax rate is 0.5 for which the security loading factor is zero and satisfies the condition (6.7) for the existence of optimal surplus level to start tax collection.

Remark 7.5. In Figure 4a, the plot of expected discounted ($\boldsymbol{\delta} = 0.04$) tax collected until ruin against threshold surplus to start tax collection validates the existence of optimal surplus level to start tax collection. Further, using Figure 4b, we explore the Eq. (6.13) to determine the optimal threshold level ($\varsigma^\star = 3.05$) which maximises the expected discounted tax collected until ruin. The corresponding coordinate values from Figures 4a and 4b are tabulated in Table 5.



(a) Ruin Probabilities (b) Laplace transform of ruin time

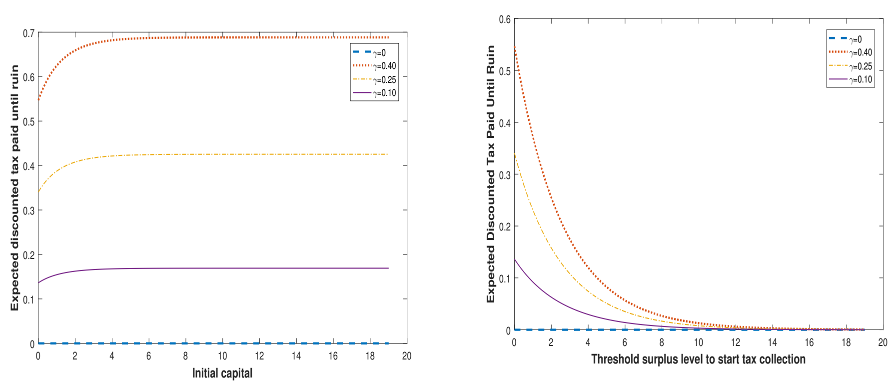
Figure 2: Ruin Probabilities and Laplace transform ruin probabilities vs Initial capital

u	$\psi_{0,0}(u)$	$\psi_{0.10,0}(u)$	$\psi_{0.25,0}(u)$	$\psi_{0.40,0}(u)$
0	0.5264	0.5460	0.5802	0.6219
10	0.1139	0.1214	0.1349	0.1521
20	0.0569	0.0612	0.0689	0.0790
40	0.0241	0.0262	0.0301	0.0353
60	0.0117	0.0128	0.0149	0.0179
80	0.0048	0.0053	0.0062	0.0076

Table 1: Ruin Probabilities Vs Initial capital for various tax rates

u	$\psi_{0,0.6}(u)$	$\psi_{0.10,0.6}(u)$	$\psi_{0.25,0.6}(u)$	$\psi_{0.40,0.6}(u)$
0	0.1835	0.1883	0.1972	0.2096
3	0.0308	0.0317	0.0334	0.0357
5	0.0061	0.0062	0.0066	0.0070
7	0.0013	0.0013	0.0014	0.0015
9	0.0002	0.0002	0.0003	0.0003

Table 2: Laplace transform of ruin time Vs Initial capital for various tax rates



(a) Vs Initial capital (b) Vs Threshold surplus level to start tax collection

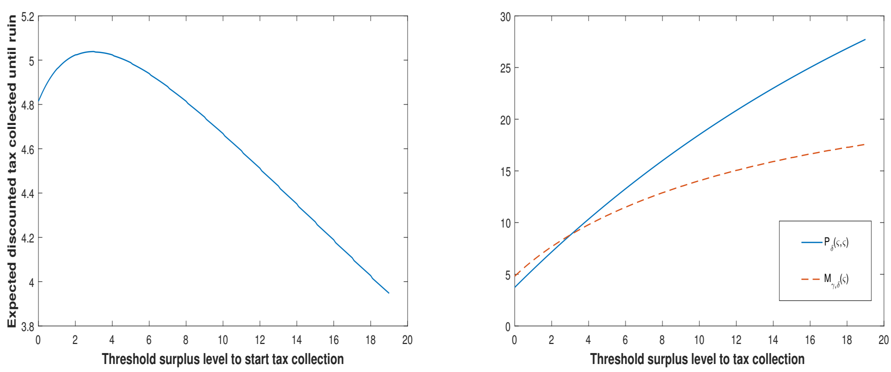
Figure 3: Expected discounted tax paid until ruin

u	$M_{0,0.6}(u)$	$M_{0.10,0.6}(u)$	$M_{0.25,0.6}(u)$	$M_{0.40,0.6}(u)$
0	0.0000	0.1363	0.3405	0.5469
5	0.0000	0.1676	0.4215	0.6818
8	0.0000	0.1689	0.4249	0.6876
12	0.0000	0.1691	0.4253	0.6882
15	0.0000	0.1691	0.4253	0.6882

Table 3: Expected discounted tax paid until ruin Vs Initial capital for various tax rates

ς	$Y_{0,0.6}(0,\varsigma)$	$Y_{0.10,0.6}(0,\varsigma)$	$Y_{0.25,0.6}(0,\varsigma)$	$Y_{0.40,0.6}(0,\varsigma)$
0	0.0000	0.1363	0.3405	0.5469
5	0.0000	0.0295	0.0741	0.1199
8	0.0000	0.0096	0.0240	0.0389
12	0.0000	0.0021	0.0054	0.0087
15	0.0000	0.0006	0.0017	0.0028
18	0.0000	0.0002	0.0005	0.0007

Table 4: Expected discounted tax paid until ruin Vs Threshold surplus level to start tax collection for various tax rates



(a) Expected discounted tax paid until ruin (b) $P_{\delta}(\varsigma,\varsigma)$ and $M_{\gamma,\delta}(\varsigma)$

Figure 4: Performance quantities Vs Threshold surplus level to start tax collection

ς	$Y_{0.5,0.04}(0,\varsigma)$	$M_{0.5,0.04}(\varsigma)$	$P_{0.5}(\varsigma,\varsigma)$
0	4.8147	4.8147	3.7371
5	5.0239	9.7982	10.3340
8	4.8805	12.2209	14.6567
12	4.5911	14.5734	19.6965
15	4.1077	15.9063	22.9989

Table 5: Performance quantities Vs Threshold surplus level to start tax collection

8 Conclusion and Remarks

In this paper, we have analysed a random income risk model with the latent tax strategy. Assuming the MAP inter-renewal times and PH sizes, we have developed the differential equations satisfied by the survival probability, the Laplace transform of ruin time and the moments of discounted tax payments up to ruin. For the single-phase cash-flow model, we established the tax identity. Further, we obtained the solution for Laplace transform of ruin time and moments of discounted tax payments up to ruin, and the optimal surplus level for starting tax collection is determined. Finally, the expressions are numerically tracked for various tax rates.

A future step to this work is to link the taxable dividends to the non-taxable capital injections, proposed by Schmidli [23] in the classical risk model.

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