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ESTIMATING PARAMETERS OF THE INVERSE GOMPERTZ DISTRIBUTION UNDER UNIFIED HYBRID CENSORING SCHEME

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Abstract In this article, the problems of point and interval estimation for the associated parameters, along with reliability function (r(x)) and the hazard rate function (h(x)) of the inverse Gompertz distribution, are considered using a unified hybrid censoring scheme. In the case of point estimation, some classical estimators, such as the maximum likelihood estimator (MLE) and the maximum product spacing estimator (MPSE), are derived. Further, Bayes estimators are considered with respect to a suitable prior for the associated parameters under the balanced loss functions(balanced Linex loss function and balanced general entropy loss function). The asymptotic confidence intervals using the MLEs and MPSEs of the parameters are derived. Furthermore, the equal-tailed and the highest posterior density credible intervals are derived using the posterior samples. The mean squared errors are used to compare the point estimators, whereas the interval estimators are compared through their coverage probability and average lengths. Finally, a real data set is taken for application purposes.

1 Introduction

The problem is to obtain point and interval estimators of the parameters involved in the inverse Gompertz distribution (IGD) using a unified hybrid censoring scheme and two balanced loss functions. The probability density function (PDF) and the cumulative distribution function (CDF) of this distribution are, respectively, given by

$$f(x;\alpha,\beta)=\frac{\alpha}{x^2}e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)+\frac{\beta}{x}},\ x,\alpha,\beta>0 \tag{1.1}$$

and

$$F(x;\alpha,\beta)=e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)},\ x,\alpha,\beta>0. \tag{1.2}$$

The reliability function (RF) and the hazard rate function (HRF) of this IGD are obtained as

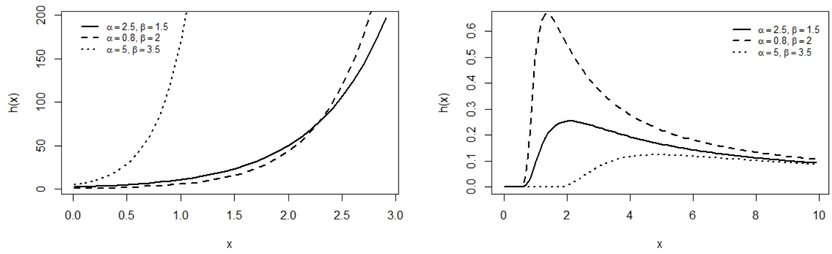
$$r(x)=1-F(x;\alpha,\beta)=1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)} \tag{1.3}$$

and

$$h(x)=\frac{f(x;\alpha,\beta)}{r(x)}=\frac{\alpha}{x^2}e^{\frac{\beta}{x}}\left(e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}-1\right)^{-1}, \tag{1.4}$$

respectively.

Gompertz [13] introduced a two-parameter lifetime distribution, called Gompertz distribution (GD), to model human behavioral and mortality related data, which is a generalization of



(a) Hazard rate plot for GD (b) Hazard rate plot for IGD

Figure 1: Hazard rate plots

the exponential distribution. This distribution has many real-life applications, especially in actuarial and medical sciences. We refer to [20], [7], and [10]) for some practical applications of GD in several fields of studies, such as environmental science, medical sciences, engineering, economics, behavioral sciences, biological sciences.

The GD has only the increasing HRF, which restricts its modeling of various phenomena in several fields. Eliwa et al. [8] introduced the two-parameter IGD with an upside-down bathtub-shape HRF. Furthermore, this IGD is capable of modeling both symmetric and skewed (positive and negative) data sets. The hazard rate function for both GD and IGD has been plotted in Figure 1 (a)-(b) for some particular choices of parameters.

Since our main focus is to obtain point and interval estimators for the parameters of the IGD under a unified hybrid censoring scheme (UHCS), we briefly discuss some results related to hybrid censoring in the following. Epstein [9] introduced the type-i hybrid censoring scheme (type-i HCS) by combining the type-i and type-ii censoring schemes. Since type-i HCS has a drawback of uncertainty about the number of observed failures, Childs et al. [6] proposed the type-ii HCS to overcome this drawback. Chandrasekar et al. [3] improved these two types of censored sampling schemes by introducing two extensions of this type, named generalized type-i HCS and generalized type-ii HCS. Further, Balakrishnan and Rasouli [2] introduced a more generalized censoring scheme, known as a UHCS, by combining these two types of generalized HCS's. Recently, Ateya [1] derived various point and interval estimators for the parameters of inverse Weibull distribution using UHCS. Panahi and Sayyareh [17] derived the maximum likelihood estimators (MLEs) of the parameters of Burr type-iii distribution using the EM-algorithm as well as the Bayes estimators using the Markov Chain Monte Carlo (MCMC) technique under UHCS. Jeon and Kang [14] derived the point and interval estimators for the parameters, reliability function, and entropy function from the Rayleigh distribution under UHCS. We refer to [14] and the references cited therein for some results on estimating parameters under UHCS.

The rest of our work can be organized as follows. The UHCS has been discussed with its special cases in Section 2. In section 3, the MLEs and associated asymptotic confidence intervals (ACIs) are derived using this UHCS. Further, the maximum product spacing estimator (MPSE) and the associated ACIs are derived in Section 4. The Bayesian estimation of the associated parameters and function of parameters are considered using the gamma prior under the balanced Linex loss (BLEL) and balanced general entropy loss (BGEL) function in Section 5. Due to the difficulty in deriving the analytical form of the Bayes estimators, the MCMC technique has been employed to generate posterior samples for the parameters. In Section 6, we compare the performances of all the proposed estimators (both point and interval) using an extensive simulation study. The point estimators are compared through their mean squared errors (MSEs). However, the interval estimators are compared in terms of coverage probability (CP) and average length (AL). In Section 7, we consider a real-life situation that has been modeled using the IGD, and further estimation methodologies have been explained.

2 Unified Hybrid Censoring

Let us consider a life-testing experiment in which n identical units are placed for life-test. Further, let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the corresponding lifetimes from an IGD with PDF (1.1). Fix $k, r \in \{1, 2, \ldots, n\}$ and $T_1 < T_2 \in (0, \infty)$ such that $k < r$. If the kth failure occurs before time T_1, terminate the experiment at min$\{$max$\{X_{r:n}, T_1\}, T_2\}$. If the kth failure occurs between T_1 and T_2, terminate the experiment at min$\{X_{r:n}, T_2\}$ and if the kth failure occurs after time T_2, then terminate the experiment at $X_{k:n}$. Under this censoring scheme, we can guarantee that the experiment will be completed at most in time T_2 with at least k failures and if not, we can guarantee exactly k failures. Thus, under this UHCS, we have the following six different cases.

Case 1: $0 < X_{k:n} < X_{r:n} < T_1 < T_2$, the experiment ends at the point of T_1.

Case 2: $0 < X_{k:n} < T_1 < X_{r:n} < T_2$, the experiment ends at the point of $X_{r:n}$.

Case 3: $0 < X_{k:n} < T_1 < T_2 < X_{r:n}$, the experiment ends at the point of T_2.

Case 4: $0 < T_1 < X_{k:n} < X_{r:n} < T_2$, the experiment ends at the point of $X_{r:n}$.

Case 5: $0 < T_1 < X_{k:n} < T_2 < X_{r:n}$, the experiment ends at the point of T_2.

Case 6: $0 < T_1 < T_2 < X_{k:n} < X_{r:n}$, the experiment ends at the point of $X_{k:n}$.

Note that in the proposed UHCS, termination time T_2 is introduced in addition to time T_1 to achieve more flexibility than the generalized type-i HCS. Let d_j denote the number of failures until time $T_j,\ j = 1, 2$. Thus, the likelihood function under UHCS is given by

$$l(\theta \mid \underline{x}) = \begin{cases} \frac{n!}{(n-d_1)!} \prod_{i=1}^{d_1} f(x_{i:n}) \Big(1 - F(T_1)\Big)^{n-d_1}, \ d_1 = d_2 = d = r, \ldots, n, \\ \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_{i:n}) \Big(1 - F(x_{r:n})\Big)^{n-r}, \ d_1 = k, \ldots, r-1; d_2 = r, \\ \frac{n!}{(n-d_2)!} \prod_{i=1}^{d_2} f(x_{i:n}) \Big(1 - F(T_2)\Big)^{n-d_2}, \ d_1 = k, \ldots, r-1; d_2 = k, \ldots, r-1; \\ \qquad\qquad d_1 \leq d_2, \\ \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_{i:n}) \Big(1 - F(x_{r:n})\Big)^{n-r}, \ d_1 = 0, \ldots, k-1; d_2 = r, \\ \frac{n!}{(n-d_2)!} \prod_{i=1}^{d_2} f(x_{i:n}) \Big(1 - F(T_2)\Big)^{n-d_2}, \ d_1 = 0, \ldots, k-1; d_2 = k, \ldots, r-1, \\ \frac{n!}{(n-k)!} \prod_{i=1}^{k} f(x_{i:n}) \Big(1 - F(x_{k:n})\Big)^{n-k}, \ d_2 = 0, \ldots, k-1. \end{cases} \tag{2.1}$$

For details of the construction of likelihood function under UHCS, we refer to [2].

3 The MLE & Asymptotic Confidence Interval

The likelihood functions (2.1) based on the UHCS can be written in a general form as

$$l(\alpha, \beta \mid \underline{x}) = \frac{n!}{(n-m)!} \prod_{i=1}^{m} f(x_{i:n})(1 - F(c))^{n-m}, \tag{3.1}$$

where

$$(m, c) = \begin{cases} (d_1, T_1) & \text{for Case 1,} \\ (r, x_{r:n}) & \text{for Case 2 and Case 4,} \\ (d_2, T_2) & \text{for Case 3 and Case 5,} \\ (k, x_{k:n}) & \text{for Case 6.} \end{cases}$$

Thus from (3.1) the likelihood function for the IGD is derived as

$$l(\alpha,\beta\mid \underline{x}) \propto \alpha^m e^{-\frac{\alpha}{\beta}\sum_{i=1}^{m}(e^{\frac{\beta}{x_{i:n}}}-1)+\beta\sum_{i=1}^{m}\frac{1}{x_{i:n}}}\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}\right)^{n-m}\prod_{i=1}^{m}\frac{1}{x_{i:n}^2}. \tag{3.2}$$

Taking logarithm on the likelihood function (3.2), the log-likelihood function can be obtain as

$$\begin{aligned} L(\alpha,\beta\mid \underline{x}) \quad &\propto \quad m\log\alpha+\beta\sum_{i=1}^{m}\frac{1}{x_{i:n}}-\frac{\alpha}{\beta}\sum_{i=1}^{m}(e^{\frac{\beta}{x_{i:n}}}-1)-2\sum_{i=1}^{m}\log x_{i:n} \\ &+(n-m)\log\left(1-e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}\right). \end{aligned} \tag{3.3}$$

The first and second order derivatives of the log-likelihood function (3.3) respect to the parameters (α,β) can be derived as follows,

$$\frac{\partial L}{\partial \alpha}=\frac{m}{\alpha}-\frac{1}{\beta}\sum_{i=1}^{m}(e^{\frac{\beta}{x_{i:n}}}-1)+\frac{(n-m)(e^{\frac{\beta}{c}}-1)}{\beta\left(e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}-1\right)}, \tag{3.4}$$

$$\frac{\partial L}{\partial \beta}=-\frac{m\alpha}{\beta^2}+\sum_{i=1}^{m}\frac{1}{x_{i:n}}+\frac{\alpha}{\beta^2}\sum_{i=1}^{m}e^{\frac{\beta}{x_{i:n}}}-\frac{\alpha}{\beta}\sum_{i=1}^{m}\frac{1}{x_{i:n}}e^{\frac{\beta}{x_{i:n}}}+\frac{(n-m)\alpha}{\beta^2}\frac{1+(\frac{\beta}{c}-1)e^{\frac{\beta}{c}}}{e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}-1}, \tag{3.5}$$

$$\frac{\partial^2 L}{\partial \alpha^2}=-\frac{m}{\alpha^2}-(n-m)\frac{(e^{\frac{\beta}{c}}-1)^2e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}}{\beta^2\left(e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}-1\right)^2}, \tag{3.6}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial\alpha\partial\beta} \quad &= \quad \frac{1}{\beta^2}\sum_{i=1}^{m}\left(e^{\frac{\beta}{x_{i:n}}}-1\right)-\frac{1}{\beta}\sum_{i=1}^{m}\frac{1}{x_{i:n}}e^{\frac{\beta}{x_{i:n}}} \\ &+ \quad (n-m)\frac{\left((\frac{\beta}{c}-1)e^{\frac{\beta}{c}}+1\right)}{\beta^2\left(e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}-1\right)}\left(1-\frac{\alpha\left(e^{\frac{\beta}{c}}-1\right)}{\beta\left(e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}-1\right)}\right) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial\beta^2} \quad &= \quad -\frac{2\alpha}{\beta^3}\sum_{i=1}^{m}\left(e^{\frac{\beta}{x_{i:n}}}-1\right)+\frac{2\alpha}{\beta^2}\sum_{i=1}^{m}\frac{1}{x_{i:n}}e^{\frac{\beta}{x_{i:n}}}-\frac{\alpha}{\beta}\sum_{i=1}^{m}\frac{1}{x_{i:n}^2}e^{\frac{\beta}{x_{i:n}}} \\ &+(n-m)\frac{\alpha\left(\left((\frac{\beta}{c})^2-\frac{2\beta}{c}+2\right)e^{\frac{\beta}{c}}-2\right)}{\beta^3\left(e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}-1\right)} \\ &-(n-m)\frac{\alpha^2\left((\frac{\beta}{c}-1)e^{\frac{\beta}{c}}+1\right)^2e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}}{\beta^4\left(e^{\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}-1\right)^2} \end{aligned} \tag{3.8}$$

The maximum likelihood estimators of the parameters α and β can be derived by solving the nonlinear equations $\frac{\partial L}{\partial\alpha}=0$ and $\frac{\partial L}{\partial\beta}=0$. From (3.4) and (3.5) it is clear that the MLEs has no closed form. So, the Newton's method can be employ to derive the MLEs numerically. Further, utilizing the obtained MLEs $\hat{\alpha}_{ml}$ and $\hat{\beta}_{ml}$ of the parameters α and β, the MLEs of the $r(x_0)$ and $h(x_0)$ can be derived easily as

$$\hat{r}_{ml}(x_0)=1-e^{-\frac{\hat{\alpha}_{ml}}{\hat{\beta}_{ml}}(e^{\frac{\hat{\beta}_{ml}}{x_0}}-1)}$$

and

$$\hat{h}_{ml}(x_0) = \frac{\hat{\alpha}_{ml}}{x_0^2} e^{\frac{\hat{\beta}_{ml}}{x_0}} \left(e^{\frac{\hat{\alpha}_{ml}}{\hat{\beta}_{ml}} (e^{\frac{\hat{\beta}_{ml}}{x_0}} - 1)} - 1 \right)^{-1}.$$

Using the MLEs of the parameters the ACIs of the parameters can be obtained by using the observed Fisher information matrix, where the observed Fisher information matrix is

$$\hat{I}(\alpha, \beta) = \begin{bmatrix} -\frac{\partial^2 L}{\partial \alpha^2} & -\frac{\partial^2 L}{\partial \beta \partial \alpha}, \\ -\frac{\partial^2 L}{\partial \alpha \partial \beta} & -\frac{\partial^2 L}{\partial \beta^2} \end{bmatrix}.$$

Using this observed Fisher information matrix the $100(1-\psi)\%$ ACIs for α, β, $r(x_0)$ and $h(x_0)$ can be obtained by applying the normal approximation of the MLEs (as the MLE follows the normal distribution $N(\theta, \sqrt{Var(\hat{\theta}_{ml})})$ as follows:

$$\hat{\alpha}_{ml} \pm z_{\psi/2} \sqrt{Var(\hat{\alpha}_{ml})},\ \hat{\beta}_{ml} \pm z_{\psi/2} \sqrt{Var(\hat{\beta}_{ml})}$$

$$\hat{r}_{ml}(x_0) \pm z_{\psi/2} \sqrt{Var(\hat{r}_{ml}(x_0))} \text{ and } \hat{h}_{ml}(x_0) \pm z_{\psi/2} \sqrt{Var(\hat{h}_{ml}(x_0))},$$

respectively, where z_ψ is the ψ^{th} percentile of the standard normal distribution; $Var(\alpha_{ml})$ and $Var(\beta_{ml})$ are the diagonal values of the variance matrix (i.e., inverse of the observe Fisher information matrix $\hat{I}(\alpha_{ml}, \beta_{ml})$); $Var(\hat{r}_{ml}(x_0))$ and $Var(\hat{h}_{ml}(x_0))$ can be derived from variance-covariance matrix by applying delta method.

Since the parameters α and β are positive valued, it is also possible to use logarithmic transformation to compute approximate confidence intervals for these parameters. We refer to [16] in this direction. They pointed out that the confidence interval obtained using the normal approximation of the log-transformed MLE method has a better coverage probability than that obtained using the normal approximation of the MLE method. The normal approximate $100(1-\psi)\%$ confidence interval of α, β, $r(x_0)$ and $h(x_0)$ for log-transformed MLE are respectively

$$\hat{\alpha}_{ml} \times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{\alpha}_{ml})}}{\hat{\alpha}_{ml}}},\ \hat{\beta}_{ml} \times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{\beta}_{ml})}}{\hat{\beta}_{ml}}},$$

$$\hat{r}_{ml}(x_0) \times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{r}_{ml}(x_0))}}{\hat{r}_{ml}(x_0)}} \text{ and } \hat{h}_{ml}(x_0) \times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{h}_{ml}(x_0))}}{\hat{h}_{ml}(x_0)}}.$$

4 The MPSE & Asymptotic Confidence Interval

Ranneby [18] introduced another classical estimation method, called the maximum product spacing (MPS) method, which gives a consistent estimator under much more general conditions than MLEs. Further, MPSEs are asymptotically normal and asymptotically are as efficient as MLEs when these exist.

The product of the spacing's under UHCS can be obtained as

$$p(\alpha, \beta) = \left\{ \prod_{i=1}^{m+1} \{F(x_{i:n}) - F(x_{i-1:n})\}\{1 - F(c)\}^{n-m} \right\}^{\frac{1}{n+1}}, \tag{4.1}$$

where $F(x_{0:n}) = 0$ and $F(x_{m+1:n}) = 1$.

Taking logarithm on both sides of the equation (4.1), we have

$$\begin{aligned} P(\alpha, \beta) \quad &= \quad \frac{1}{n+1} \Bigg[\log F(x_{1:n}) + \sum_{i=2}^{m} \log \{F(x_{i:n}) - F(x_{i-1:n})\} \\ &\quad + \log\{1 - F(x_{m:n})\} + (n-m) \log\{1 - F(c)\} \Bigg]. \end{aligned} \tag{4.2}$$

The first and second order partial derivatives of $P(\alpha,\beta)$ (given in (4.2)) with respect to α and β are

$$\frac{\partial P}{\partial\theta_k}=\frac{1}{n+1}\left[\frac{F_{\theta_k}(x_{1:n})}{F(x_{1:n})}-\frac{F_{\theta_k}(x_{m:n})}{1-F(x_{m:n})}+(n-m)\frac{F_{\theta_k}(c)}{1-F(c)}+\sum_{i=2}^{m}\frac{F_{\theta_k}(x_{i:n})-F_{\theta_k}(x_{i-1:n})}{F(x_{i:n})-F(x_{i-1:n})}\right],$$

$$\begin{aligned}\frac{\partial^2 P}{\partial\theta_k\partial\theta_l} &= \frac{1}{n+1}\Bigg[\frac{F(x_{1:n})F_{\theta_k\theta_l}(x_{1:n})-F_{\theta_k}(x_{1:n})F_{\theta_l}(x_{1:n})}{(F(x_{1:n}))^2}+\sum_{i=2}^{m}\frac{F_{\theta_k\theta_l}(x_{i:n})-F_{\theta_k\theta_l}(x_{i-1:n})}{F(x_{i:n})-F(x_{i-1:n})}\\ &-\frac{(1-F(x_{m:n}))F_{\theta_k\theta_l}(x_{m:n})+F_{\theta_k}(x_{m:n})F_{\theta_l}(x_{m:n})}{(1-F(x_{m:n}))^2}\\ &+(n-m)\frac{(1-F(c))F_{\theta_k\theta_l}(c)+F_{\theta_k}(c)F_{\theta_l}(c)}{(1-F(c))^2}\\ &+\sum_{i=2}^{m}\frac{(F_{\theta_k}(x_{i:n})-F_{\theta_k}(x_{i-1:n}))(F_{\theta_l}(x_{i:n})-F_{\theta_l}(x_{i-1:n}))}{(F(x_{i:n})-F(x_{i-1:n}))^2}\Bigg]\end{aligned}$$

where $k,l\in\{1,2\},\ \theta_1=\alpha,\ \theta_2=\beta,$

$$F_{\alpha}(x)=-\frac{1}{\beta}(e^{\frac{\beta}{x}}-1)e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)},$$

$$F_{\alpha\alpha}(x)=\frac{1}{\beta^2}(e^{\frac{\beta}{x}}-1)^2e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)},$$

$$F_{\beta}(x)=\frac{\alpha}{\beta^2}\{(1-\frac{\beta}{x})e^{\frac{\beta}{x}}-1\}e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)},$$

$$F_{\beta\beta}(x)=\frac{\alpha}{\beta^3}\left[\frac{\alpha}{\beta}\{(1-\frac{\beta}{x})e^{\frac{\beta}{x}}-1\}^2+\frac{\beta}{x}(2-\frac{\beta}{x})e^{\frac{\beta}{x}}+2(1-e^{\frac{\beta}{x}})\right]e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{x}}-1)}.$$

As similar to the derivation of the MLEs, the MPSEs of the parameters can be obtained by solving the system of non-linear equations $\frac{\partial P}{\partial\alpha}=0$ and $\frac{\partial P}{\partial\beta}=0$, using the Newtons method. Further, the obtained MPSEs $\hat{\alpha}_{mp}$ and $\hat{\beta}_{mp}$ of the parameters α and β, respectively, can be used to derive the MPSEs $\hat{r}_{mp}(x_0)$ and $\hat{h}_{mp}(x_0)$ of the RF $(r(x_0))$ and HRF $(h(x_0))$, respectively, in a similar way as described in Section 3 for the MLEs of $r(x_0)$ and $h(x_0)$.

According to Chen and Mi [5], the MPSEs also follows the asymptotic properties as MLEs. So, as similar to the MLEs the $100(1-\psi)\%$ asymptotic confidence intervals of α, β, $r(x_0)$ and $h(x_0)$ can be obtained as follows:

(i) using normal approximation of the MPSEs, we have the confidence intervals as

$$\hat{\alpha}_{mp}\pm z_{\psi/2}\sqrt{Var(\hat{\alpha}_{mp})},\ \hat{\beta}_{mp}\pm z_{\psi/2}\sqrt{Var(\hat{\beta}_{mp})}$$

$$\hat{r}_{mp}(x_0)\pm z_{\psi/2}\sqrt{Var(\hat{r}_{mp}(x_0))}\text{ and }\hat{h}_{mp}(x_0)\pm z_{\psi/2}\sqrt{Var(\hat{h}_{mp}(x_0))},$$

(ii) using normal approximation of the log transformed MPSEs, the confidence intervals are obtained as

$$\hat{\alpha}_{mp}\times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{\alpha}_{mp})}}{\hat{\alpha}_{mp}}},\ \hat{\beta}_{mp}\times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{\beta}_{mp})}}{\hat{\beta}_{mp}}},$$

$$\hat{r}_{mp}(x_0)\times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{r}_{mp}(x_0))}}{\hat{r}_{mp}(x_0)}}\text{ and }\hat{h}_{mp}(x_0)\times e^{\frac{z_{\psi/2}\sqrt{Var(\hat{h}_{mp}(x_0))}}{\hat{h}_{mp}(x_0)}}.$$

5 Bayesian Estimation

In this section, the Bayes estimators for α, β, $r(x)$ and $h(x)$ are derived by using the MCMC method under the BLEL and BGEL functions, where the loss functions are given respectively as

$$L_{blel}(\theta, \hat{\delta}) = \omega\{e^{\nu(\hat{\delta}_0 - \hat{\delta})} - \nu(\hat{\delta}_0 - \hat{\delta}) - 1\} + (1-\omega)\{e^{\nu(\hat{\delta}-\theta)} - \nu(\hat{\delta}-\theta) - 1\},\ \nu \neq 0 \quad (5.1)$$

and

$$L_{bgel}(\theta, \hat{\delta}) = \omega\{(\frac{\hat{\delta}_0}{\hat{\delta}})^{\mu} - \mu \ln(\frac{\hat{\delta}_0}{\hat{\delta}}) - 1\} + (1-\omega)\{(\frac{\hat{\delta}}{\theta})^{\mu} - \mu \ln(\frac{\hat{\delta}}{\theta}) - 1\},\ \mu \neq 0, \quad (5.2)$$

where $\hat{\delta}_0$ is a general target estimator of θ, which is considered here the MLE of θ (see [12]). It has been assumed that, the parameters α and β are independent and follow gamma distributions G($a_1,\ b_1$) and G($a_2,\ b_2$) respectively. The density functions of the populations G($a_1,\ b_1$) and G($a_2,\ b_2$) are given by

$$\pi_1(\alpha) \propto \alpha^{a_1-1} e^{-\frac{\alpha}{b_1}},\ \alpha > 0, \quad (5.3)$$

and

$$\pi_2(\beta) \propto \beta^{a_2-1} e^{-\frac{\beta}{b_2}},\ \beta > 0, \quad (5.4)$$

respectively. Here the hyper-parameters a_1, b_1, a_2 and b_2 are assumed to be known and non-negative.

From (3.2), (5.3) and (5.4) the joint posterior density function can be obtained as follows,

$$\begin{aligned} \pi(\alpha, \beta|\underline{x}) \quad \propto \quad & \alpha^{m+a_1-1}\beta^{a_2-1} e^{-\alpha\left\{\frac{1}{b_1}+\frac{1}{\beta}\sum_{i=1}^{m}(e^{\frac{\beta}{x_{i:n}}}-1)\right\} - \beta\left\{\frac{1}{b_2} - \sum_{i=1}^{m}\frac{1}{x_{i:n}}\right\}} \\ & \times\left(1 - e^{-\frac{\alpha}{\beta}(e^{\frac{\beta}{c}}-1)}\right)^{n-m} \prod_{i=1}^{m} \frac{1}{x_{i:n}^2}. \end{aligned} \quad (5.5)$$

We adopt the Metropolis-Hastings algorithm (MHA) to generate posterior samples for α and β from the posterior density function (5.5). The algorithm for generating posterior samples using MHA can be described as follows.

(i) Consider the initial values (α_0, β_0) for the parameters (α, β).

(ii) For $i-th$ iteration generate $\alpha_* \sim N(\alpha_{i-1}, \sigma_\alpha)$ and $\beta_* \sim N(\beta_{i-1}, \sigma_\beta)$.

(iii) Compute $A = \min\left\{1, \frac{\pi(\alpha_*, \beta_*|\underline{x})}{\pi(\alpha_{i-1}, \beta_{i-1}|\underline{x})}\right\}$.

(iii) Generate $u \sim Uniform(0, 1)$,

(iv) If $u \leq A$, update (α_i, β_i) by (α_*, β_*), otherwise update (α_i, β_i) by $(\alpha_{i-1}, \beta_{i-1})$,

(v) Repeat (ii)-(iv) for $i = 1, 2, \ldots, N$, for a large positive integer N.

Thus the posterior samples for α and β can be obtained as $(\alpha_1, \alpha_2, \ldots \alpha_N)$ and $(\beta_1, \beta_2, \ldots, \beta_N)$. The Bayes estimators of the parametric function $\kappa(\alpha, \beta)$ under the loss functions BLEL and BGEL are respectively given by

$$\hat{\kappa}_{blel}(\alpha, \beta) = -\frac{1}{\nu} \ln\left\{\omega e^{-\nu\hat{\kappa}(\alpha,\beta)} + \frac{1-\omega}{N-N_0} \sum_{i=N_0+1}^{N} e^{-\nu\kappa(\alpha_i,\beta_i)}\right\} \quad (5.6)$$

and

$$\hat{\kappa}_{bgel}(\alpha, \beta) = \left\{\omega(\hat{\kappa}(\alpha, \beta))^{-\mu} + \frac{1-\omega}{N-N_0} \sum_{i=N_0+1}^{N} (\kappa(\alpha_i, \beta_i))^{-\mu}\right\}^{-\frac{1}{\mu}}. \quad (5.7)$$

Note that, for $\omega = 1$, the estimators are the MLEs, whereas, for $\omega = 0$, the estimators are the Bayes estimators under the asymmetric loss functions, such as the Linex loss and general entropy loss, respectively. Further, it is interesting to note that, for $\nu \to 0$, the Linex loss function behaves like the squared error loss function (SEL). Also, for $\mu = -1$ and $\mu = 1$, the general entropy loss function becomes the SEL and the weighted SEL, respectively. Further, using the posterior samples, the credible and the HPD intervals have been obtained using the method of Chen and Shao [4].

Remark 5.1. The Bayes estimators under the Lindley [15] and Tierney and Kadane [19] approximations do not perform better than the proposed MCMC method in terms of MSE. However, for some combinations of sample sizes and censoring schemes, the Bayes estimators under Tierney and Kadane approximation compete with the MCMC method, which has been seen from our simulation study. Unlike the MCMC method, these two approximations do not help to derive confidence intervals; hence we have not included the details of these two approximation methods here.

6 Simulation Study

In this section, the performances of all the proposed point and interval estimators of α, β, $r(x)$ and $h(x)$ from IGD under UHCS has been compared numerically. In this regard, an extensive simulation study has been conducted with the help of the Monte Carlo simulation procedure with $10,000$ replications for various combinations of sample sizes and censoring schemes. The point estimators are compared using their corresponding mean squared error (MSE's), which is given as $MSE(\hat{\delta}) = E(\hat{\delta} - \theta)^2$. The interval estimators are compared using their corresponding coverage probability (CP) and average length (AL).

The choice of the hyper-parameters for gamma priors are chosen suitably as $a_1 = 1.8$, $b_1 = 0.5$, $a_2 = 4$ and 0.5. In MH-algorithm we have considered $N = 10,000$ and $N_0 = 1000$. All the results obtained from the simulation are given in Tables 1-6. In Tables 1-2, the notations (ν_1, ν_2, ν_3, μ_1, μ_2, μ_3) are used to denote ($\nu = -1.0$, $\nu = -0.2$, $\nu = 1.0$, $\mu = -1.0$, $\mu = -0.2$, $\mu = 1.0$), respectively. In Tables 3-6, each column of the interval estimators for a particular censoring scheme contains two values, which correspond to CP and AL of the corresponding interval estimators under that censoring scheme. Further, in the Tables 3-6, the notations "NA" and "NAL" stand for the ACI with respect to normal approximation of the associated estimator and log-transformed estimator, respectively.

In our study, we have considered the lower acceptable threshold CP as 0.90 for the confidence level 0.95 and 0.95 for the confidence level 0.99. The following observations were made from our simulation study regarding the performances of point and interval estimators.

(i) For fixed r, k, T_1 the MSEs of the estimators decrease as T_2 increases. Further, similar type of results observed for the cases, r, k, T_2 fixed and T_1 increases; r, T_1, T_2 fixed and k increases; k, T_1, T_2 fixed and r increases.

(ii) For a particular censoring scheme, the MSEs of the Bayes estimators increase as the value of ω increases under BLEL and BGEL.

(iii) For a particular censoring scheme, the MSEs of the Bayes estimators of α and $r(x)$ under BLEL and BGEL decrease as the value of ν and μ increase, respectively. A similar type of observation has been noticed for the Bayes estimators of β and $h(x)$ under the BLEL function, whereas, under the BGEL function, the MSEs of the Bayes estimators become smaller for $\mu \to 0$.

(iv) The Bayes estimators under BLEL (with $\nu \approx 0$) and the Bayes estimators under BGEL (with $\mu = -1$) perform quite similarly in terms of MSEs.

(v) The Bayes estimators perform better than the MLEs in estimating α, β, $r(x)$ and $h(x)$. However, in estimating β the MPSE dominates the MLE. It is also observed that the Bayes estimators under BLEL (with $\omega \approx 0$ and $\nu = 1.0$) perform better than other estimators for α, β and $r(x)$. Further, the Bayes estimators under BGEL (with $\omega \approx 0$ and $\mu = 1.0$) perform better than other estimators in estimating $h(x)$.

(vi) From Tables 3-6 it has been observed that the AL of the interval estimators decrease as the sample size increase.

(vii) From Tables 3-6, we have observed that only credible and HPD intervals attain the nominal level. Among these qualified intervals, the HPD interval has the shortest AL among all other intervals. Thus HPD intervals perform better than others for interval estimation of α, β, $r(x)$ and $h(x)$.

7 Application with a Real Data

In this section, a real data set reported by Fuller Jr et al. [11], is considered for application purpose. The data contains the strengths of polished glass used in the aircraft window.
Data: 18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

We have conducted a goodness of fit test, such as the Kolmogorov-Smirnov (K-S) test, to fit the IGD to this data set. The obtained test static and the p-value of the K-S test are 0.13923 and 0.539, respectively. So, this real data is well fitted with the IGD model. Further, the real data is analyzed using the UHCS's,

$$\text{CS}_1:\ r = 25,\ k = 20,\ T1 = 20,\ T2 = 30$$

$$\text{CS}_2:\ r = 27,\ k = 24,\ T1 = 30,\ T2 = 50\ t0 = 20.$$

Using these UHCS's the obtained results for the proposed estimators are given in Table 7 for the real data. Based on our computational results, we recommend to use (1.537, 79.644, 0.972493, 0.01886) as the point estimates for (α, β, $r(x)$, $h(x)$), respectively under CS_1. However, under CS_2 the recommended estimates are (1.166, 85.597, 0.977916, 0.16017).

The preferred confidence intervals for α, β, $r(x)$ and $h(x)$ with $x = 20$, are the HPD intervals as given in the Table 7.

Concluding Remarks

In this article, various point and interval estimators have been investigated for the parameters, RF and HRF associated with the IGD under UHCS.

Several point estimators such as MLE, MPSE, and the Bayes estimators are derived for the parameters, HRF and RF using balanced loss functions. These estimators are computed numerically using Newton's method, as the MLEs and MPSEs have no closed-form expressions. Further, the Bayes estimators are considered using independent gamma prior for the parameters under BLEL and BGEL functions. As the Bayes estimators also have no closed-form expressions, the Bayes estimators are evaluated numerically using the posterior samples for the parametric function of interest with the help of the MH algorithm. All the point estimators are compared through MSEs.

Several interval estimators such as the ACIs, credible, and HPD intervals are derived numerically for the parameters, RF and HRF. Further, the ACIs are derived using the MLEs, log-transformed MLEs, MPSEs, and log-transformed MPSEs, whereas the credible intervals and HPD intervals are derived using the posterior samples. The interval estimators are compared in terms of CP and AL. Finally, a real-life data set has been considered for application purposes. This data set has been satisfactorily modeled using the IGD. We hope the present study will shed some light on this direction and motivate the researchers to investigate IGD, which has been overlooked despite its many applications. The Bayesian prediction for future observations from the IGD is still an open problem to investigate.

Table 1: MSEs of the point estimators for the parameters α and β for different censoring schemes

n	(k, r)	(T_1, T_2)	ML	MPS	BLEL						BGEL					
					ω=0.2			ω=0.8			ω=0.2			ω=0.8		
					ν_1	ν_2	ν_3	ν_1	ν_2	ν_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
$\alpha = 0.8$																
30	(15, 20)	(3, 5)	0.087	0.116	0.075	0.066	0.059	0.083	0.081	0.079	0.065	0.063	0.062	0.081	0.080	0.079
30	(15, 20)	(3, 8)	0.085	0.113	0.074	0.066	0.058	0.081	0.080	0.077	0.064	0.062	0.061	0.079	0.079	0.078
30	(15, 20)	(7, 10)	0.084	0.111	0.073	0.065	0.058	0.081	0.079	0.077	0.064	0.062	0.061	0.079	0.078	0.077
30	(18, 20)	(7, 10)	0.082	0.107	0.071	0.063	0.056	0.078	0.077	0.075	0.062	0.060	0.060	0.076	0.076	0.075
30	(18, 27)	(7, 10)	0.082	0.106	0.071	0.063	0.056	0.078	0.077	0.075	0.062	0.060	0.060	0.076	0.076	0.075
50	(25, 35)	(5, 8)	0.050	0.061	0.046	0.043	0.039	0.049	0.048	0.047	0.042	0.041	0.041	0.048	0.048	0.047
50	(25, 35)	(7, 10)	0.049	0.060	0.045	0.042	0.039	0.048	0.047	0.047	0.042	0.041	0.041	0.047	0.047	0.047
50	(30, 35)	(7, 10)	0.050	0.061	0.046	0.043	0.039	0.048	0.048	0.047	0.042	0.041	0.041	0.048	0.047	0.047
50	(30, 35)	(9, 10)	0.050	0.061	0.046	0.043	0.039	0.048	0.048	0.047	0.042	0.041	0.041	0.047	0.047	0.047
50	(35, 45)	(9, 10)	0.049	0.060	0.045	0.042	0.039	0.048	0.047	0.046	0.042	0.041	0.041	0.047	0.047	0.047
$\beta = 2$																
30	(15, 20)	(3, 5)	0.482	0.434	0.477	0.390	0.307	0.478	0.458	0.421	0.373	0.364	0.444	0.453	0.448	0.466
30	(15, 20)	(3, 8)	0.480	0.432	0.469	0.386	0.306	0.474	0.456	0.420	0.370	0.363	0.443	0.451	0.447	0.455
30	(15, 20)	(7, 10)	0.457	0.423	0.449	0.372	0.299	0.452	0.435	0.402	0.357	0.350	0.430	0.430	0.427	0.444
30	(18, 20)	(7, 10)	0.452	0.411	0.448	0.370	0.296	0.449	0.431	0.398	0.354	0.347	0.424	0.426	0.422	0.439
30	(18, 27)	(7, 10)	0.452	0.411	0.445	0.369	0.296	0.447	0.430	0.398	0.353	0.347	0.424	0.425	0.422	0.439
50	(25, 35)	(5, 8)	0.253	0.248	0.250	0.223	0.197	0.251	0.245	0.235	0.217	0.215	0.233	0.244	0.242	0.245
50	(25, 35)	(7, 10)	0.251	0.245	0.248	0.221	0.196	0.249	0.243	0.233	0.216	0.214	0.231	0.242	0.241	0.244
50	(30, 35)	(7, 10)	0.242	0.237	0.239	0.213	0.188	0.240	0.234	0.224	0.207	0.206	0.223	0.233	0.232	0.234
50	(30, 35)	(9, 10)	0.241	0.236	0.239	0.212	0.188	0.240	0.234	0.224	0.207	0.205	0.222	0.232	0.231	0.234
50	(35, 45)	(9, 10)	0.240	0.234	0.237	0.210	0.115	0.239	0.232	0.222	0.205	0.203	0.219	0.231	0.230	0.232

Table 2: MSEs of the point estimators for the RF $r(x)$ and HRF $h(x)$, at $x = 3$ for different censoring schemes (tabulated results are obtained by multiplying 10^3 with the computed results)

n	(k, r)	(T_1, T_2)	ML	MPS	BLEL						BGEL					
					ω=0.2			ω=0.8			ω=0.2			ω=0.8		
					ν_1	ν_2	ν_3	ν_1	ν_2	ν_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
								$r(x)$								
30	(15, 20)	(3, 5)	5.169	5.170	3.886	3.842	3.685	4.823	4.815	4.801	3.831	3.751	3.973	4.813	4.719	4.816
30	(15, 20)	(3, 8)	5.093	5.095	3.872	3.813	3.637	4.764	4.658	4.632	3.800	3.657	3.833	4.748	4.603	4.719
30	(15, 20)	(7, 10)	5.062	5.094	3.857	3.803	3.532	4.738	4.726	4.708	3.790	3.572	3.828	4.723	4.518	4.698
30	(18, 20)	(7, 10)	5.039	5.068	3.826	3.772	3.520	4.714	4.702	4.684	3.759	3.540	3.795	4.699	4.493	4.671
30	(18, 27)	(7, 10)	4.847	4.854	3.710	3.660	3.405	4.542	4.451	4.415	3.649	3.416	3.703	4.529	4.417	4.508
50	(25, 35)	(5, 8)	2.526	2.975	2.158	2.107	1.959	2.429	2.351	2.303	2.107	1.982	1.929	2.416	2.208	2.359
50	(25, 35)	(7, 10)	2.502	2.945	2.121	2.079	1.820	2.402	2.322	2.276	2.069	1.819	1.886	2.389	2.173	2.329
50	(30, 35)	(7, 10)	2.473	2.910	2.096	2.055	1.796	2.373	2.295	2.248	2.045	1.808	1.866	2.361	2.151	2.303
50	(30, 35)	(9, 10)	2.291	2.651	1.906	1.844	1.776	2.190	2.173	2.146	1.906	1.804	1.776	2.190	2.143	2.246
50	(35, 45)	(9, 10)	2.203	2.636	1.833	1.793	1.735	2.106	2.096	2.080	1.783	1.702	1.705	2.093	2.071	2.135
								$h(x)$								
30	(15, 20)	(3, 5)	3.605	3.870	2.746	2.698	2.628	3.379	3.364	3.342	2.686	2.592	2.462	3.361	3.330	3.281
30	(15, 20)	(3, 8)	3.220	3.392	2.456	2.415	2.357	3.018	3.006	2.988	2.406	2.329	2.225	3.003	2.978	2.937
30	(15, 20)	(7, 10)	3.195	3.259	2.452	2.414	2.269	3.002	2.912	2.873	2.315	2.114	2.045	2.988	2.762	2.726
30	(18, 20)	(7, 10)	3.122	3.253	2.414	2.375	2.219	2.935	2.845	2.807	2.265	1.953	1.891	2.921	2.693	2.659
30	(18, 27)	(7, 10)	3.029	3.158	2.316	2.277	2.222	2.841	2.751	2.712	2.168	1.860	1.796	2.827	2.600	2.565
50	(25, 35)	(5, 8)	1.611	1.720	1.378	1.368	1.355	1.551	1.531	1.544	1.366	1.339	1.338	1.547	1.509	1.506
50	(25, 35)	(7, 10)	1.606	1.702	1.371	1.362	1.350	1.545	1.527	1.525	1.360	1.337	1.337	1.542	1.509	1.502
50	(30, 35)	(7, 10)	1.603	1.638	1.358	1.349	1.336	1.532	1.523	1.519	1.327	1.311	1.310	1.539	1.502	1.498
50	(30, 35)	(9, 10)	1.529	1.577	1.310	1.300	1.286	1.473	1.470	1.466	1.298	1.281	1.262	1.469	1.464	1.456
50	(35, 45)	(9, 10)	1.402	1.526	1.173	1.164	1.152	1.342	1.340	1.336	1.162	1.150	1.128	1.339	1.336	1.330

Table 3: CP and AL of the interval estimators for the parameter $\alpha = 0.8$ for different censoring schemes

			95%						99%					
			ACI						ACI					
n	(k, r)	(T_1, T_2)	ML		MPS		CrI	HPD	ML		MPS		CrI	HPD
			NA	NAL	NA	NAL			NA	NAL	NA	NAL		
30	(15, 20)	(3, 5)	0.910	0.962	0.959	0.949	0.970	0.955	0.954	0.993	0.982	0.985	0.993	0.992
			1.132	1.237	1.270	1.382	1.058	1.016	1.487	1.731	1.670	1.928	1.412	1.361
30	(15, 20)	(5, 8)	0.919	0.952	0.959	0.948	0.965	0.957	0.954	0.994	0.978	0.990	0.995	0.989
			1.110	1.209	1.238	1.341	1.038	0.999	1.459	1.688	1.627	1.866	1.386	1.337
30	(15, 20)	(7, 10)	0.891	0.955	0.942	0.939	0.961	0.954	0.942	0.993	0.980	0.991	0.996	0.987
			1.110	1.208	1.234	1.337	1.036	0.996	1.459	1.686	1.622	1.859	1.381	1.334
30	(18, 20)	(7, 10)	0.901	0.955	0.954	0.950	0.972	0.953	0.950	0.993	0.986	0.991	0.994	0.993
			1.102	1.200	1.226	1.329	1.033	0.993	1.448	1.675	1.612	1.848	1.379	1.330
30	(20, 27)	(7, 10)	0.885	0.934	0.939	0.935	0.950	0.950	0.944	0.985	0.972	0.984	0.989	0.989
			1.105	1.203	1.228	1.331	1.032	0.993	1.452	1.680	1.614	1.851	1.378	1.330
50	(25, 35)	(5, 8)	0.970	0.964	0.977	0.922	0.972	0.980	0.992	0.991	0.995	0.982	0.995	0.997
			0.928	0.975	0.997	1.046	0.883	0.858	1.219	1.327	1.311	1.422	1.168	1.139
50	(25, 35)	(7, 10)	0.959	0.951	0.970	0.928	0.965	0.970	0.983	0.993	0.995	0.984	0.997	0.994
			0.920	0.966	0.988	1.036	0.877	0.853	1.209	1.316	1.298	1.408	1.164	1.134
50	(30, 35)	(7, 10)	0.959	0.964	0.973	0.930	0.971	0.969	0.990	0.990	0.996	0.982	0.994	0.996
			0.918	0.964	0.986	1.034	0.874	0.850	1.206	1.313	1.296	1.405	1.155	1.127
50	(30, 35)	(9, 10)	0.971	0.972	0.983	0.948	0.982	0.983	0.994	0.995	0.996	0.986	0.998	1.000
			0.925	0.971	0.992	1.040	0.879	0.855	1.215	1.322	1.304	1.413	1.162	1.134
50	(35, 45)	(9, 10)	0.962	0.962	0.973	0.932	0.973	0.972	0.989	0.991	0.996	0.982	0.995	0.996
			0.919	0.965	0.986	1.034	0.876	0.852	1.208	1.314	1.296	1.405	1.158	1.130

Table 4: CP and AL of the interval estimators for the parameter $\beta = 2$ for different censoring schemes

			95%						99%					
			ACI						ACI					
n	(k, r)	(T_1, T_2)	ML		MPS		CrI	HPD	ML		MPS		CrI	HPD
			NA	NAL	NA	NAL			NA	NAL	NA	NAL		
30	(15, 20)	(3, 5)	0.954	0.910	0.938	0.975	0.967	0.959	0.992	0.962	0.990	0.992	0.993	0.992
			2.506	2.668	2.508	2.811	2.316	2.299	3.294	3.668	3.296	4.032	3.014	2.993
30	(15, 20)	(5, 8)	0.955	0.918	0.943	0.980	0.965	0.960	0.991	0.972	0.986	0.994	0.991	0.989
			2.434	2.588	2.431	2.714	2.258	2.241	3.198	3.556	3.195	3.879	2.937	2.914
30	(15, 20)	(7, 10)	0.943	0.899	0.927	0.969	0.956	0.948	0.986	0.954	0.988	0.987	0.992	0.987
			2.438	2.592	2.434	6.275	2.264	2.247	3.204	3.560	3.199	4.909	2.945	2.926
30	(18, 20)	(7, 10)	0.935	0.908	0.949	0.964	0.953	0.954	0.994	0.950	0.989	0.994	0.995	0.989
			2.443	2.597	2.440	2.709	2.270	2.254	3.211	3.568	3.207	3.785	2.954	2.936
30	(20, 27)	(7, 10)	0.930	0.898	0.924	0.962	0.950	0.947	0.991	0.943	0.989	0.988	0.991	0.989
			2.452	2.608	2.446	2.735	2.271	2.256	3.222	3.584	3.215	3.807	2.954	2.936
50	(25, 35)	(5, 8)	0.951	0.940	0.916	0.984	0.952	0.950	0.990	0.978	0.983	0.993	0.991	0.989
			1.888	1.962	1.881	1.986	1.805	1.792	2.481	2.652	2.473	2.715	2.356	2.341
50	(25, 35)	(7, 10)	0.939	0.922	0.928	0.974	0.946	0.938	0.990	0.968	0.984	0.991	0.989	0.988
			1.889	1.961	1.882	1.984	1.807	1.794	2.482	2.648	2.474	2.701	2.359	2.347
50	(30, 35)	(7, 10)	0.952	0.933	0.923	0.981	0.961	0.957	0.992	0.976	0.989	0.994	0.991	0.990
			1.890	1.961	1.883	1.982	1.805	1.794	2.483	2.648	2.474	2.703	2.359	2.344
50	(30, 35)	(9, 10)	0.960	0.938	0.934	0.983	0.962	0.959	0.995	0.980	0.990	0.994	0.994	0.994
			1.887	1.959	1.879	1.977	1.802	1.790	2.479	2.644	2.469	2.695	2.358	2.343
50	(35, 45)	(9, 10)	0.949	0.930	0.926	0.980	0.956	0.955	0.990	0.975	0.984	0.992	0.991	0.988
			1.885	1.957	1.878	1.978	1.803	1.790	2.478	2.642	2.469	2.697	2.356	2.341

Table 5: CP and AL of the interval estimators for the reliability function $r(x)$ at $x = 3$ for different censoring schemes

			95%						99%					
			ACI						ACI					
n	(k, r)	(T_1, T_2)	ML		MPS		CrI	HPD	ML		MPS		CrI	HPD
			NA	NAL	NA	NAL			NA	NAL	NA	NAL		
30	(15, 20)	(3, 5)	0.930	0.961	0.957	0.953	0.980	0.966	0.979	0.989	0.988	0.986	0.994	0.992
			0.276	0.286	0.281	0.290	0.271	0.269	0.363	0.385	0.369	0.390	0.352	0.348
30	(15, 20)	(5, 8)	0.933	0.955	0.957	0.945	0.973	0.966	0.978	0.991	0.986	0.989	0.992	0.990
			0.271	0.280	0.275	0.284	0.267	0.264	0.356	0.378	0.362	0.382	0.346	0.342
30	(15, 20)	(7, 10)	0.925	0.950	0.943	0.947	0.970	0.956	0.973	0.991	0.988	0.987	0.996	0.995
			0.270	0.279	0.274	0.283	0.267	0.264	0.355	0.376	0.360	0.380	0.345	0.341
30	(18, 20)	(7, 10)	0.932	0.960	0.955	0.943	0.982	0.972	0.984	0.986	0.987	0.982	0.996	0.992
			0.270	0.279	0.274	0.283	0.266	0.262	0.355	0.376	0.360	0.380	0.343	0.340
30	(20, 27)	(7, 10)	0.909	0.943	0.933	0.937	0.970	0.950	0.967	0.984	0.980	0.978	0.995	0.993
			0.270	0.279	0.274	0.282	0.266	0.263	0.355	0.376	0.360	0.380	0.344	0.340
50	(25, 35)	(5, 8)	0.974	0.955	0.963	0.931	0.989	0.988	0.996	0.986	0.993	0.983	1.000	1.000
			0.218	0.222	0.219	0.223	0.207	0.206	0.286	0.295	0.288	0.297	0.269	0.267
50	(25, 35)	(7, 10)	0.969	0.948	0.961	0.934	0.990	0.991	0.994	0.993	0.996	0.983	1.000	1.000
			0.217	0.221	0.218	0.222	0.207	0.205	0.285	0.294	0.287	0.296	0.268	0.266
50	(30, 35)	(7, 10)	0.962	0.955	0.963	0.933	0.983	0.985	0.994	0.983	0.992	0.974	0.999	0.999
			0.217	0.221	0.218	0.222	0.206	0.204	0.285	0.294	0.287	0.296	0.267	0.265
50	(30, 35)	(9, 10)	0.977	0.968	0.973	0.943	0.993	0.990	0.997	0.989	0.996	0.986	0.999	0.999
			0.217	0.221	0.219	0.223	0.206	0.204	0.286	0.295	0.288	0.296	0.266	0.264
50	(35, 45)	(9, 10)	0.971	0.955	0.961	0.932	0.986	0.986	0.995	0.986	0.993	0.980	0.998	0.999
			0.217	0.221	0.218	0.222	0.206	0.204	0.285	0.294	0.287	0.295	0.267	0.265

Table 6: CP and AL of the interval estimators for the hazard rate function $h(x)$ at $x = 3$ for different censoring schemes

			95%						99%					
			ACI						ACI					
n	(k, r)	(T_1, T_2)	ML		MPS		CrI	HPD	ML		MPS		CrI	HPD
			NA	NAL	NA	NAL			NA	NAL	NA	NAL		
30	(15, 20)	(3, 5)	0.960	0.955	0.924	0.782	0.948	0.947	0.994	0.993	0.984	0.910	0.989	0.989
			0.166	0.227	0.160	0.162	0.153	0.151	0.218	0.302	0.211	0.214	0.199	0.198
30	(15, 20)	(5, 8)	0.952	0.957	0.921	0.778	0.947	0.947	0.990	0.992	0.976	0.920	0.986	0.986
			0.161	0.222	0.155	0.157	0.149	0.147	0.212	0.296	0.204	0.207	0.194	0.193
30	(15, 20)	(7, 10)	0.948	0.951	0.909	0.784	0.948	0.947	0.996	0.997	0.972	0.908	0.986	0.985
			0.159	0.221	0.154	0.155	0.147	0.146	0.210	0.295	0.202	0.205	0.193	0.191
30	(18, 20)	(7, 10)	0.968	0.964	0.937	0.779	0.962	0.957	0.994	0.994	0.984	0.923	0.992	0.992
			0.160	0.221	0.154	0.155	0.148	0.147	0.210	0.294	0.203	0.205	0.194	0.192
30	(20, 27)	(7, 10)	0.954	0.957	0.926	0.764	0.947	0.951	0.993	0.992	0.982	0.916	0.988	0.985
			0.160	0.221	0.154	0.155	0.148	0.147	0.210	0.294	0.203	0.205	0.194	0.192
50	(25, 35)	(5, 8)	0.959	0.956	0.941	0.790	0.954	0.959	0.993	0.989	0.987	0.903	0.988	0.987
			0.161	0.221	0.155	0.156	0.149	0.148	0.211	0.295	0.204	0.207	0.195	0.193
50	(25, 35)	(7, 10)	0.962	0.961	0.925	0.801	0.960	0.958	0.993	0.994	0.979	0.927	0.988	0.987
			0.159	0.221	0.154	0.155	0.147	0.146	0.209	0.295	0.202	0.205	0.193	0.191
50	(30, 35)	(7, 10)	0.976	0.964	0.940	0.803	0.969	0.968	0.996	0.994	0.988	0.925	0.992	0.991
			0.160	0.221	0.154	0.155	0.148	0.147	0.210	0.294	0.203	0.205	0.194	0.192
50	(30, 35)	(9, 10)	0.965	0.965	0.924	0.783	0.954	0.953	0.993	0.988	0.987	0.924	0.987	0.985
			0.160	0.220	0.154	0.155	0.148	0.147	0.210	0.294	0.203	0.205	0.194	0.192
50	(35, 45)	(9, 10)	0.970	0.962	0.930	0.791	0.961	0.955	0.996	0.993	0.990	0.921	0.994	0.992
			0.159	0.221	0.154	0.155	0.148	0.147	0.209	0.294	0.202	0.205	0.193	0.192

Table 7: Point and interval estimates for the real data with two different censoring scheme (the values for the $r(x)$ and $h(x)$ are presented by multiplying with 10^3)

Point Estimates

	UHCS	ML	MPS	BLEL $\omega=0.2$ ν_1	BLEL $\omega=0.2$ ν_2	BLEL $\omega=0.2$ ν_3	BLEL $\omega=0.8$ ν_1	BLEL $\omega=0.8$ ν_2	BLEL $\omega=0.8$ ν_3	BGEL $\omega=0.2$ μ_1	BGEL $\omega=0.2$ μ_2	BGEL $\omega=0.2$ μ_3	BGEL $\omega=0.8$ μ_1	BGEL $\omega=0.8$ μ_2	BGEL $\omega=0.8$ μ_3
α	CS_1	2.557	3.575	2.028	1.820	1.537	2.449	2.383	2.190	1.769	1.645	1.438	2.360	2.298	2.140
	CS_2	1.497	2.068	1.400	1.296	1.166	1.474	1.448	1.404	1.272	1.196	1.067	1.441	1.417	1.360
β	CS_1	99.298	86.871	154.026	132.539	79.644	152.640	125.627	81.030	111.299	110.702	109.832	102.299	102.057	101.738
	CS_2	114.590	102.836	159.664	140.121	85.597	158.278	133.280	86.983	120.545	120.066	119.357	116.079	115.940	115.746
$r(x)$	CS_1	974.382	956.128	972.906	972.744	972.493	974.013	973.972	973.909	972.703	972.528	972.258	973.962	973.918	973.850
	CS_2	981.845	967.254	978.285	978.141	977.916	980.956	980.919	980.861	978.104	977.947	977.704	980.909	980.869	980.806
$h(x)$	CS_1	24.085	31.566	25.903	25.838	25.741	24.540	24.524	24.499	25.821	23.310	18.860	24.519	23.890	22.525
	CS_2	21.305	29.932	23.420	23.353	23.253	21.834	21.817	21.791	23.336	20.630	16.017	21.813	21.134	19.680

Interval Estimates

			ACI ML NA	ACI ML NAL	ACI MPS NA	ACI MPS NAL	CrI	HPD
α	CS_1	95%	(-0.518, 5.632)	(0.768, 8.511)	(-0.498, 7.647)	(1.144,1 1.169)	(0.564, 3.117)	(0.515, 2.971)
		99%	(-1.484, 6.598)	(0.527, 12.419)	(-1.778, 8.927)	(0.800, 15.977)	(0.339, 3.698)	(0.258, 3.354)
	CS_2	95%	(-0.285, 3.279)	(0.455, 4.922)	(-0.284, 4.420)	(0.663, 6.449)	(0.434, 2.442)	(0.277, 2.172)
		99%	(-0.845, 3.839)	(0.313, 7.154)	(-1.023, 5.159)	(0.464, 9.220)	(0.303, 3.025)	(0.277, 2.822)
β	CS_1	95%	(63.993, 134.604)	(69.587, 141.695)	(52.578, 121.165)	(58.538, 128.920)	(90.650, 141.178)	(90.942, 141.378)
		99%	(52.899, 145.697)	(62.231, 158.443)	(41.802, 131.941)	(51.709, 145.946)	(82.395, 152.750)	(80.212, 149.566)
	CS_2	95%	(80.935, 148.244)	(85.427, 153.708)	(69.987, 135.684)	(74.717, 141.536)	(96.976, 150.120)	(95.674, 147.435)
		99%	(70.360, 158.820)	(77.896, 168.568)	(59.666, 146.006)	(67.582, 156.480)	(88.619, 160.133)	(88.334, 159.320)
$r(x)$	CS_1	95%	(932.498, 1016.265)	(933.385, 1017.178)	(895.830, 1016.426)	(897.692, 1018.368)	(909.593, 996.948)	(925.117, 999.533)
		99%	(919.337, 1029.426)	(920.863, 1031.010)	(876.883, 1035.373)	(880.078, 1038.750)	(878.428, 998.738)	(892.309, 999.533)
	CS_2	95%	(950.086, 1013.604)	(950.594, 1014.123)	(918.614, 1015.894)	(919.817, 1017.138)	(920.575, 998.017)	(933.742, 999.440)
		99%	(940.106, 1023.583)	(940.981, 1024.483)	(903.330, 1031.178)	(905.397, 1033.338)	(872.690, 999.206)	(893.035, 999.701)
$h(x)$	CS_1	95%	(-1.385, 49.555)	(8.366, 69.345)	(5.244, 57.888)	(13.711, 72.671)	(5.792, 59.946)	(3.499, 54.163)
		99%	(-9.388, 57.559)	(6.000, 96.677)	(-3.026, 66.159)	(10.551, 94.440)	(2.744, 69.778)	(1.800, 66.663)
	CS_2	95%	(-4.216, 46.825)	(6.430, 70.584)	(1.327, 58.538)	(11.511, 77.837)	(4.147, 60.583)	(1.595, 52.085)
		99%	(-12.235, 54.844)	(4.413, 102.843)	(-7.662, 67.526)	(8.525, 105.099)	(2.340, 72.780)	(0.865, 69.422)

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