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DEPENDENCE STRESS-STRENGTH RELIABILITY ESTIMATION OF BIVARIATE XGAMMA EXPONENTIAL DISTRIBUTION UNDER COPULA APPROACH

A. James and N. Chandra

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Abstract In general, the characteristics determination of stress-strength reliability based on statistical independence are well attempted in the literature. In contrast, assuming independence assumptions may not stands in reality because most of the occasions the stress and strength random variables are somehow linked or inter correlated. We therefore propose the copula based dependence stress-strength reliability model. We assume stress (Y) follows the exponential distribution and strength (X) follows an Xgamma distribution. A Farlie-Gumbel-Morgenstern Bivariate Xgamma Exponential distribution is proposed, and its several statistical properties, including reliability characteristics, are derived. We estimated stress-strength reliability (R) and dependence parameter using maximum likelihood estimation, inference function margin, and semi-parametric methods. Moreover, the confidence interval and coverage probability of the dependence parameter are also reported. In addition, a Monte Carlo simulation study is conducted to evaluate the effectiveness of the various estimators, and finally, two real data sets is presented to illustrate the results.

1 Introduction

The gamma and exponential are the classical lifetime distributions frequently used to model lifetime data and a number of new life distributions are induced by combining these two distributions. Sen et al. [2] proposed the Xgamma (XG) distribution, as a finite mixture of exponential and gamma distributions. Several interesting features about the distribution make it useful for analyzing time-to-event data sets. Estimation of the XG distribution using type-II progressive censoring has been carried out by Sen et al. [3]. Later, several versions of XG distribution has developed including stress-strength reliability modelling with quasi XG (Sen et al. [4]), weighted XG (Sen et al. [5]) and inverse XG (Yadav et al. [1]). The probability density function (p.d.f) and cumulative distribution function (c.d.f) of the XG distribution are as follows:

$$f_X(x;\theta) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x},\ x>0,\ \theta>0, \tag{1.1}$$

and

$$F_X(x;\theta) = 1-\frac{(1+\theta+\theta x+\frac{\theta^2 x^2}{2})}{(1+\theta)}e^{-\theta x},\ x>0,\ \theta>0. \tag{1.2}$$

The above p.d.f can be expressed as

$$f_X(x;\theta) = \phi_1 f_1(x) + \phi_2 f_2(x), \tag{1.3}$$

where $f_1(x)$= exp(θ) and $f_2(x)$= gamma$(3,\theta)$ with mixing proportions $\phi_1 = \frac{\theta}{1+\theta}$ and $\phi_2 = (1-p) = \frac{1}{1+\theta}$ respectively.

Stress-strength models are commonly used in reliability analysis to describe the useful life of a component having random stress (Y) and strength (X). It is assumed that the system will perform as long as the stress is less than the strength. Therefore, $R = P(Y < X)$ is a probability that determines the component's reliability. Stress-strength models are used in many fields of research, particularly in engineering, for example, to model the deterioration of concrete pressure vessels, the degeneration of rocket motors, static fatigue of ceramic components, and fatigue failure of aircraft structures (Nadarajah [6]).

A considerable amount of works on stress-strength reliability based on independent assumptions is available in the literature. For example, Sen et al. [7] derived the expression for the stress-strength reliability when X and Y follow two-parameter XG distributions. Krishnamoorthy et al. [8] investigated the expression of R for two-parameter exponential marginals. Jana et al. [9] examined the properties of R for exponential distributions with different values for location and scale parameters. Yadav et al. [1] proposed a new IXG distribution and its several reliability measures including stress-strength reliability function are derived. An extensive review on stress-strength reliability is given in the monograph of Kotz et al. [10].

However, in many cases the stress-strength variables are associated in some way. Some real life situations, where the dependence between stress and strength are taken in to account are given below:

(i) In electronic systems, a coherent system is configured with components in a series setup and probability that it will perform its intended function (strength) subjected to bearing the accumulated stresses of each components on it. If stress is higher than the strength of any one of its components lead to failure of a system. Hence assessing dependence relation between them is an essential task for estimating the desired life.

(ii) In human psychological mental health, it is essential to maintain mental health of an individual for surviving a normal life. There are many stress factors that are adversely affect the mental health (strength) of individual, including anxiety, poor social support, family problems, financial issues etc. and these stress factors are associated up to some extent. If these stresses leave you with little mental impact, then R measures the probability that an individual will survives naturally and living the normal life without affecting mental health issue. Otherwise, they may lead to competing health issues like., social isolation, depression, loneliness, sleeping disorders, and health issues like heart disease, diabetes and cancer.

(iii) In the case of a pharmaceutical manufacturing company that launches a new drug, the random variable Y represents the remission time of a patient A treated with the new drug while X represents the remission time of A treated with an existing drug. Then $R = P(Y < X)$ measures the probability that the new drug is more effective than the existing one.

In the stress-strength dependence assumption, a reasonable amount of work attempted by several authors for estimation of R by assuming stresses are dependent on strength or vice versa, where stresses or strengths follow bivariate continuous life distributions, like, bivariate Marshall-Olkin exponential by Chandra and Pandey [11], bivariate gamma distribution by Nadarajah [12] and references therein. These attempts were mainly focused on estimation of R rather than exploring the level (low or high) of dependence relation between either stresses or strengths. However, ignoring such dependence relation may leads over or under estimation of reliability function. Hence, some attempts in this direction is sought to establish the dependence stress-strength model for life predition.

In this paper, we consider copulas technique to model the dependence relationship between two or more dimensional random variables. Farlie–Gumbel–Morgenstern (FGM) copulas are among the most widely used family of copulas because of their simplicity. Domma and Giordano [13] considered FGM and generalized FGM copula to estimate R with Burr system of margins. Domma and Giordano [14] developed the stress-strength model to calculate household financial fragility when X and Y follow the Dangum marginals. Barbiero [15] studied the stress-strength reliability estimation using the extended FGM and Ali-Mikhail-Haq copula with exponential distribution. The stress-strength model is investigated by Patil and Naik-Nimbalkar [16] with stress and strength marginal's are belonging to the Pareto family and the dependence is represented using four different types of copulas and the asymptotic properties of R as well as the dependence parameter are also derived. Recently, Bai et al. [17] have attempted both independent and dependent stress-strength reliability model for multi-state system.

In literature, a large amount of work on stress-strength reliability assumes statistical independence between two or more random variables, follow a family of life distributions. Besides, considering different distributional assumption of stress and strength is one of the another scope of this attempt. In reality, however, it is more practical to choose marginal distributions of X and Y from two different parametric families of distributions. In this study, estimation of dependent stress-strength reliability parameters are considered by assuming that stress follows exponential and strength follows XG marginals using copula approach. Also, we proposed Farlie-Gumbel-Morgenstern Bivariate Xgamma Exponential (FGMBXE) distribution and its several statistical properties, including reliability characteristics are derived. Further, the length of the asymptotic confidence interval as well as the coverage probability of the dependence parameter are computed. A numerical study, is performed to evaluate the effectiveness of the various methods proposed by using simulation and real data sets.

The remaining sections of the paper are organised as follows. In Section 2, we proposed FGMBXE distribution. In Section 3, we derived some statistical properties of FGMBXE distribution. The dependence stress-strength reliability and associated properties are derived in Section 4. In Section 5, we derived estimates of R and dependence parameter α by using maximum likelihood estimation (MLE), Inference Function Margins (IFM), and semi-parametric (SP) methods. Asymptotic confidence intervals are presented in Section 6. A Monte Carlo simulation study is performed in Section 7. Two real data sets is analysed in Section 8. Finally, the study is concluded in Section 9.

2 FGM Bivariate Xgamma Exponential Distribution

The c.d.f and p.d.f of Morgenstern family of bivariate distribution is given by

$$F_{(XY)}(x,y) = F_X(x)G_Y(y)\big[1+\alpha(1-F_X(x))(1-G_Y(y))\big],\ -1 \le \alpha \le 1, \tag{2.1}$$

and

$$f_{(XY)}(x,y) = f_X(x)g_Y(y)\big[1+\alpha(1-2F_X(x))(1-2G_Y(y))\big],\ -1 \le \alpha \le 1, \tag{2.2}$$

where F_X and G_Y denotes the marginal c.d.f's and f_X and g_Y are the marginal p.d.f's of X and Y respectively with dependence parameter α. When $\alpha = 0$, then X and Y will act as an independent case.

Since the Kendall's tau coefficient of Morgenstren family is given by

$$\tau = \frac{2\alpha}{9},\ -0.222 \ \le\ \tau\ \le\ 0.222,$$

which indicates that FGM copula describes a weak dependence between X and Y.

Suppose that X follows XG distribution and Y follows exponential distribution and the corresponding p.d.f's are given as

$$f_X(x;\theta) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x},\ x>0,\ \theta>0, \tag{2.3}$$

$$g_Y(y;\lambda) = \lambda e^{-\lambda y},\ y>0,\ \lambda>0, \tag{2.4}$$

and the corresponding c.d.f's are given by

$$F_X(x;\theta) = 1-\frac{(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)}e^{-\theta x},\ x>0,\ \theta>0, \tag{2.5}$$

$$G_Y(y;\lambda) = 1 - e^{-\lambda y},\ x > 0,\ \lambda > 0. \tag{2.6}$$

Then the joint c.d.f and p.d.f of FGMBXE distribution are obtained as

$$F_{(XY)}(x,y) = (1 - e^{-\lambda y})\Big(1 - \frac{(1+\theta+\theta x+\frac{\theta^2 x^2}{2})}{1+\theta}e^{-\theta x}\Big)$$

$$\times \Big(1 + \alpha\frac{e^{-\lambda y}(1+\theta+\theta x+\frac{\theta^2 x^2}{2})e^{-\theta x}}{1+\theta}\Big),\ -1 \leq \alpha \leq 1,\ x, y, \lambda, \theta > 0, \tag{2.7}$$

$$f_{(XY)}(x,y) = \frac{\lambda e^{-\lambda y}\theta^2(1+\frac{\theta^2 x^2}{2})e^{-\theta x}}{(1+\theta)}\Big(1 + \alpha(2e^{-\lambda y} - 1)(2\frac{(1+\theta+\theta x+\frac{\theta^2 x^2}{2})}{1+\theta}e^{-\theta x} - 1)\Big),$$
$$-1 \leq \alpha \leq 1,\ x, y, \lambda, \theta > 0. \tag{2.8}$$

A plot of p.d.f and c.d.f of FGMBXE distribution for different choices of parameter values are given in the following Figure 1 and 2 respectively.

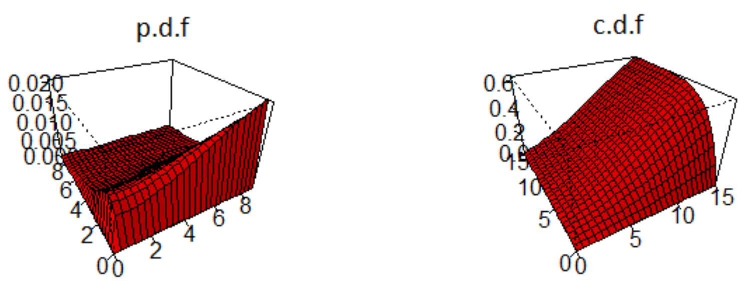

Figure 1. plot of FGMBXE distribution for θ= 0.2, λ= 0.6 and α= 0.9

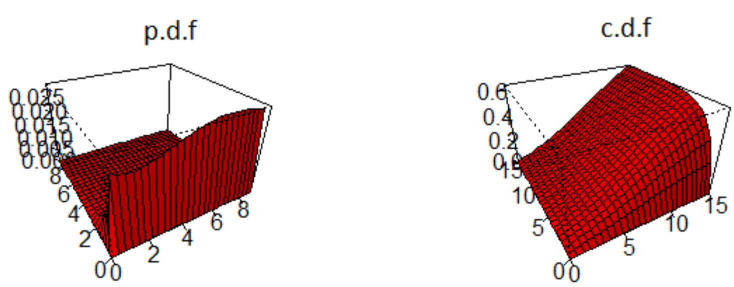

Figure 2. plot of FGMBXE distribution for θ= 0.2, λ= 0.6 and α= -0.9

3 Some statistical properties of FGM Bivariate Xgamma Exponential distribution

In this section, we obtain some important statistical properties of FGMBXE distribution, such as the conditional distribution, moment generating function and positive quadrant dependence.

3.1 Conditional Distribution

The conditional c.d.f of X given Y= y of FGMBXE distribution is given by

$$F_{X|Y}(x|y) = \big(1-\alpha(2e^{-\lambda y}-1)\big)\big(1-\frac{(1+\theta+\theta x+\frac{\theta^2x^2)}{2}}{(1+\theta)}e^{-\theta x}\big)$$
$$-\alpha(2e^{-\lambda y}-1)\big(1-\frac{(1+\theta+\theta x+\frac{\theta^2x^2}{2})^2}{(1+\theta)^2}e^{-2\theta x}\big), \tag{3.1}$$

and the corresponding conditional p.d.f is given as

$$f_{X|Y}(x|y) = \frac{\theta^2(1+\frac{\theta}{2}x^2)e^{-\theta x}}{(1+\theta)}\big(1+\alpha\big(\frac{2(1+\theta+\theta x+\frac{\theta^2x^2)}{2}}{(1+\theta)}e^{-\theta x}-1)(2e^{-\lambda y}-1)\big)\big). \tag{3.2}$$

Then the conditional expectation of X given $Y=y$ is obtained as

$$E[X|Y=y] = \frac{\theta^2}{(1+\theta)}\left(\frac{3+\theta}{\theta^3}+\alpha\left(2e^{-\lambda y}\right)\right.$$
$$\left.\times\left(\frac{2}{(1+\theta)}\left(\frac{8\theta^2(2+\theta)+2(2+9\theta+13\theta^2+15\theta)}{34\theta^4}\right)-\frac{3+\theta}{\theta^3}\right)\right). \tag{3.3}$$

Similarly, we can derive the expressions of $F_{Y|X}(y|x)$, $f_{Y|X}(y|x)$ and E[Y|X=x].

3.2 Moment Generating Function

Let (X,Y) be a two-dimensional random variable with joint p.d.f $f_{(XY)}(x,y)$, then the moment generating function (m.g.f) of (X,Y) is defined as

$$M_{(XY)}(t_1,t_2) = E(e^{t_1x}e^{t_2y}) \tag{3.4}$$

$$= \int_0^\infty\int_0^\infty e^{t_1x}e^{t_2y}f_{(XY)}(x,y)dydx, \tag{3.5}$$

where (t_1,t_2) are real parameters.

The m.g.f of FGMBXE distribution is obtained as

$$M_{XY}(t_1,t_2) = \frac{\theta^2(t_1^2-2\theta t_1+\theta^2+\theta)\lambda}{(\theta+1)(t_1-\theta)^3(\lambda-t_2)}+\alpha\left(\frac{\lambda\theta^2t_2}{(1+\theta)(t_2-\lambda)(2\lambda-t_2)}\right.$$

$$\left.\times\left(\frac{A(\theta,t_1)}{2(1+\theta)}-\frac{(t_1^2-2\theta t_1+\theta^2+\theta)}{(t_1-\theta)^3}\right)\right), \tag{3.6}$$

where

$$A(\theta,t_1) = \frac{4\theta(t_1-2\theta)^3-4(\theta+1)(t_1-2\theta)^4 4\theta(2\theta+1)(t_1-2\theta)^2+12\theta^2(t_1-2\theta)-24\theta^3}{(t_1-2\theta)^5}.$$

3.3 Positive Quadrant Dependence (PQD)

Lehmann[18] defined the Positive quadrant dependence property as

$$PQD(X,Y) = P(X>x,Y>y) \geq P(X>x)P(Y>y),\ \forall\, x,y>0. \tag{3.7}$$

A reverse inequality of (3.7) defines negative quadrant dependence (NQD). Then the following theorem gives the inequality condition of PQD (NQD) of FGMBXE distribution based dependence parameter.

Theorem 1 FGMBXE distribution is PQD (NQD) for positive (negative) value of α.

Proof: Consider

$$\begin{aligned} S(x,y)-S(x)S(y) &= P(X>x,Y>y)-P(X>x)P(Y>y) \\ &= \Big(\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})e^{-\lambda y}}{(1+\theta)}\Big)\alpha\big(1-e^{-\lambda y}\big)\Big(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)}\Big) \\ &= \alpha\phi(x,y), \end{aligned}$$

where

$$\phi(x,y)=\Big(\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})e^{-\lambda y}}{(1+\theta)}\Big)(1-e^{-\lambda y})\Big(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)}\Big)$$

$$= S(x)S(y)F(x)G(y)\ \geq\ 0;\ \ S(.),F(.)\ \geq 0,\forall\, x,y\ \geq\ 0,$$

and thus the following inequalities of $\alpha\phi(x,y)$ is given by

$$\alpha\phi(x,y)\begin{cases} \geq\ 0;\ \alpha>0,\forall\, x,y\geq 0, \\ \leq\ 0;\ \alpha<0,\forall\, x,y\geq 0, \end{cases} \tag{3.8}$$

which implies the condition given in (3.7). Therefore, FGMBXE distribution has the properties PQD (NQD) for positive (negative) values of α.

4 Reliability Measures

In this section, we derived some important reliability characteristics of FGMBXE distribution, which includes dependence stress-strength reliability, survival function, hazard rate function, mean residual life, vitality function, totally positive of order 2 or reverse rule of order $2(TP_2$ or $RR_2)$, right-tail increasing and left-tail decreasing and mean time to failure.

4.1 Reliability for dependence stress and strength

In this section, we assume that the two dimensional random variable (X,Y) follow FGMBXE distribution with dependence parameter α, then the corresponding R is derived as

$$R=P(Y<X)=\int_0^\infty\int_0^x f_{(XY)}(x,y)dydx$$

$$=1-\frac{\theta^2((\lambda+\theta)^2+\theta)}{(1+\theta)(\lambda+\theta)^3}+\alpha\bigg(\frac{2\theta^2}{(1+\theta)^2(2\theta+\lambda)^5}((2\theta+\lambda)^4(1+\theta)+(2\theta+\lambda)^3\theta$$

$$+(2\theta+\lambda)^2(\theta+2\theta^2)+3\theta^2(2\theta+\lambda)+6\theta^3)-\frac{\theta^2}{8(1+\theta)^2(\theta+\lambda)^5}(8(\theta+\lambda)^4$$

$$(1+\theta)+4\theta(\theta+\lambda)^3+2(\theta+3\theta^2)(\theta+\lambda)^2+3\theta^2(\theta+\lambda)+6\theta^3)$$

$$-\frac{\theta^2((\lambda+\theta)^2+\theta)}{(1+\theta)(\lambda+\theta)^3}+\frac{\theta^2((2\lambda+\theta)^2+\theta)}{(1+\theta)(2\lambda+\theta)^3}\bigg), \tag{4.1}$$

when the dependence parameter α = 0, then the stress-strength reliability given in (4.1) will reduce to independence case of X and Y. Additionally, estimate of R can be obtained by replacing the estimates of parameters in (4.1) using invariance property.

4.2 Survival Function

Survival function of Morgenstren family is of the form

$$S(x,y) = (1 - F_X(x))(1 - G_Y(y))(1 + \theta F_X(x)G_Y(y)). \tag{4.2}$$

Hence by using (2.5), (2.6) and (4.2), the survival function of FGMBXE distribution is obtained as

$$\begin{aligned} S(x,y) =& \frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})e^{-\lambda y}}{(1+\theta)} \\ & \times \left(1+\alpha(1-e^{-\lambda y})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right). \end{aligned} \tag{4.3}$$

4.3 Hazard Rate Function

Basu [23] proposed the bivarite hazard rate function of the form

$$h(x,y) = \frac{f(x,y)}{S(x,y)}. \tag{4.4}$$

Using (2.8) and (4.4) hazard rate function of FGMBXE distribution is obtained as

$$h(x,y) = \frac{\theta^2(1+\frac{\theta}{2}x^2)\left(1+\alpha(2e^{-\lambda y}-1)(\frac{2e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)}-1)\right)}{(1+\theta+\theta x+\frac{\theta^2x^2}{2})\left(1+\alpha(1-e^{-\lambda y})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right)}. \tag{4.5}$$

Johnson and Kotz [24] defined a hazard rate function in a vector form, as shown below

$$h_V(x,y) = \left(\frac{-\partial \ln S(x,y)}{\partial x}, \frac{-\partial \ln S(x,y)}{\partial y}\right), \tag{4.6}$$

where $S(x,y)$ denote the bivariate survival function. From (4.3) we get hazard components as

$$h_1(x,y) = \theta - \frac{\theta^2x+\theta}{(1+\theta+\theta x+\frac{\theta^2x^2}{2})} - \frac{\frac{\alpha}{(1+\theta)}(1-e^{-\lambda y})\theta^2(1+\frac{\theta}{2}x^2)}{\left(1+\alpha(1-e^{-\lambda y})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right)}, \tag{4.7}$$

$$h_2(x,y) = \lambda - \frac{\alpha\lambda(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})e^{-\lambda y}}{\left(1+\alpha(1-e^{-\lambda y})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right)}. \tag{4.8}$$

The following theorem reveals the hazard characteristcs of FGMBXE distribution.

Theorem 1. The hazard function of FGMBXE distribution is increasing (decreasing) for positive (negative) values of the dependence parameter α.

Proof: To prove FGMBXE distribution is IFR (increasing faliure rate) for positive values of α, it is sufficient to show that (4.7) and (4.8) are increasing functions in x and y respectively.

Consider

$$
\begin{aligned}
-\frac{\partial \ln S(x,y)}{\partial x} &= -\frac{\partial}{\partial x}\ln(\bar{F_X}\ \bar{G_Y}(1+\alpha F_X G_y)), \text{where } \bar{F}=1-F \\
&= -\frac{\partial}{\partial x}\ln \bar{F_X} \ - \ \frac{\partial}{\partial x}\ln \bar{G_Y} \ - \ \frac{\partial}{\partial x}\ln(1+\alpha F_X G_Y) \\
&= h_X(1-(\bar{F_X}^{-1}((\alpha G_Y)^{-1}+1)-1)^{-1}),
\end{aligned}
$$

where h_X is the hazard rate function of X. For $0 \leq \alpha \leq 1$, $\alpha^{-1} \geq 1$ which implies $((\alpha G_Y)^{-1} + 1) > 1$, because $(G_Y)^{-1} \geq 1$. Therefore $(1-(\bar{F_X})^{-1}((\alpha G_Y)^{-1}+1)-1)^{-1})$ is positive increasing function in x because F_x is an increasing function in x. Further, the h_X is an increasing function in x. Hence $-\frac{\partial \ln S(x,y)}{\partial x}$ is an increasing function in x. In a Similar way, we can show that FGMBXE distribution is DFR for negative values of α.

4.4 Mean Residual Life

Bivariate mean residual life (m.r.l) function proposed by Shanbag and Kotz [25] is of the form

$$r(x,y) = (r_1(x,y), r_2(x,y)), \tag{4.9}$$

where

$$r_1(x,y) = E(X-x|X \geq x, Y \geq y), \tag{4.10}$$

and

$$r_2(x,y) = E(Y-y|X \geq x, Y \geq y). \tag{4.11}$$

The expression for $r_1(x,y)$ and $r_2(x,y)$ of FGMBXE distribution is obtained as

$$r_1(x,y) = \frac{\frac{(3+\theta+2\theta x+\frac{\theta^2x^2}{2})}{\theta} - \alpha\left(\frac{(3+\theta+2\theta x+\frac{\theta^2x^2}{2})}{\theta} - \frac{(1-e^{-\lambda y})A(\theta,x)}{16\theta(1+\theta x)}\right)}{(1+\theta+\theta x+\frac{\theta^2x^2}{2})\left(1+\alpha(1-e^{-\lambda y})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right)}, \tag{4.12}$$

$$r_2(x,y) = \frac{\frac{1}{\lambda}\left((1+\alpha(1-\frac{e^{-\lambda y}}{2})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right)}{\left(1+\alpha(1-e^{-\lambda y})\left(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)}\right)\right.}, \tag{4.13}$$

where $A(\theta,x) = (8(1+\theta+\theta x+\frac{\theta^2x^2}{2})+2(2+\theta x)+4((1+\theta+\theta x+\frac{\theta^2x^2}{2})+(1+\theta x)^2+6(1+\theta x)+3)$.

By combining (4.12), (4.13) and (4.9), the expression of m.r.l for FGMBXE distribution can be obtained.

4.5 Vitality Function

Bivariate vitality function proposed by Sankaran and Nair [26] is given by

$$V(x,y) = (V_1(x,y), V_2(x,y)), \tag{4.14}$$

where

$$V_1(x,y) = E(X|X \geq x, Y \geq y), \tag{4.15}$$

$$V_2(x,y) = E(X|X \geq x, Y \geq y). \tag{4.16}$$

Further, the bivariate vitality function $V_i(x,y)$ is related to the mean residual life function $r(x,y)$ with the following relation as

$$V_i(x,y) = x + r_i(x,y),\ i = 1,2. \tag{4.17}$$

$V_1(x,y)$ and $V_2(x,y)$ of FGMBXE distribution is obtained as

$$V_1(x,y) = x + \frac{\frac{(3+\theta+2\theta x+\frac{\theta^2x^2}{2})}{\theta} - \alpha\left(\frac{(3+\theta+2\theta x+\frac{\theta^2x^2}{2})}{\theta} - \frac{(1-e^{-\lambda y})A(\theta,x)}{16\theta(1+\theta x)}\right)}{(1+\theta+\theta x+\frac{\theta^2x^2}{2})\left(1+\alpha(1-e^{-\lambda y})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right)}, \tag{4.18}$$

$$V_2(x,y) = y + \frac{\frac{1}{\lambda}\left(1+\alpha(1-\frac{e^{-\lambda y}}{2})(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right)}{\left(1+\alpha(1-e^{-\lambda y})\left(1-\frac{e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)}\right)\right.}. \tag{4.19}$$

Hence the vitality function of FGMBXE distribution can be obtained by combining (4.18), (4.19) and (4.14).

4.6 Totally Positive of order 2 or Reverse Rule of order $2(TP_2$ or $RR_2)$

Let (X,Y) be a two dimensional continuous random pair with joint p.d.f $f_{(XY)}(x,y)$ is said to be TP_2 or RR_2 if

$$f(x,y)f(u,v) \geq (\leq) f(x,v)f(u,y),\ x<u,\ y<v. \tag{4.20}$$

Then the local dependence function of FGMBXE distribution is defined as

$$r_f(x,y) = \frac{\partial^2}{\partial x \partial y}\ln f_{(XY)}(x,y) = \frac{\frac{4\alpha\lambda e^{-\lambda y}\theta^2(1+\frac{\theta}{2})x^2e^{-\theta x}}{(1+\theta)}}{\left(1+\alpha(2e^{-\lambda y}-1)(\frac{2e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)}-1)\right)}, \tag{4.21}$$

$r_f(x,y) \geq (\leq) 0$ according as $\alpha \geq (\leq) 0$. Thus $f_{(XY)}(x,y)\ is\ TP_2(RR_2),$ if $\alpha \geq (\leq)0$.

4.7 Right Tail Increasing and Left Tail Decreasing

Let (X,Y) be a two-dimensional random vector with c.d.f $F_{(XY)}(x,y)$. Y is right-tail increasing (RTI) in X if

$$RTI(Y|X) = P(Y>y|X>x) = \frac{\overline{F}_{(XY)}(x,y)}{\overline{F}_X(x)} \uparrow x\ \ \forall y, \tag{4.22}$$

where $\overline{F}_{(XY)}(x,y) = 1 - F_{(XY)}(x,y)\ and\ \overline{F}_X(x) = 1 - F(x)$.

For FGMBXE distribution,

$$P(Y>y|X>x) = e^{-\lambda y}\left(1+\alpha(1-e^{-\lambda y})(\frac{1-e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right), \tag{4.23}$$

it is clear that $P(Y>y|X>x) > 1$ for $\alpha > 0$ which implies $RTI(Y|X)$. Similarly, Y is left-tail decreasing(LTD) in X if

$$LTD(Y|X) = P(Y \leq y|X \leq x) = \frac{F_{(XY)}(x,y)}{F_X(x)} \downarrow x\ \ \forall y. \tag{4.24}$$

For FGMBXE distribution,

$$P(Y \leq y|X \leq x) = (1-e^{-\lambda y})\left(1+\alpha(1-e^{-\lambda y})(\frac{1-e^{-\theta x}(1+\theta+\theta x+\frac{\theta^2x^2}{2})}{(1+\theta)})\right), \tag{4.25}$$

it is observed from the expression in (4.25) that $P(Y \leq |X \leq x) < 1$ for $\alpha < 0$ which implies $LTD(Y|X)$.

4.8 Mean Time To Failure

Let (X,Y) be a two-dimensional random variable with joint survival function S(x,y), then the mean time to failure of (X,Y) is defined as

$$MTTF = \int_{\mu_1}^{\infty}\int_{\mu_2}^{\infty} S(x,y)dydx. \tag{4.26}$$

Using (4.26) the MTTF of FGMBXE distribution is obtained as

$$MTTF = \frac{1}{\theta\lambda(1+\theta)}\Big(3+\alpha(\frac{3}{2}-\frac{(2+\theta)(6+4\theta)+3}{16(1+\theta)})\Big). \tag{4.27}$$

5 Parameter Estimation

In this section, we consider three different estimation procedures which includes, MLE, IFM and SP for estimating the dependence stress-strength reliability parameters.

5.1 Maximum Likelihood Estimation

Suppose that a bivariate random sample $(x_i, y_i),\ i = 1, 2, .., n$ of size n drawn from the FGM-BXE distribution. Then the log-likelihood function can be expressed as

$$\begin{aligned}\ell =& n\ln(\lambda) - \lambda\sum_{i=1}^{n} y_i + 2n\ln(\theta) - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln(1+\frac{\theta^2 x_i^2}{2}) + \ln\sum_{i=1}^{n}(1+\alpha(2e^{-\lambda y_i}-1)\\ &(2\frac{(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2})}{1+\theta}e^{-\theta x_i}-1)).\end{aligned} \tag{5.1}$$

The following maximum likelihood equations can be obtained by partially differentiating the log-likelihood function with respect to the unknown parameters and equating it to zero, we have

$$\begin{aligned}\frac{\partial\ln\ell}{\partial\theta} =& \frac{2n}{\theta} - \frac{n}{(1+\theta)} + \sum_{i=1}^{n}\frac{x_i^2}{2(1+\frac{\theta}{2})x_i^2} - \sum_{i=1}^{n} x_i\\ &+\sum_{i=1}^{n}\frac{1}{D(\theta,\lambda,\alpha)}(2e^{-\lambda y_i})\Big((\frac{-2(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2})}{1+\theta})\theta e^{-\theta y_i} + \frac{2e^{-\theta y_i}}{(1+\theta)^2}\\ &((1+\theta(1+y_i+\theta y_i^2)) - (1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2})1+\theta)\Big) = 0,\end{aligned} \tag{5.2}$$

$$\frac{\partial\ln\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} y_i - 2\alpha\lambda\sum_{i=1}^{n}\frac{e^{-\lambda y_i}(\frac{2e^{-\theta x_i}(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2})}{(1+\theta)}-1)}{D(\theta,\lambda,\alpha)} = 0, \tag{5.3}$$

$$\frac{\partial\ln\ell}{\partial\alpha} = \sum_{i=1}^{n}\frac{(2e^{-\lambda y_i}-1)(\frac{2e^{-\theta x_i}(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2})}{(1+\theta)}-1)}{D(\theta,\lambda,\alpha)} = 0, \tag{5.4}$$

where $D(\theta,\lambda,\alpha) = 1+\alpha(2e^{-\lambda y_i}-1)(\frac{2e^{-\theta x_i}(1+\theta+\theta x_i+\frac{\theta^2 x_i^2}{2})}{(1+\theta)}-1)$.

All the likelihood equations mentioned above are appeared in nonlinear form and cannot be solved analytically. Hence, It is numerically solved using Newton-Raphson method with R software with optim function.

5.2 Inference Function Margin

The inference function margin method was originally proposed by Xu [30] and Joe [31] as a two-stage estimation process in which we estimate the parameters of the marginal distributions separately in the first stage

$$\ell_1 = \sum_{i=1}^{n} \ln f(x_i, \delta_1)\ ; \ \ \ell_2 = \sum_{i=1}^{n} \ln g(y_i, \delta_2). \tag{5.5}$$

Next, according to the previous step, the joint density is optimized by using the dependence parameter α by considering the ML estimates obtained in the previous step of the marginals $\hat{F}(x_i, \delta_1)$ and $\hat{G}(y_i, \delta_2)$. The log-likelihood equations of XG and exponential distributions can be defined as follows:

$$\frac{\partial \ell_1}{\partial \theta} = 0, \qquad and \qquad \frac{\partial \ell_2}{\partial \lambda} = 0. \tag{5.6}$$

By solving the above likelihood equations simultaneously, we obtain the maximum likelihood estimate of θ and λ.

Based on the earlier step, the IFM estimate for a FGMBXE distribution is as follows:

$$\ell_{IFM} = \sum_{i=1}^{n} \ln\left(1 + \alpha(1 - 2\hat{F}(x_i))(1 - 2\hat{G}(y_i))\right). \tag{5.7}$$

By differentiating the above log-likelihood function with respect to α and equating to zero we get the likelihood equation as given below

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{(2e^{-\lambda y_i} - 1)(2\frac{(1 + \theta + \theta x_i + \frac{\theta^2 x_i^2}{2})}{(1+\theta)} e^{-\theta x_i} - 1)}{(1 + \alpha(2e^{-\lambda y_i} - 1)(2\frac{(1 + \theta + \theta x_i + \frac{\theta^2 x_i^2}{2})}{(1+\theta)} e^{-\theta x_i} - 1))} = 0. \tag{5.8}$$

MLE of α has no closed-form expression and cannot be solved analytically. Hence, we can numerically obtain the MLE of α by employing iterative procedure of Newton Raphson technique or any suitable iterative methods.

5.3 Semi-parametric method

Kim et al. [32] proposed the semi-parametric estimation method, in which marginal distributions are estimated non-parametrically by transforming observations into pseudo-observations using sample empirical distributions

$$\widetilde{F}_i(x) = \frac{\sum_{j=1}^{n} I(X_{i,j} \le x_i)}{n+1};\ i = 1, 2. \tag{5.9}$$

Then, α is estimated by the maximizer of the pseudo log-likelihood,

$$\ell_{SP} = \sum_{i=1}^{n} \ln\left[c\left(\widetilde{F}(x_i), \widetilde{G}(y_i); \alpha\right)\right], \tag{5.10}$$

by considering (5.10), the log-likelihood function is given by

$$\ell_{SP} = \sum_{i=1}^{n} \ln\left[1 + \alpha(1 - 2\widetilde{F}(x_i)(1 - 2\widetilde{G}(y_i)))\right]. \tag{5.11}$$

The likelihood equation under semi-parametric is given by

$$\frac{\partial \ell_{SP}}{\partial \alpha} = \sum_{i=1}^{n} \frac{(1 - 2\widetilde{F}(x_i)(1 - 2\widetilde{G}(y_i))}{[1 + \alpha(1 - 2\widetilde{F}(x_i)(1 - 2\widetilde{G}(y_i)))]} = 0. \tag{5.12}$$

Equation (5.12) does not have a closed form solution. It is difficult to obtain explicit expression of MLE of α. Hence, as an alternatively, we can obtain estimate numerically by using iterative algorithm such as Newton Raphson method or any appropriate iterative techniques.

6 Asymptotic Confidence Interval

In this section, we present an asymptotic confidence interval using MLE, IFM, and SP methods. To begin, we will first find $I_O(\hat{\Theta})$, the Fishers information matrix as given below

$$I_O(\hat{\Theta}) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \lambda} & \frac{\partial^2 \ell}{\partial \theta \alpha} \\ \frac{\partial^2 \ell}{\partial \lambda \theta} & \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \alpha} \\ \frac{\partial^2 \ell}{\partial \alpha \theta} & \frac{\partial^2 \ell}{\partial \alpha \lambda} & \frac{\partial^2 \ell}{\partial \alpha^2} \end{bmatrix}\Bigg|_{(\hat{\theta},\hat{\lambda},\hat{\alpha})} = -H(\Theta)\Big|_{\Theta=\hat{\Theta}}, \tag{6.1}$$

where H is the Hessian matrix.

Hence the $100(1-\gamma)\%$ confidence interval for Θ where $\hat{\Theta} = (\hat{\theta}, \hat{\lambda}, \hat{\alpha})$ can be obtained as

$\hat{\Theta} \pm Z_{\gamma/2}\sqrt{Var(\hat{\Theta})}$,

where $Var(\hat{\Theta})$ is the diagonal entries of the inverse of observed Fisher information matrix and $Z_{\gamma/2}$ is the upper percentile of standard normal variate.

7 Simulation study

This section presents a numerical study to investigate the performance of the stress-strength parameter R, and dependence parameter α provided in the Sections 4.1 and 5. The goal of the study is to assess the influence of the dependence parameter α on R. We compare the estimates of R under MLE and IFM methods for different sample sizes based on the mean square error (MSE). A Monte Carlo simulation study is performed by generating data sets from FGMBXE distribution for three different sets of assumed parameters $(\theta, \lambda) = ((0.2, 0.4), (0.3, 0.2), (0.2, 0.5))$ with six different possible values -0.9, -0.5, -0.1, 0.1, 0.5 and 0.9 of α in the parameter space. Where 1000 data sets were simulated for three different sample sizes 50, 100, and 200. Average estimate and mean squared error of R are presented in Table 7, while the average estimate, length of the confidence interval (L.CI) and coverage probability (CP) for the dependence parameter α are shown in Table 7. All computations are performed in R software using the maxLik and copula packages.

From Table 1 and 2 we arrive at the following conclusions

- From Table 7 , we notice that the performance of the semi parametric method for estimating the dependence parameter α are better than the corresponding IFM and MLE estimates on the basis of MSE's.
- With respect to MSE, IFM performs better than MLE method of estimation for estimating the reliability parameter R.
- The MSE's of the MLE, IFM and SP estimates of the dependence parameter α are decreases as sample sizes increases and a similar trend have been observed in the estimates of reliability R.
- From Table 7, we observe that as true value of R increases when the dependence parameter α increases and then the MSE's of the estimators are also decreases. It means that the efficiency of the estimator R will be maximum for higher value of the α.
- The Length of the Confidence Interval (L.CI) of dependence parameter α is also decreases as the sample size increases.

Table 1. Estimates of R for different combination of parameter values against the varied values of α.

n		(θ, λ) α	-0.9	-0.5	-0.1	0.1	0.5	0.9
	MLE	(0.2, 0.4)	0.9081	0.9150	0.9219	0.9253	0.9322	0.9409
			0.9379	0.9375	0.9536	0.9525	0.9570	0.9576
			0.0065	0.0059	0.0058	0.0054	0.0051	0.0045
		(0.3, 0.2)	0.6522	0.6721	0.6921	0.7021	0.7220	0.7419
			0.6486	0.6758	0.6744	0.7275	0.7487	0.7872
			0.0099	0.0094	0.0090	0.0085	0.0080	0.0071
		(0.2, 0.5)	0.9143	0.9320	0.9497	0.9585	0.9762	0.9839
			0.9217	0.9474	0.9779	0.9765	0.9718	0.9857
			0.0097	0.0082	0.0060	0.0055	0.0050	0.0040
50								
	IFM	(0.2, 0.4)	0.9081	0.9150	0.9219	0.9253	0.9322	0.9409
			0.9055	0.9116	0.9220	0.9259	0.9334	0.9396
			0.0046	0.0038	0.0028	0.0024	0.0021	0.0020
		(0.3, 0.2)	0.6522	0.6721	0.6921	0.7021	0.7220	0.7419
			0.6283	0.6600	0.6889	0.7026	0.7237	0.7446
			0.0097	0.0090	0.0085	0.0076	0.0070	0.0065
		(0.2, 0.5)	0.9143	0.9320	0.9497	0.9585	0.9762	0.9839
			0.9005	0.9266	0.9503	0.9607	0.9789	0.9982
			0.0089	0.0071	0.0051	0.0046	0.0040	0.0030
	MLE	(0.2, 0.4)	0.9081	0.9150	0.9219	0.9253	0.9322	0.9409
			0.9379	0.9476	0.9450	0.9526	0.9570	0.9498
			0.0036	0.0035	0.0030	0.0028	0.0019	0.0022
		(0.3, 0.2)	0.6522	0.6721	0.6921	0.7021	0.7220	0.7419
			0.6600	0.6762	0.6859	0.7181	0.7392	0.7546
			0.0077	0.0069	0.0065	0.0058	0.0055	0.0045
		(0.2, 0.5)	0.9143	0.9320	0.9497	0.9585	0.9762	0.9839
			0.0089	0.9594	0.9781	0.9763	0.9714	0.9756
			0.0070	0.0068	0.0058	0.0050	0.0045	0.0030
100								
	IFM	(0.2, 0.4)	0.9081	0.9150	0.9219	0.9253	0.9322	0.9409
			0.9065	0.9134	0.9205	0.9268	0.9329	0.9407
			0.0031	0.0029	0.0028	0.0022	0.0020	0.0011
		(0.3, 0.2)	0.6522	0.6721	0.6921	0.7021	0.7220	0.7419
			0.6589	0.6677	0.6912	0.7181	0.7237	0.7416
			0.0066	0.0063	0.0060	0.0058	0.0056	0.0050
		(0.2, 0.5)	0.9143	0.9320	0.9497	0.9585	0.9762	0.9839
			0.9165	0.9395	0.9489	0.9600	0.9781	0.9971
			0.0068	0.0062	0.0046	0.0040	0.0035	0.0020
	MLE	(0.2, 0.4)	0.9081	0.9150	0.9219	0.9253	0.9322	0.9409
			0.9184	0.9278	0.9349	0.9327	0.9372	0.9504
			0.0026	0.0024	0.0019	0.0014	0.0010	0.0008
		(0.3, 0.2)	0.6522	0.6721	0.6921	0.7021	0.7220	0.7419
			0.6502	0.6754	0.6975	0.7016	0.7302	0.7440
			0.0025	0.0023	0.0020	0.0017	0.0010	0.0008
		(0.2, 0.5)	0.9143	0.9320	0.9497	0.9585	0.9762	0.9839
			0.9181	0.9497	0.9582	0.9559	0.9717	0.9861
			0.0046	0.0036	0.0032	0.0030	0.0025	0.0010
200								
	IFM	(0.2, 0.4)	0.9081	0.9150	0.9219	0.9253	0.9322	0.9409
			0.9069	0.9137	0.9212	0.9255	0.9331	0.9404
			0.0015	0.0012	0.0010	0.0009	0.0008	0.0005
		(0.3, 0.2)	0.6522	0.6721	0.6921	0.7021	0.7220	0.7419
			0.6561	0.6693	0.6996	0.7077	0.7249	0.7411
			0.0039	0.0035	0.0030	0.0025	0.0010	0.006
		(0.2, 0.5)	0.9143	0.9320	0.9497	0.9585	0.9762	0.9839
			0.9104	0.9304	0.9491	0.9605	0.9779	0.9983
			0.0044	0.0030	0.0041	0.0036	0.0020	0.0005

The values presented in rows: first- true value, second- estimates and third- MSE for R.

Table 2. Estimates of α for different combination of parameter values against the varied values of α.

n		(θ, λ) α	-0.9	-0.5	-0.1	0.1	0.5	0.9
	MLE	(0.2, 0.4)	-0.8755	-0.5367	-0.0997	0.5627	0.5321	0.8918
			0.1904	0.1695	0.0837	0.1192	0.0754	0.0911
			1.0740	1.4486	1.2210	1.3881	0.9648	1.0262
			0.9190	0.9150	0.9030	0.9260	0.9280	0.9070
		(0.3, 0.2)	-0.9963	-0.5488	-0.1103	0.1143	0.5419	0.9521
			0.5443	0.3474	0.1499	0.0858	0.0858	0.1251
			1.6191	1.7414	1.3657	1.0966	1.0013	1.2561
			0.8900	0.8920	0.9180	0.9250	0.9250	0.9152
		(0.2, 0.5)	-0.9205	-0.5211	-0.1212	0.0990	0.5151	0.9185
			0.2116	0.1762	0.1338	0.0931	0.1513	0.1703
			1.2978	1.4471	1.3324	1.1558	1.5568	0.9224
			0.9130	0.9060	0.9300	0.9270	0.9080	0.9010
	IFM	(0.3, 0.2)	-0.9205	-0.5129	-0.1271	0.1127	0.4953	0.8988
			0.0943	0.1309	0.1329	0.0728	0.0430	0.0845
			1.0838	1.3301	1.3049	1.0591	0.7887	1.6860
			0.9675	0.9140	0.9320	0.9361	0.9460	0.9287
50		(0.3, 0.2)	-0.9582	-0.5133	-0.1034	0.1034	0.4901	0.9125
			0.2359	0.1979	0.1028	0.0876	0.0939	0.0921
			1.4867	1.5841	1.1867	0.9190	1.8045	1.2154
			0.8990	0.9150	0.9280	0.9100	0.9190	0.9052
		(0.2, 0.5)	-0.9241	-0.4994	-0.0950	0.0981	0.4943	0.8924
			0.1389	0.1285	0.0899	0.0644	0.0464	0.0859
			1.2170	1.3268	1.1624	0.9847	0.8098	0.9897
			0.9010	0.9270	0.9310	0.9180	0.9159	0.9117
	SP	(0.2, 0.4)	-0.9156	-0.5150	-0.1168	0.1116	0.5125	0.8977
			0.0980	0.1319	0.1343	0.0710	0.0936	0.0932
			1.0817	1.3223	1.3004	1.0636	1.7870	1.6704
			0.9010	0.9110	0.8980	0.9489	0.9340	0.9155
		(0.3, 0.2)	-0.9573	-0.4929	-0.0993	0.1133	0.4929	0.9241
			0.2540	0.1920	0.0982	0.0992	0.0736	0.0854
			1.4867	1.5784	1.1858	0.9193	1.1057	1.0214
			0.8990	0.9290	0.9370	0.9280	0.9190	0.9125
		(0.2, 0.5)	-0.9326	-0.4982	-0.0950	0.0987	0.4937	0.8975
			0.1462	0.1345	-0.0950	0.0630	0.0457	0.0869
			1.2071	1.3212	1.1586	0.9829	0.8072	0.8874
			0.9020	0.9210	0.9140	0.9380	0.9195	0.9027
	MLE	(0.2, 0.4)	-0.8755	-0.4916	-0.1199	0.1104	0.5368	0.9245
			0.0904	0.1081	0.0621	0.0844	0.0636	0.0674
			1.0640	1.2047	1.3104	1.1219	0.9194	0.8382
			0.9290	0.9310	0.9190	0.9370	0.9230	0.9150
		(0.3, 0.2)	-0.8904	-0.4960	-0.1202	0.1130	0.5341	0.9121
			0.0940	0.1080	0.0869	0.0695	0.0733	0.0585
			0.9756	1.0071	1.1420	0.9195	0.9900	0.9564
			0.9340	0.9200	0.9300	0.9240	0.9260	0.9214
		(0.2, 0.5)	-0.9139	-0.5061	-0.1234	0.1036	0.5268	0.9716
			0.0940	0.0985	0.0980	0.0685	0.0678	0.0882
			1.0747	1.2020	1.0057	1.0172	0.9639	0.9994
			0.9280	0.9280	0.9400	0.9430	0.9340	0.9250
	IFM	(0.2, 0.4)	-0.9105	-0.4807	-0.1070	0.1111	0.5037	0.9073
			0.0843	0.0996	0.0936	0.0579	0.0579	0.0518
			1.0038	1.1177	1.1598	0.9496	0.7498	0.6724
			0.9270	0.9190	0.9270	0.9530	0.9460	0.9300

The values presented in rows: first- estimates, second-MSE, third-L.CI and fourth - coverage probability for α.

100		(0.3, 0.2)	-0.9215	-0.4950	-0.1016	0.1199	0.4944	0.9121
			0.0848	0.0899	0.0888	0.0571	0.0515	0.0564
			1.0060	1.0135	1.0638	0.8666	0.7944	0.9854
			0.9260	0.9300	0.9400	0.9430	0.9330	0.9124
		(0.3, 0.2)	-0.9060	-0.5053	-0.1182	0.1027	0.4922	0.9025
			0.0858	0.0866	0.0723	0.0689	0.0494	0.0523
			1.0035	1.1119	1.0651	0.8669	0.7866	0.9535
			0.9160	0.9380	0.9480	0.9420	0.9440	0.9270
	SP	(0.2, 0.4)	-0.9166	-0.4795	-0.1104	0.1134	0.5033	0.9062
			0.0880	0.0985	0.0904	0.0586	0.0579	0.0628
			1.0017	1.1126	1.1592	0.9500	0.7491	0.6692
			0.9070	0.9140	0.9340	0.9480	0.9440	0.9240
		(0.3, 0.2)	-0.9220	-0.4967	-0.1099	0.1286	0.4970	0.9124
			0.0866	0.0906	0.0862	0.0480	0.0618	0.0789
			1.0004	1.1119	1.0587	0.8684	0.7937	0.8954
			0.9110	0.9280	0.9490	0.9410	0.9370	0.9254
		(0.2, 0.5)	-0.9060	-0.5069	-0.1178	0.1044	0.4931	0.9033
			0.0858	0.0874	0.0723	0.0490	0.0393	0.0630
			1.0035	1.1100	1.0630	0.8673	0.7865	0.9526
			0.9160	0.9330	0.9480	0.9420	0.9420	0.9350
	MLE	(0.3, 0.2)	-0.9034	-0.5027	-0.1074	0.1003	0.5078	0.9061
			0.0417	0.0471	0.0367	0.0368	0.0345	0.0381
			0.7374	0.8449	0.9346	0.9680	0.7681	0.5558
			0.9410	0.9430	0.9490	0.9350	0.9440	0.9270
		(0.3, 0.2)	-0.9059	-0.5024	-0.1098	0.1050	0.5041	0.9012
			0.0418	0.0451	0.0316	0.0397	0.0341	0.3561
			0.7380	0.8570	0.9393	0.9769	0.7775	0.8541
			0.9410	0.9530	0.9390	0.9480	0.9440	0.9321
		(0.2, 0.5)	-0.9028	-0.5020	-0.1092	0.1061	0.5064	0.9012
			0.0309	0.0326	0.0383	0.0366	0.0399	0.0331
			0.7357	0.8471	0.9318	0.9684	0.8600	0.7862
			0.9380	0.9300	0.9410	0.9520	0.9530	0.9570
	IFM	(0.3, 0.2)	-0.9021	-0.5096	-0.0980	0.0991	0.9440	0.9088
			0.0361	0.0334	0.0447	0.0382	0.0323	0.0290
			0.6936	0.7844	0.8245	0.8247	0.6059	0.6145
			0.9330	0.9350	0.9460	0.9370	0.9540	0.9420
		(0.3, 0.2)	-0.9042	-0.5060	-0.1007	0.1069	0.5015	0.9012
			0.0368	0.0324	0.0377	0.0328	0.0393	0.7451
			0.6910	0.7872	0.8244	0.8238	0.5861	0.8944
			0.9280	0.9480	0.9490	0.9550	0.9590	0.9321
200		(0.2, 0.5)	-0.8994	-0.4991	-0.1089	0.1051	0.4958	0.9047
			0.0300	0.0462	0.0444	0.0484	0.0317	0.0397
			0.6932	0.7880	0.8245	0.8247	0.7044	0.6095
			0.9320	0.9360	0.9320	0.9540	0.9690	0.9558
	SP	(0.3, 0.2)	-0.9026	-0.5008	-0.0990	0.1097	0.4943	0.9096
			0.0374	0.0433	0.0342	0.0287	0.0220	0.0198
			0.6929	0.7825	0.8239	0.8240	0.7055	0.5125
			0.9320	0.9320	0.9500	0.9400	0.9520	0.9390
		(0.3, 0.2)	-0.9018	-0.5058	-0.1008	0.1078	0.5054	0.9055
			0.0274	0.0226	0.0274	0.0224	0.0297	0.0214
			0.6887	0.0226	0.8235	0.8249	0.6868	0.8941
			0.9240	0.9440	0.9530	0.9590	0.9500	0.9411
		(0.2, 0.5)	-0.9009	-0.4992	-0.1082	0.1061	0.4968	0.8996
			0.0211	0.0269	0.0242	0.0288	0.0224	0.0202
			0.6906	0.7851	0.8244	0.8236	0.7042	0.8077
			0.9310	0.9360	0.9500	0.9510	0.9570	0.9536

The values presented in rows: first- estimates, second-MSE, third-L.CI and fourth - coverage probability for α.

8 Real data analysis

In this section, we consider two real data sets to validate the methodologies proposed in the previous sections.

8.1 Data set I

In this section, we used the data set originally reported by Lawless [19]. Which represents the failure times in minutes for two different types of electrical insulation in an experiment in which the insulation is subjected to a continuously increasing voltage stress. Mokhlis et al. [20] estimated the R for the negative exponential distribution using the same data. Recently, Yazgan et al. [21] used the data set for the calculation of fuzzy stress-strength reliability for weighted exponential distribution. The correlation coefficient and test of correlation for real data set are reported in the following table

Table 3. The correlation coefficient and test of correlation for real data

	Corr	P-value
Pearson's	-0.1621	0 0.6146
Kendall's	-0.2121	0.3807

First, we fit the data sets for XG and exponential distribution using Kolmogorov-Smirnov test, the results are presented in Table 4. Further, plots of empirical and theoretical c.d.f's and P-P plots for XG as well as exponential distribution are shown in Figures 3 and 4, respectively.

Table 4. Goodness of fit test for XG and exponential distributions

	X				Y			
	D	P-value	AIC	BIC	D	P-value	AIC	BIC
XG	0.23908	0.4318	131.0786	131.56353	-	-	-	-
Expo	-	-	-	-	0.19345	0.6921	127.0109	127.4958
Weibull	0.18015	0.7688	132.0086	132.9785	0.13535	0.9596	125.87758	126.8473
Gen.Exp	0.17242	0.8106	131.6783	132.6482	0.13064	0.9701	125.9381	126.9079

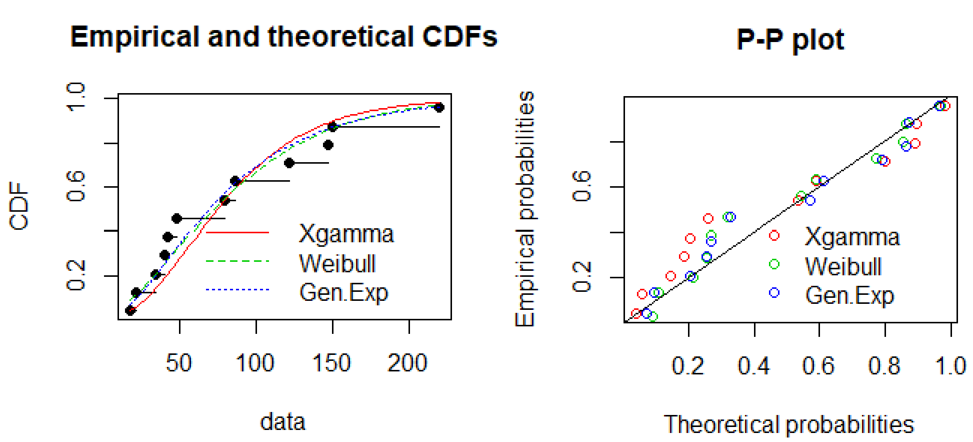

Figure 3. The plots of empirical and theoretical c.d.f's and P–P plot for data X

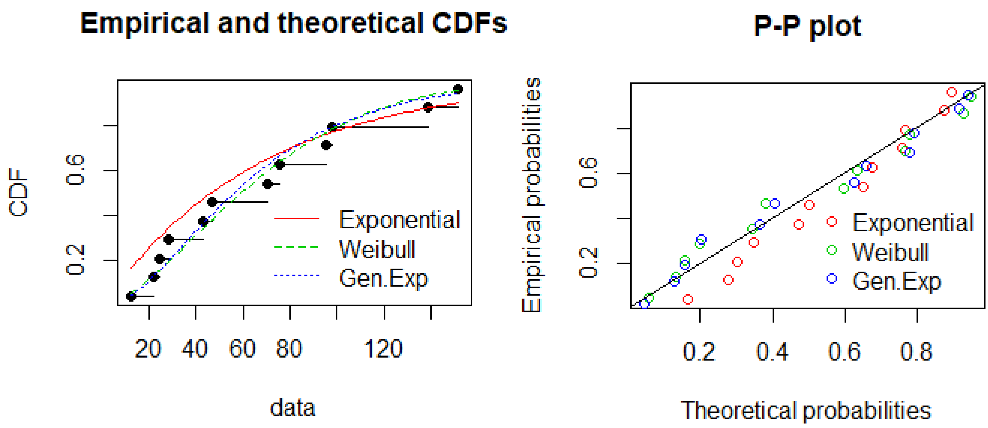

Figure 4. The plots of empirical and theoretical c.d.f's and P–P plot for data Y

Next, a goodness of fit for the copula family proposed by Genet et al. [27] is performed, and the results are reported in Table 5.

Table 5. Goodness of fit test for FGM copula

	Statistic	$\hat{\alpha}$	p-value
Anderson-Darling-type(R_n)	0.54487	-0.8563	0.1953

Finally, we fit the XG exponential distribution based on the FGM copula to the data. We compared the proposed model with FGMBW distribution discussed by Ehab et al. [28] and FGM-BGE distribution discussed by Al turk et al. [29] based on Akaike's Information Criteria (AIC) and Bayesian Information Criteria (BIC).

Table 6. The estimates of the parameters of bivariate FGM distributions

	Estimates	AIC	BIC
FGM-Xgamma Exp	(0.019967, 0.018436 , -0.7860)	173.3825	179.807
FGM-Weibull	(0.1788, 1.6372, 0.1200, 0.1929, -0.6878)	369.0863	371.5109
FGM-Gen.Exp	(2.5129, 0.02160, 0.5790, 0.0042, -0.7619)	268.3908	270.8153

Table 7. The estimates and the corresponding standard deviation of parameters of FGMBXE distribution

Methods		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$
MLE	Estimate	0.0199	0.0184	-0.7860
	(std)	(0.0076)	(0.0140)	(0.7259)
IFM	Estimate	0.0348	0.0148	-0.6157
	(std)	(0.0058)	(0.0042)	(0.7400)
SP	Estimate	-	-	-0.6397
	(std)			(0.6610)

8.2 Data set II

The second data set contains records for rainfall at the Los Angeles Civic Center from 1943 to 2018 given in [22]. Based on the Weibull records, they used the data to estimate stress-strength reliability R in the presence of inter-record times. The correlation coefficient and test of correlation for real data set are reported in the following Table 8

Table 8. The correlation coefficient and test of correlation for real data

	Corr	P-value
Pearson's	0.1159	00.332
Kendall's	0.1064	0.1861

The Kolmogorov-Smirnov test is used to check the data set coming from the XG and exponential distribution, and the results are presented in Table 9. Also, the plots of empirical and theoretical c.d.fs and P-P plots for XG as well as exponential distributions are shown in Figures 5 and 6, respectively.

Table 9. Goodness of fit test for XG and exponential distributions

	X				Y			
	D	P-value	AIC	BIC	D	P-value	AIC	BIC
XG	0.1364	0.2364	320.414	322.6907	-	-	-	-
Expo	-	-	-	-	0.0794	0.7531	275.0612	277.3378
Weibull	0.0656	0.9151	316.5952	321.1485	0.0652	0.9194	275.7419	280.2952
Gen.Exp	0.0630	0.9371	316.5726	321.126	0.0730	0.8369	276.4605	281.0139

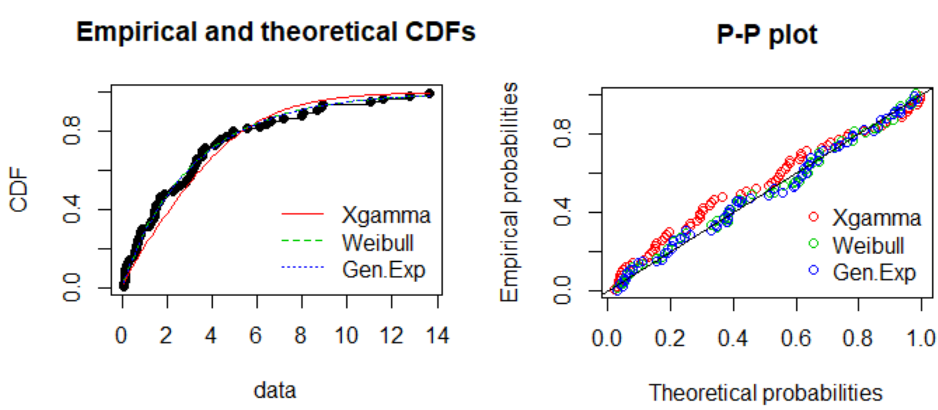

Figure 5. The plots of empirical and theoretical c.d.f's and P–P plot for data X

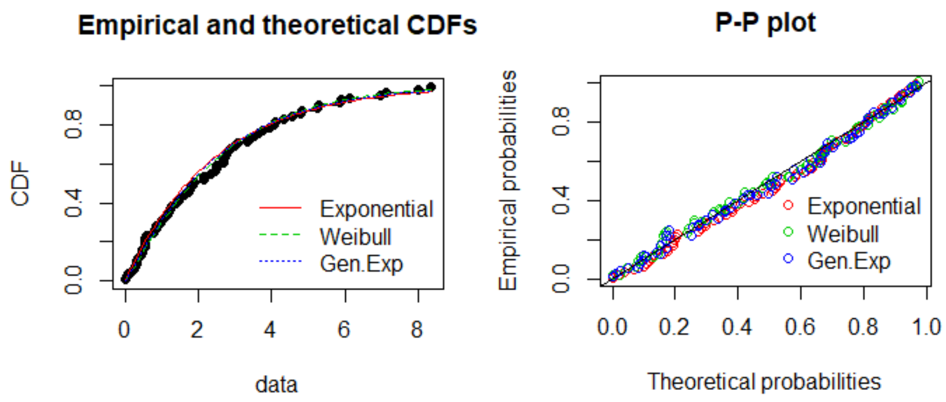

Figure 6. The plots of empirical and theoretical c.d.f's and P–P plot for data Y

Next, a goodness of fit for the copula family proposed by Genet et al. [27] is performed, and the results are reported in Table 10. Finally, we fit the XG exponential distribution based on the

Table 10. Goodness of fit test for FGM copula

	Statistic	$\hat{\alpha}$	p-value
Anderson-Darling-type(R_n)	0.1974	0.5881	0.5909

FGM copula to the data. We compared the proposed model with FGMBW distribution discussed by Ehab et al. [28] and FGMBGE distribution discussed by Al turk et al. [29] based on AIC and BIC.

Table 11. The estimates of the parameters of bivariate distributions

	Estimates	AIC	BIC
FGM-Xgamma Exp	(0.3755, 0.3648, 0.4428)	499.3806	514.7639
FGM-Weibull	(0.8816, 2.4856, 1.0946, 3.3566, 0.4467)	602.424	613.8073
FGM-Gen.Exp	(0.8965, 0.3195, 1.2267, 0.3974, 0.5882)	525.7348	537.1182

Table 12. The estimates and the corresponding standard deviation of parameters of FGMBXE distribution

Methods		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$
MLE	Estimate	0.3755	0.3648	0.4428
	(std)	(0.0393)	(0.0628)	(0.4359)
IFM	Estimate	0.3855	0.3558	0.5274
	(std)	(0.0458)	(0.0342)	(0.2947)
SP	Estimate	-	-	0.5702
	(std)			(0.3246)

9 Concluding remarks

In this study, we consider dependence stress-strength reliability model using FGM copula function. We proposed FGMBXE model to assess the correlation between stress and strength and its variability or dependency impact in survival of a system reliability. We also consider maximum likelihood estimation, inference function margin, and semi-parametric methods for estimating the dependence parameter α, where the first two methods are used for estimating reliability parameter R.

Further, the said estimators are numerically examined using Monte-Carlo simulation for different sample sizes. It is observed that these estimators behave consistently. Overall, the results indicate that semi-parametric dominates the other two methods in estimating α where as IFM gives better reliability estimates than MLE based on MSEs. Also, we observed that as the true value of R increases when the dependence parameter α increases, and the MSEs of the proposed estimators are also decreasing. We analyzed two real data sets to examine the applicability of the proposed model which have the better fit for the data sets. As a future perspectives, other extensions of FGM copula can be used for modeling stronger dependence stress-strength reliability.

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Author information

A. James and N. Chandra, Department of Statistics, Ramanujan School of Mathematical Sciences, Pondicherry University, Puducherry - 605 014, India.
E-mail: `nc.stat@gmail.com (Corresponding E-mail)`