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A TRIANGULAR QUADRATURE FOR NUMERICAL INTEGRATION OF ANALYTIC FUNCTIONS

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Abstract An efficient adaptive scheme based on a triangular quadrature of precision nine for approximate evaluation of line integral of analytic functions has been constructed. The Triangular quadrature $SM_T(f)$ is formed by a suitable linear combination of three quadrature rules lower precisions namely: Boole's transformed rule, Gauss-Legendre 3-point transformed rule and Clenshaw-Curtis seven point transformed rule with precisions five, five and seven respectively. An adaptive quadrature scheme is designed. Some test integrals having analytic function integrands have been evaluated using the triangular quadrature and its constituent rules in non-adaptive mode. The same set of test integrals have been evaluated using those rules as base rules in the adaptive scheme. Basing on the error analysis as well as numerical evidence the triangular quadrature is found to be a better choice. Also, the triangular quadrature based adaptive scheme is found to be the most effective.

1 Introduction

Despite the simple nature of the problem and the practical value of its method, numerical integration has been of great interest to both pure and applied mathematicians like Archimedes, Kepler, Huygens, Newton, Euler, Gauss, Jacobi, Chebyshev, Markhoff, Fejer, Polyya, Szego, Schoenberg and Sobolov.

There are several rules [4, 11] for the approximate evaluation of real definite integral

$$I(f) = \int_a^b f(x)dx \tag{1.1}$$

However, there are only few quadrature rules for evaluating an integral of type

$$I(f) = \int_L f(z)dz \tag{1.2}$$

where L is a directed line segment from the point $(z_0 - h)$ to $(z_0 + h)$ in the domain of f. Using the transformation $z = z_0 + ht$, $t \in [-1, 1]$ (due to [6] lether (1976)), we transformed the integral (1.2) to the form

$$h \int_{-1}^{1} f(z_0 + ht)dt$$

and made the approximation of the integral by applying standard quadrature rule meant for approximate evaluation of real definite integral (1.1). The rules so formed are termed as TRANSFORMED RULES for numerical integration of (1.2). The integral (1.1) have been successfully approximated by several authors [5, 7, 8, 9, 10, 11] by using mixed quadrature rule in the complex plane. In literature, precision of a quadrature rule has been enhanced through Richardson extrapolation and Kronrod extension [2, 10, 11]. These methods of precision enhancement are very much cumbersome and each having single base rule. But the enhancement of precision by mixed quadrature approach is very much simple with the aid of two rules and easy to handle.
In this paper, keeping in view the improvement of precision method proposed by earlier authors,

a Trianular quadrature rule of precision nine has been designed by a suitable linear combination of following three rules at a time.

I Boole's transformed quadrature rule.

II Gauss-Legendre 3-point transformed quadrature rule.

III Clenshaw-Curtis 7-point transformed quadrature rule.

This paper is organized as follows.
Section 1 is an introductory one. Section 2 describes about the constituent rules and their error analysis. Section 3 describes the construction of Triangular quadrature rule. Section 4 speaks about the truncation error of the Triangular quadrature rule and its analytic dominance over constituent rules. Section 5 deals with the numerical evaluation of test integrals using the Triangular rule and its constituents in non-adaptive mode and interpretations of relative dominance through tables and graphs. Section 6 includes adaptive quadrature routine and table of numerical values of test integrals obtained by applying Triangular quadrature and its ingredients in adaptive environment. Section 7 contains the conclusion.

2 Constituent rules of the Triangular quadrature rule

For construction of the Triangular quadrature rule let us consider following three quadrature rules.

2.1 Boole's transformed quadrature rule

The Boole's transformed rule [1, 2, 3, 12] is given by

$$I(f) \approx BL(f) = \frac{h}{45}\Big[7f(z_0-h)+32f(z_0-\frac{h}{2})+12f(z_0)+32f(z_0+\frac{h}{2})+7f(z_0+h)\Big] \quad (2.1)$$

Appling Taylors theorem (2.1) becomes

$$BL(f) = 2h\Big[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{6\times 6!}f^{vi}(z_0) + \frac{57}{45\times 8}\frac{h^8}{8!}f^{viii}(z_0) + \frac{5}{32}\frac{h^{10}}{10!}f^{x}(z_0) + \frac{897}{45\times 128}\frac{h^{12}}{12!}f^{xii}(z_0) + \cdots\Big] \quad (2.2)$$

The exact value of the integral due to Taylor

$$I(f) = 2h\Big[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{7!}f^{vi}(z_0) + \frac{h^8}{9!}f^{viii}(z_0) + \frac{h^{10}}{11!}f^{x}(z_0) + \frac{h^{12}}{13!}f^{xii}(z_0) + \cdots\Big] \quad (2.3)$$

Lemma 2.1. *If $f(z)$ is analytic in the given domain $\Omega \supset [z_0-h, z_0+h]$, then the rule $BL(f)$ is of precision-5 and the truncation error due to the rule is $EBL(f) = O(h^7)$.*

Proof. Let us denote truncation error due to the rule $BL(f)$ is by $EBL(f)$.We have

$$EBL(f) = I(f) - BL(f) \quad (2.4)$$

Using (2.2) and (2.3) in (2.4), we get

$$EBL(f) = -\frac{1}{3}\frac{h^7}{7!}f^{vi}(z_0) - \frac{17}{20}\frac{h^9}{9!}f^{viii}(z_0) - \frac{23}{16}\frac{h^{11}}{11!}f^{x}(z_0) + \frac{1967}{15\times 128}\frac{h^{13}}{13!}f^{xii}(z_0) + \cdots \quad (2.5)$$

The error term shows that the degree of precision of the rule $BL(f)$ is five and $EBL(f) = O(h^7)$. □

2.2 The Gauss-Legendre 3-point transformed rule

The Gauss-Legendre 3-point transformed rule [1, 3, 7, 10] is given by

$$I(f) \approx GL_3(f) = \frac{h}{9}\left[5f\left(z_0 - h\sqrt{\frac{3}{5}}\right) + 8f(z_0) + 5f\left(z_0 + h\sqrt{\frac{3}{5}}\right)\right] \tag{2.6}$$

Appling Taylors theorem (2.6) becomes

$$GL_3(f) = 2h\Big[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{3}{5^2}\frac{h^6}{5!}f^{vi}(z_0) + \frac{3^2}{5^3}\frac{h^8}{8!}f^{viii}(z_0) + \frac{3^3}{5^4}\frac{h^{10}}{10!}f^{x}(z_0) + \frac{3^4}{5^5}\frac{h^{12}}{12!}f^{xii}(z_0) + \cdots\Big] \tag{2.7}$$

Lemma 2.2. *If $f(z)$ is analytic in the given domain $\Omega \supset [z_0 - h, z_0 + h]$, then the rule $GL_3(f)$ is of precision-5 and the truncation error due to the rule is $EGL_3(f) = O(h^7)$.*

Proof. We have

$$EGL_3(f) = I(f) - GL_3(f) \tag{2.8}$$

Now using (2.3) and (2.7) in (2.8), we get

$$EGL_3(f) = \frac{5}{5^2}\frac{h^7}{7!}f^{vi}(z_0) + \frac{88}{5^3}\frac{h^9}{9!}f^{viii}(z_0) + \frac{656}{5^4}\frac{h^{11}}{11!}f^{x}(z_0) + \frac{4144}{5^5}\frac{h^{13}}{13!}f^{xii}(z_0) + \cdots \tag{2.9}$$

The truncation error (2.9) shows that the degree of precision of the rule $GL_3(f)$ is five and $GL_3(f) = O(h^7)$. □

2.3 Clenshaw-Curtis 7-point transformed rule

The Clenshaw-Curtis 7-point transformed rule [8, 11] is given by

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)dz \approx CC_7(f) = \frac{h}{315}\Big[9f(z_0 - h) + 80f(z_0 - \frac{\sqrt{3}}{2}h) + 144f(z_0 - \frac{h}{2}) + 164f(z_0) + 144f(z_0 + \frac{h}{2}) + 80f(z_0 + \frac{\sqrt{3}}{2}h) + 9f(z_0 + h)\Big] \tag{2.10}$$

Applying Taylor's theorem (2.10) becomes

$$CC_7(f) = 2h\Big[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{7!}f^{vi}(z_0) + \frac{31}{280}\frac{h^8}{8!}f^{viii}(z_0) + \frac{5}{56}\frac{h^{10}}{10!}f^{x}(z_0) + \cdots\Big] \tag{2.11}$$

Lemma 2.3. *If $f(z)$ is analytic in the given domain $\Omega \supset [z_0 - h, z_0 + h]$, then the rule $CC_7(f)$ is of precision-7 and the truncation error due to the rule is $ECC_7(f) = O(h^9)$.*

Proof. We have

$$ECC_7(f) = I(f) - CC_7(f) \tag{2.12}$$

Using (2.3) and (2.11) on (2.12), the truncation error due to the rule $CC_7(f)$ is

$$ECC_7(f) = \frac{1}{40}\frac{h^9}{9!}f^{viii}(z_0) + \frac{1}{28}\frac{h^{11}}{11!}f^{x}(z_0) + \cdots \tag{2.13}$$

(2.13) indicates that the degree of precision of the rule $CC_7(f)$ is seven and $ECC_7(f) = O(h^9)$. □

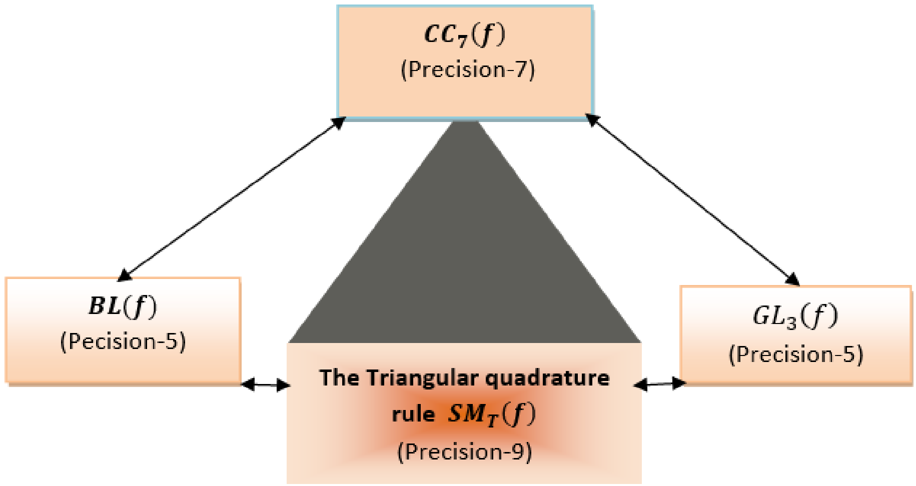


Figure 1: Diagrammatic Representation of construction of the rule

3 Formulation of the Triangular quadrature rule

Theorem-3.1 describes the formulation of the proposed Triangular quadrature rule.

Theorem 3.1. *If $f(z)$ is analytic in the given domain $\Omega \supset [z_0 - h, z_0 + h]$, then the Triangular quadrature $SM_T(f)$ and truncation error due to the rule $ESM_T(f)$ are given by* $SM_T(f) = \frac{1}{441}\Big[392CC_7(f) + 25GL_3(f) + 24BL(f)\Big]$ *and* $ESM_T(f) = \frac{1}{441}\big[392ECC_7(f) + 25EGL_3(f) + 24EBL(f)\Big]$

Proof. Resuming

$$I(f) = CC_7(f) + ECC_7(f) \tag{3.1}$$

$$I(f) = GL_3(f) + EGL_3(f) \tag{3.2}$$

$$I(f) = BL(f) + EBL(f) \tag{3.3}$$

□

adding 392 times of (3.1) 25 times of (3.2) and 24 times of (3.3), we get

$$441I(f) = 392CC_7(f) + 25GL_3(f) + 24BL(f) + 392ECC_7(f) + 25EGL_3(f) + 24EBL(f)$$

$$\Rightarrow I(f) = \frac{1}{441}\Big[392CC_7(f) + 25GL_3(f) + 24BL(f)\Big]$$

$$+\frac{1}{441}\big[392ECC_7(f) + 25EGL_3(f) + 24EBL(f)\big]$$

$$\Rightarrow I(f) = SM_T(f) + ESM_T(f)$$

Where

$$SM_T(f) = \frac{1}{441}\big[392CC_7(f) + 25GL_3(f) + 24BL(f)\big] \tag{3.4}$$

and

$$ESM_T(f) = \frac{1}{441}\big[392ECC_7(f) + 25EGL_3(f) + 24EBL(f)\big] \tag{3.5}$$

(3.4) is the required Triangular quadrature rule and (3.5) is the truncation error associated with the rule.

Table 1: The values of the test integrals obtained using Triangular quadrature rule and its three constituent rules.

Integral	Values obtained by different quadrature rules			
$I(f)$	$GL_3(f)$	$BL(f)$	$CC_7(f)$	$SM_T(f)$
$I_1 = \int_{-i}^{i} \cos z dz$	2.3503369286800113i	2.350470903569372i	2.3504023666962997i	2.3504023869560423875283446712018i
$I_2 = \int_{-i}^{i} e^z dz$	1.682942833605352i	1.68294107072143i	1.6829419695490799i	1.6829419696160712943310657596372i
$I_3 = \int_{-\frac{i}{3}}^{\frac{i}{3}} \cosh z dz$	0.6543894225254678i	0.654389363469878i	0.654389393591309492i	0.6543893935923063216870748299319i
$I_4 = \int_{-\sqrt{3}i}^{\sqrt{3}i} z^8 dz$	20.2026406194833i	44.4271032141417025i	31.0655684128960673i	31.1769145362397823i
$I_5 = \int_0^i e^{-z^2} dz$	1.46240971147732195i	1.46290943897296967i	1.46265137023528938i	1.4626517153163668i

4 Error Analysis

An error analysis of the constructed Triangular quadrature rule $SM_T(f)$ has been given by the following Theorems.

Theorem 4.1. *If f(z) is analytic in the given domain* $\Omega \supset [z_0 - h, z_0 + h]$, *then the truncation error due to the Triangular quadrature rule* $SM_T(f)$ *is* $ESM_T(f) = \frac{41}{3150}\frac{h^{11}}{11!}f^{x}(z_0) + \cdots$.

Proof. Now using Lemma-2.1, Lemma-2.2 and Lemma-2.3 on (3.5), the error due to the constructed Triangular quadrature rule became

$$ESM_T(f) = \frac{1}{441}\left[\left(\frac{25\times 8}{5^2} - \frac{24}{3}\right)\frac{h^7}{7!}f^{vi}(z_0) + \left(\frac{392}{140} + \frac{88\times 25}{5^3} - \frac{17\times 24}{20}\right)\frac{h^9}{9!}f^{viii}(z_0)\right.$$
$$\left. + \left(\frac{392}{28} + \frac{656\times 25}{5^5} - \frac{23\times 24}{16}\right)\frac{h^{11}}{11!}f^{x}(z_0) + \cdots\right]$$

$$\Rightarrow ESM_T(f) = \frac{41}{3150}\frac{h^{11}}{11!}f^{x}(z_0) + \cdots$$

The error term established that the degree of precision of the Triangular quadrature rule $SM_T(f)$ is nine. □

Theorem 4.2. *The error committed due to the quadrature rule* $SM_T(f)$ *is less than its constituent rules.*

Proof.

$$Using\ Lemma - 2.1\ and\ Theorem - 4.1,\ \mid ESM_T(f) \mid \leq \mid EBL(f) \mid$$
$$UsingLemma - 2.2\ and\ Theorem - 4.1,\ \mid ESM_T(f) \mid \leq \mid EGL_3(f) \mid$$
$$UsingLemma - 2.3\ and\ Theorem - 4.1,\ \mid ESM_T(f) \mid \leq \mid ECC_7(f) \mid$$

□

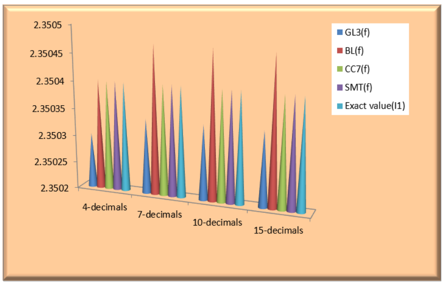
5 Numerical verification

Remark 5.1. From table-2 and figure-2a to figure-2e we have

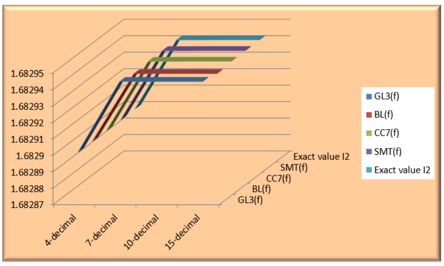
- The values obtained from the triangular quadrature rule $SM_T(f)$ covers the exact value $I_1(f)$ upto nine decimal places but the constituent rules fails after 4-7 decimal places.

Table 2: The absolute values of truncation errors due to the rules for different test integrals.

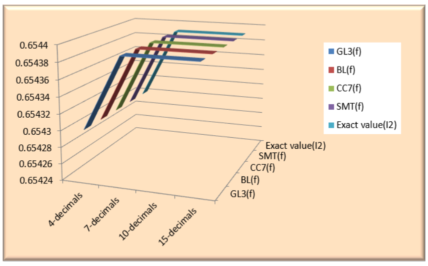
Integral	Exact value of the integral	|Error| due to quadrature rules			
	$I(f)$	$\|EGL_3(f)\|$	$\|EBL(f)\|$	$\|ECC_7(f)\|$	$\|ESM_T(f)\|$
$I_1(f)$	2.350402387287602913i	0.000065458607591613	0.000068516281769087	0.000000020591303213	0.00000000033156052547165532
$I_2(f)$	1.6829419696157930133i	0.0000008639895589867	0.0000008988943630133	0.000000000066713113	0.00000000000027828103106575
$I_3(f)$	0.65438939359230448i	0.00000002893316332	0.00000003012242648	0.000000000000994988	0.000000000000000184168707482
$I_4(f)$	31.176914536239791283494i	10.97427391675649128349	13.2501886779019112165	0.111346123343723983494	0.0000000000000008983494
$I_5(f)$	1.462651745907182i	0.00024203442986005	0.00025769306578767	0.00000037567189262	0.0000000305908152



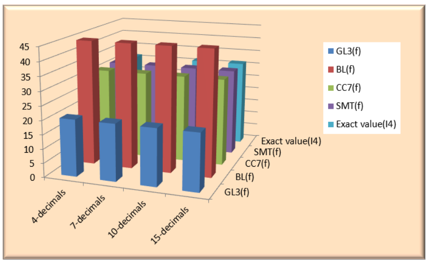
(a) Values of $I_1(f)$ obtained by different quadrature rules.



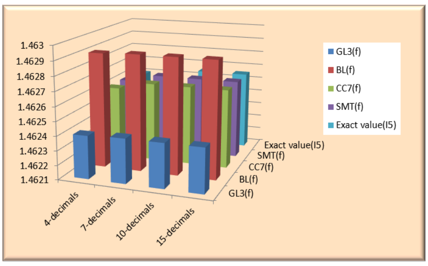
(b) Values of $I_2(f)$ obtained by different quadrature rules.



(c) Values of $I_3(f)$ obtained by different quadrature rules.



(d) Values of $I_4(f)$ obtained by different quadrature rules.



(e) Values of $I_5(f)$ obtained by different quadrature rules.

Figure 2: Values obtained by different quadrature rules for the integrals $I_1(f)$ to $I_5(f)$.

- The values obtained from the rule $SM_T(f)$ covers the exact value $I_2(f)$ upto 12 decimal places but the constituent rules fails after 6-10 decimal places.

Table 3: Approximation of the test integrals as in Table-1 using adaptive quadrature routine with Triangular quadrature and its constituent rules as base rules.

Let us consider the prescribed tolerance $\in = 1.0 \times 10^{-8}$.

Integrals	$GL_3(f)$			$CC_7(f)$		
	Approximate value(P)	No of steps	$\|Error\| = \|P - I\|$	Approximate value(P)	No of steps	$\|Error\| = \|P - I\|$
I_1	2.35040238700356998i	07	2.84×10^{-10}	2.3504023872872526i	03	1.923×10^{-13}
I_2	1.68294196961897917i	07	3.186×10^{-12}	1.68294196961553835i	01	2.546×10^{-13}
I_3	0.654389393599268328i	03	6.963×10^{-12}	0.654389393592300641i	01	3.838×10^{-15}
I_4	31.1769145352672449i	59	9.725×10^{-10}	31.1769145362138676i	15	2.592×10^{-11}
I_5	1.46265174581817542i	13	8.9×10^{-11}	1.46265174589728381i	03	9.898×10^{-12}

- The values obtained from the rule $SM_T(f)$ covers the exact value $I_3(f)$ upto 14 decimal places but the constituent rules fails after 7-12 decimal places.
- the values obtained from the rule $SM_T(f)$ covers the exact value $I_4(f)$ upto 14 decimal places but the constituent rules fails to a single decimal places.
- The values obtained from the rule $SM_T(f)$ covers the exact value $I_4(f)$ upto 7 decimal places but the constituent rules fails after 3-6 decimal places.

6 Application of the quadrature rule in Adaptive quadrature routines

An efficient adaptive strategy is given in following Algorithm [1, 7, 8, 10, 13]

Algorithm

The input to this scheme is $a, b, \in, n, f$. The output is $P \equiv \int_a^b f(x)dx$ with error less than $\in$, n is the number of intervals initially chosen. The adaptive strategy is outlined in the following four steps.

- An approximation I_1 to $I = \int_a^b f(x)dx$ is computed.
- The interval is divided into pieces, [a, c] and [c, b] where $c = \frac{a+b}{2}$, and then $I_2 \approx \int_a^c f(x)dx$ and $I_3 \approx \int_c^b f(x)dx$ are computed.
- $I_2 + I_3$ is compared with I_1, to estimate error in $I_2 + I_3$.
- If |estimated error|$\leq \frac{\in}{2}$ (termination criterion), then $I_2 + I_3$ is accepted as an approximation to $\int_a^b f(x)dx$. Otherwise, the same procedure is applied to [a, c] and [c, b], allowing each piece to a tolerance of $\frac{\in}{2}$.

Applying quadrature routines to the proposed quadrature rule to each of the sub intervals covering [a, b] until the termination criterion is satisfied. If the termination criterion is not satisfied in one or more of the sub intervals, then those sub intervals must be further subdivided and entire process repeated.

Let us consider the prescribed tolerance $\in = 1.0 \times 10^{-8}$.

Integrals	$BL(f)$			Triangular quadrature rule $SM_T(f)$		
	Approximate value(P)	No of steps	$\|Error\| = \|P - I\|$	Approximate value(P)	No of steps	$\|Error\| = \|P - I\|$
I_1	2.35040238742138 061i	11	1.337 $\times 10^{-10}$	2.35040238728724 239i	01	3.605 $\times 10^{-13}$
I_2	1.68294196961247 414i	07	3.318 $\times 10^{-12}$	1.68294196961517 9338i	01	2.665 $\times 10^{-16}$
I_3	0.65438939358505 0732i	03	7.253 $\times 10^{-12}$	0.65438939359230 449i	01	1.025 $\times 10^{-17}$
I_4	31.1769145370958 593i	63	8.56 $\times 10^{-10}$	31.1769145362397 876i	01	3.629 $\times 10^{-15}$
I_5	1.46265174599991 758i	13	9.273 $\times 10^{-11}$	1.46265174590116 61i	03	6.539 $\times 10^{-14}$

7 Conclusions

From the tables it is evident that the results of the test integrals obtained using Triangular quadrature rule are comparatively much better than those obtained using constituent rules (Gauss-Legendre 3- point, Boole's and Clenshaw-Curti's 7-point transformed rules)when computed in non adaptive mode. In adaptive scheme also, this Triangular quadrature rule $SM_T(f)$ not only gives better results than its constituent rules but also greatly reduces the number of steps of iteration for achieving desired accuracy.

Declaration

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