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ON NONLOCAL TERMINAL VALUE PROBLEM IN GENERALIZED FRACTIONAL SENSE

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Abstract In present work, we investigate the terminal value differential problem using the Hilfer-Katugampola derivative with a non-local condition. Using fixed point theory, we explore existence of unique solution result. Some relevant examples are provided to justify our findings.

1 Introduction

Differential problems of arbitrary order (FDEs) has a glorious history of more than three centuries. Until mid-twentieth century, the calculus of arbitrary order (FC) was developed as a theoretical field of applied mathematics research. During this subjective growth, numerous differential and their corresponding integral operators came into existence and were used by applied researchers in fractional sense. The Riemann-Liouville, Grunwald-Letnikov, Liouville-Caputo, Weyl, Reisz, Hilfer, Hadamard become famous in engineering and sciences applications. For details on theory and application of FC, see [4, 6, 13, 14, 18, 27]. Since last five decades, the FC has played a vital role and proved to be an adequate tool to investigate complex phenomena in the nature. The topic of qualitative properties of fractional differential equations (FDEs) and dynamical systems are always at the peak of the theoretical investigations. For some recent works on FDEs, we refer [2, 3, 7, 9, 11, 16, 19, 24, 26, 32], survey articles [1, 2, 31] and monographs [18, 21].

A terminal value problems (TVPs) for differential equations are increasingly used to simulate a wide range of phenomena in numerous fields. The TVPs arise naturally in the modelling of procedures that be measured later-ultimately after the process has begun. Several researchers have explored the theory of existence for classical TVPs in [15, 22, 23]. The development of TVPs for arbitrary order differential equations is more complex and getting a lot of attention. We look at a generalized-Katugampola (H-K) derivative as it interpolates the well-known Hilfer-Hadamard, Hilfer, Caputo-Hadamard, Hadamard, Caputo, Riemann-Liouville, Caputo-type, Katugampola differential operators [25]. We recall the following TVP [20]:

$$\begin{cases} \left({}^{\omega}D^{\mu,\delta}_{a_+}z\right)(\tau) = g(\tau, z(\tau)); & \tau \in J = (a, T], \\ \left({}^{\omega}I^{1-\beta}_{a_+}z\right)(T) = c, & c \in \mathbb{R}, \beta = \mu + \delta(1-\mu), \end{cases} \tag{1.1}$$

for H-K FDE solved using Banach fixed-point theorem and it's Volterra integral equation (VIEq):

$$z(\tau) = \frac{c}{\Gamma(\beta)}\left(\frac{\tau^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} + \int_a^{\tau}\left(\frac{\tau^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} p^{\omega-1}\frac{g(p, z(p))}{\Gamma(\mu)}dp. \tag{1.2}$$

Recently, efforts have taken to investigate the comparison principle for the TVPs in [5, 6, 10, 29, 30]. Motivated by importance of TVP and applicability of H-K differential operator, in present work, we explore the following TVP:

$$\begin{cases} \left({}^{\omega}D^{\mu,\delta}_{a_+}z\right)(\tau) = g\big(t, z(\tau)\big), & 0<\mu<1,\ 0\le\delta\le 1,\ \tau\in(a,T], \\ \left({}^{\omega}I^{1-\beta}_{a_+}z\right)(T) = \sum_{\kappa=1}^{m}\eta_\kappa z(\xi_\kappa), & \mu\le\beta=\mu+\delta(1-\mu),\ \xi_\kappa\in(a,T], \end{cases} \tag{1.3}$$

where ${}^{\omega}D^{\mu,\delta}_{a_+}$ is the H-K derivative of fractional order $\mu\in(0,1)$ and type $\delta\in[0,1]$, and ${}^{\omega}I^{1-\beta}_{a_+}$ is the Katugampola integral with $(\omega>0)$. Here, ξ_κ are points such that $0<a<\xi_1\le\cdots\le\xi_m<T$, $g:(a,T]\times\mathbb{R}\to\mathbb{R}$ is a given function and $\eta_\kappa\in\mathbb{R}$, $\kappa=1,2,\cdots,m$.

We discuss the criteria for existence of solution to nonlocal terminal value problem (NTVP) (1.3). We emmploy the fixed point theorem due to Banach and Leray-Schauder to achieve well-posedness results for NTVP (1.3). In the beginning, we obtain a nonlinear mixed-type equivalent volterra integral equation (VIEq):

$$\begin{aligned} z(\tau) = {} & \frac{K}{\Gamma(\mu)}\left(\frac{\tau^\omega-a^\omega}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^\omega-p^\omega}{\omega}\right)^{\mu-1} g(p,z(p))dp \\ & + \frac{1}{\Gamma(\mu)}\int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega-p^\omega}{\omega}\right)^{\mu-1} g(p,z(p))dp, \end{aligned} \tag{1.4}$$

where

$$K = \left[\Gamma(\beta) - \sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{\xi_\kappa^\omega-a^\omega}{\omega}\right)^{\beta-1}\right]^{-1}, \tag{1.5}$$

for NTVP (1.3) in $\zeta_{1-\beta,\omega}[a,T]$ which is defined in next section.

2 Preliminaries

Some definitions and lemmas are given for development of our results. Consider following definitions of the (Euler's) Gamma and Beta functions as follows:

$$\Gamma(\mu) = \int_0^\infty x^{\mu-1}e^{-x}dx, \quad \mathbf{B}(\mu,\delta) = \int_0^1 (1-x)^{\mu-1}x^{\delta-1}dx,\ \mu>0,\delta>0.$$

It is well known that $\mathbf{B}(\mu,\delta)=\frac{\Gamma(\mu)\Gamma(\delta)}{\Gamma(\mu+\delta)}$, for $\mu>0,\delta>0$, see [18]. Throughout the work, we suppose $[a,T]$, $(0<a<T<\infty)$, be interval on $\mathbb{R}^+$ which is finite and $\omega>0$.

Definition 2.1. [18] The space $X^q_c(a,T)$, $(c\in\mathbb{R}, 1\le q\le\infty)$, consists Lebesgue measurable functions g on (a,T) which are real-valued for which $\|g\|_{X^q_c}<\infty$, where

$$\|g\|_{X^q_c} = \left(\int_a^b |t^c g(\tau)|^q\frac{dt}{t}\right)^{\frac{1}{q}},\ 1\le q\le\infty,$$

and when $c=\frac{1}{q}$, we observe that $X^c_{1/q}(a,T)=L_q(a,T)$.

Definition 2.2. [18] We signify by $\zeta[a,T]$ a space of continuous functions g on $(a,T]$ with the norm

$$\|g\|_\zeta = \sup_{\tau\in[a,T]}|g(\tau)|.$$

The weighted space $\zeta_{\beta,\omega}[a,T]$, $0\le\beta<1$, on $(a,T]$ of functions g is defined by

$$\zeta_{\beta,\omega}[a,T] = \left\{g:(a,T]\to\mathbb{R}: \left(\frac{\tau^\omega-a^\omega}{\omega}\right)^\beta g(\tau)\in\zeta[a,T]\right\}, \tag{2.1}$$

having the norm

$$\|g\|_{\zeta_{\beta,\omega}} = \|\left(\frac{\tau^\omega-a^\omega}{\omega}\right)^\beta g(\tau)\|_\zeta = \sup_{\tau\in[a,T]}|\left(\frac{\tau^\omega-a^\omega}{\omega}\right)^\beta g(\tau)|,$$

and $\zeta_{0,\omega}[a,T]=\zeta[a,T]$.

Definition 2.3. [20] Let $\delta_\omega = \left(\tau^{\omega-1}\frac{d}{d\tau}\right)$, $0 \le \beta < 1$. Denote $\zeta^n_{\delta_\omega,\beta}[a,T]$ the Banach space of continuously differentiable functions g with δ_ω, on $[a,T]$ upto $(n-1)$ $(n \in \mathbb{N})$, order and it's derivative $\delta^n_\omega g$ on $(a,T]$ such that $\delta^n_\omega g \in \zeta_{\beta,\omega}[a,T]$:

$$\zeta^n_{\delta_\omega,\beta}[a,T] = \left\{g : (a,T] \to \mathbb{R} : \delta^k_\omega g \in \zeta[a,T], k = 0,1,\cdots,n-1,\ \delta^n_\omega g \in \zeta_{\beta,\omega}[a,T]\right\},$$

having the norm

$$\|g\|_{\zeta^n_{\delta_\omega,\beta}} = \sum_{k=0}^{n-1}\|\delta^k_\omega g\|_\infty + \|\delta^n_\omega g\|_{\zeta_{\beta,\omega}}, \quad \|g\|_{\zeta^n_{\delta_\omega}} = \sup_{t\in\Omega}\left|\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^\beta g(\tau)\right|.$$

As the case for $n = 0$, $\zeta^0_{\delta_\omega,\beta}[a,T] = \zeta_{\beta,\omega}[a,T]$.

Definition 2.4. [17] Let $\mu > 0$ and $g \in X^q_c(a,T)$. The Katugampola integral of left kind ${}^\omega I^\mu_{a_+}$ of order μ can be given as:

$$ {}^\omega I^\mu_{a_+} g(\tau) = \int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{\mu-1}\frac{g(p)}{\Gamma(\mu)}dp, \quad \tau > a. \tag{2.2}$$

Definition 2.5. [17] Let $\mu \in \mathbb{R}^+\backslash\mathbb{N}$ and $n = [\mu] + 1$, here the whole part of $[\mu]$ is μ. The Katugampola derivative of left kind ${}^\omega D^\mu_{a_+}$ is given as:

$$\begin{aligned} {}^\omega D^\mu_{a_+} g(\tau) &= \delta^n_\omega({}^\omega I^{n-\mu}_{a_+} g(p))(\tau) \\ {}^\omega D^\mu_{a_+} g(\tau) &= \left(\tau^{\omega-1}\frac{d}{d\tau}\right)^n \int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{n-\mu-1}\frac{g(p)}{\Gamma(n-\mu)}dp. \end{aligned}\tag{2.3}$$

Definition 2.6. [28] Let $n-1 < \mu < n$ and $0 \le \delta \le 1$, $n \in \mathbb{N}$. The H-K derivative, $\omega > 0$, of a function $g \in \zeta_{\beta,\omega}[a,T]$, given by

$$({}^\omega D^{\mu,\delta}_{a_+} g)(\tau) = ({}^\omega I^{\delta(1-\mu)}_{a_+}\delta^n_\omega\ {}^\omega I^{(1-\delta)(n-\mu)}_{a_+} g)(\tau), \tag{2.4}$$

for the functions for which rhs expression exists.

Lemma 2.7. *[21] Suppose that $\mu > 0, \delta > 0, 1 \le q \le \infty$ and $\omega, c \in \mathbb{R}$ such that $\omega \ge c$. Then, for $g \in X^q_c(a,T)$, Katugampola integral satisfies semi-group property:*

$$ {}^\omega I^\mu_{a_+}\,{}^\omega I^\delta_{a_+} g(\tau) = {}^\omega I^{\mu+\delta}_{a_+} g(\tau). \tag{2.5}$$

Lemma 2.8. *[9] Consider that $\mu > 0, 0 \le \beta < 1$, and $g \in \zeta_{\beta,\omega}[a,T]$. Then, $\forall \tau \in (a,T]$,*

$$ {}^\omega D^\mu_{a_+}\,{}^\omega I^\mu_{a_+} g(\tau) = g(\tau).$$

Lemma 2.9. *[9] Consider that $\mu > 0$, $g \in \zeta_{\beta,\omega}[a,T]$, $0 \le \beta < 1$ and ${}^\omega I^{1-\mu}_{a_+} g \in \zeta^1_{\beta,\omega}[a,T]$, then*

$$ {}^\omega I^\mu_{a_+}\,{}^\omega D^\mu_{a_+} g(\tau) = g(\tau) - \frac{{}^\omega I^{1-\mu}_{a_+} g(a)}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\mu-1}.$$

Lemma 2.10. *[9] Consider ${}^\omega I^\mu_{a_+}$ and ${}^\omega D^\mu_{a_+}$ are as defined in above definition. Then*

$$\begin{aligned} {}^\omega I^\mu_{a_+}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\sigma-1} &= \frac{\Gamma(\sigma)}{\Gamma(\sigma+1)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\mu+\sigma-1}, \quad \mu \ge 0,\ \sigma > 0,\ \tau > a. \\ {}^\omega D^\mu_{a_+}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\mu-1} &= 0, \qquad 0 < \mu < 1. \end{aligned}$$

Remark 2.11. [9] For $0 < \mu < 1, 0 \le \delta \le 1$, the H-K derivative ${}^\omega D^{\mu,\delta}_{a_+}$ having property

$$ {}^\omega D^{\mu,\delta}_{a_+} f = {}^\omega I^{\delta(n-\mu)}_{a_+}\delta^n_\omega\,{}^\omega I^{n-\beta}_{a_+} = {}^\omega I^{\delta(n-\mu)}_{a_+}\ {}^\omega D^\beta_{a_+}, \qquad \beta = \mu + \delta(n-\mu).$$

Lemma 2.12. *[20] Let* $\mu > 0, g \in \zeta_{1-\beta,\omega}[a,b]$ *and* $0 < \beta \leq 1$. *If* $\mu > \beta$, *then*

$$({}^{\omega}I^{\mu}_{a_+}g)(a) = \lim_{x \to a_+} ({}^{\omega}I^{\mu}_{a_+}g)(\tau) = 0.$$

To discuss the existence for solution to NTVP (1.3), we consider following spaces :

$$\zeta^{\mu,\delta}_{1-\beta,\omega}[a,T] = \{g \in \zeta_{1-\beta,\omega}[a,T] : {}^{\omega}D^{\mu,\delta}_{a_+}g \in \zeta_{1-\beta,\omega}[a,T]\},\ 0 < \beta \leq 1, \tag{2.6}$$

and

$$\zeta^{\beta}_{1-\beta,\omega}[a,T] = \{g \in \zeta_{1-\beta,\omega}[a,T] : {}^{\omega}D^{\beta}_{a_+}g \in \zeta_{1-\beta,\omega}[a,T]\},\ 0 < \beta \leq 1.$$

Since ${}^{\omega}D^{\mu,\delta}_{a_+}g = {}^{\omega}I^{\delta(1-\mu)\omega}_{a_+}D^{\beta}_{a_+}g$, clearly $\zeta^{\beta}_{1-\beta,\omega}[a,T] \subset \zeta^{\mu,\delta}_{1-\beta,\omega}[a,T] \subset \zeta_{1-\beta,\omega}[a,T]$.

Lemma 2.13. *[17] Let* $\mu > 0, \delta > 0$ *and* $\beta = \mu + \delta - \mu\delta$. *If* $g \in \zeta^{\beta}_{1-\beta,\omega}[a,T]$, *then*

$${}^{\omega}I^{\beta}_{a_+}\, {}^{\omega}D^{\beta}_{a_+}g(\tau) = {}^{\omega}I^{\mu}_{a_+}\, {}^{\omega}D^{\mu,\delta}_{a_+}g(\tau) = {}^{\omega}D^{\delta(1-\mu)}_{a_+}g(\tau).$$

Lemma 2.14. *[12] Let* F *be a Banach space and* $\Omega \subset F$ *is closed and convex. Assume that* G *is relatively open subset of* Ω *with* $0 \in G$ *and* $N : \bar{G} \to \Omega$ *is compact and continuous mapping. Then either:*
(a) N *having fixed point in* $\bar{G}$ *or*
(b) $\exists\, y \in \delta G$ *such as* $z = \lambda N z$ *for some* $\lambda \in (0,1)$, *where* δG *is boundary of* G.

3 Main results

In this part, we prove the equivalence of NTVP (1.3) and VIEq (1.4).

Lemma 3.1. *[9] Consider* $0 < \mu < 1$, $0 \leq \delta \leq 1$ *and* $\beta = \mu + \delta - \mu\delta$. *If* $g : (a,T] \times \mathbb{R} \to \mathbb{R}$ *is a function such as* $g(\cdot, z(\cdot)) \in \zeta_{1-\beta,\omega}[a,T]$ *for any* $z(\cdot) \in \zeta_{1-\beta,\omega}[a,T]$, *then* $z \in \zeta^{\beta}_{1-\beta,\omega}[a,T]$ *satisfies NTVP* (1.3) *iff* z *satisfies nonlinear VIEq.* (1.4).

Proof. $\Rightarrow$ Let us rewrite (1.2) as

$$z(\tau) = \frac{{}^{\omega}I^{1-\beta}_{a_+}z(T)}{\Gamma(\beta)}\left(\frac{\tau^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} + \int_a^{\tau} p^{\omega-1}\left(\frac{\tau^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\frac{g(p,z(p))}{\Gamma(\mu)}dp. \tag{3.1}$$

A substitution $\tau = \xi_\kappa$ in (3.1) yields

$$z(\xi_\kappa) = \frac{{}^{\omega}I^{1-\beta}_{a_+}z(T)}{\Gamma(\beta)}\left(\frac{{\xi_\kappa}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} + \int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\frac{g(p,z(p))}{\Gamma(\mu)}dp, \tag{3.2}$$

and after multiplying η_κ on each sides of (3.2),

$$\eta_\kappa z(\xi_\kappa) = \frac{{}^{\omega}I^{1-\beta}_{a_+}z(T)}{\Gamma(\beta)}\eta_\kappa\left(\frac{{\xi_\kappa}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} + \eta_\kappa\int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\frac{g(p,z(p))}{\Gamma(\mu)}dp.$$

By terminal condition of NTVP (1.3),

$$\begin{aligned}
{}^{\omega}I^{1-\beta}_{a_+}a_+(T) &= \sum_{\kappa=1}^{m}\eta_\kappa z(\xi_\kappa) \\
&= \frac{{}^{\omega}I^{1-\beta}_{a_+}z(T)}{\Gamma(\beta)}\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} \\
&\quad + \sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\frac{g(p,z(p))}{\Gamma(\mu)}dp
\end{aligned}$$

which implies

$$
\begin{aligned}
{}^{\omega}I_{a_+}^{1-\beta}z(T)\left(\Gamma(\beta)-\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}\right)&\\
=\frac{\Gamma(\beta)}{\Gamma(\mu)}\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp&
\end{aligned}
$$

$$
i.e.\qquad {}^{\omega}I_{a_+}^{1-\beta}z(T)=\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp, \tag{3.3}
$$

where K is the same as in (1.5). Substitute (3.3) into (3.1), the integral equation (1.4) is obtained.
$\Leftarrow$ Employing ${}^{\omega}I_{a_+}^{1-\beta}$ on each sides of equation (1.4),

$$
\begin{aligned}
{}^{\omega}I_{a_+}^{1-\beta}z(\tau)={}^{\omega}I_{a_+}^{1-\beta}\left(\frac{\tau^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}K\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}\frac{g(p,z(p))}{\Gamma(\mu)}dp&\\
+{}^{\omega}I_{a_+}^{1-\beta\omega}I_{a_+}^{\mu}g(\tau,z(\tau)),&
\end{aligned}
$$

using Lemmas 2.7 and 2.10,

$$
{}^{\omega}I_{a_+}^{1-\beta}z(\tau)=\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp+{}^{\omega}I_{a_+}^{1-\delta(1-\mu)}g(\tau,z(\tau)).
$$

Since $1-\beta<1-\delta(1-\mu)$, using Lemma 2.12 and $t\to a_+$ yields

$$
{}^{\omega}I_{a_+}^{1-\beta}z(a)=\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp. \tag{3.4}
$$

A substitution $\tau=\xi_\kappa$ in (1.4) gives

$$
\begin{aligned}
z(\xi_\kappa)=\frac{K}{\Gamma(\mu)}\left(\frac{{\xi_\kappa}^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp&\\
+\frac{1}{\Gamma(\mu)}\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp.&
\end{aligned}
$$

Further

$$
\begin{aligned}
\sum_{\kappa=1}^{m}\eta_\kappa z(\xi_\kappa)=\frac{K}{\Gamma(\mu)}\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}&\\
+\sum_{\kappa=1}^{m}\eta_\kappa\frac{1}{\Gamma(\mu)}\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp&
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\kappa=1}^{m}\eta_\kappa z(\xi_\kappa)=\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}\frac{g(p,z(p))}{\Gamma(\mu)}dp\left[1+K\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}\right]&\\
=\frac{\Gamma(\beta)}{\Gamma(\mu)}K\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa}p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}g(p,z(p))dp.&
\end{aligned} \tag{3.5}
$$

Now, linking (3.4) and (3.5), it follows that

$$
\omega I_{a_+}^{1-\beta}z(a_+)=\sum_{\kappa=1}^{m}\eta_\kappa z(\xi_\kappa).
$$

Applying ${}^{\omega}D^{\beta}_{a_+}$ to each sides of (1.4), from Lemmas 2.10 and 3.1,

$$ {}^{\omega}D^{\beta}_{a_+} z(\tau) = {}^{\omega}D^{\delta(1-\mu)}_{a_+} g(\tau, z(\tau)). \tag{3.6} $$

Since $z \in \zeta^{\beta}_{1-\beta,\omega}[a,T]$, from definition of $\zeta^{\beta}_{1-\beta,\omega}[a,T]$, we have

$$ {}^{\omega}D^{\beta}_{a_+} z \in \zeta_{1-\beta,\omega}[a,T] \quad \text{then} \quad {}^{\omega}D^{\delta(1-\mu)}_{a_+} f = \delta_{\omega}\, {}^{\omega}I^{1-\delta(1-\mu)}_{a_+} g \in \zeta_{1-\beta,\omega}[a,T]. $$

For $g \in \zeta_{1-\beta,\omega}[a,T]$, obviously ${}^{\omega}I^{1-\delta(1-\mu)}_{a_+} g \in \zeta_{1-\beta,\omega}[a,T]$, then ${}^{\omega}I^{1-\delta(1-\mu)}_{a_+} g \in \zeta^{\delta_{\omega}}_{1-\beta,\omega}[a,T]$. This means g and ${}^{\omega}I^{1-\delta(1-\mu)}_{a_+} g$ assures Lemma 2.9. Finally, applying ${}^{\omega}I^{1-\delta(1-\mu)}_{a_+}$ on each sides of (3.6), Lemma 2.9 supports to give

$$ {}^{\omega}D^{\mu,\delta}_{a_+} z(\tau) = g(\tau, z(\tau)) - \frac{{}^{\omega}I^{1-\delta(1-\mu)}_{a_+} g(a)}{\Gamma(\delta(1-\mu))} \left(\frac{\tau^{\omega} - a^{\omega}}{\omega} \right)^{\delta(1-\mu)-1}. $$

By Lemma 2.12, it is observe that ${}^{\omega}I^{1-\delta(1-\mu)}_{a_+} g(a) = 0$. Hence

$$ {}^{\omega}D^{\mu,\delta}_{a_+} z(\tau) = g(\tau, z(\tau)). $$

This completes the proof. □

Let us state and prove the existence of unique solution for NTVP (1.3). Our considerations are as follows:

(H_1) $g : (a,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(\cdot, z(\cdot)) \in \zeta^{\delta(1-\mu)}_{1-\beta,\omega}[a,T]$ for any $z \in \zeta_{1-\beta,\omega}[a,T]$ and $\exists$ a constant $L > 0$ s. t. $\forall z, \bar{z} \in \mathbb{R}$,

$$ |g(\tau, z(\tau)) - g(\tau, \bar{z}(\tau))| \leq L|z - \bar{z}|. $$

(H_2) The constant

$$ \phi = \frac{|K|}{\Gamma(\mu)} \sum_{\kappa=1}^{m} \eta_{\kappa}(\xi_{\kappa}) + \left(\frac{\tau^{\omega} - a^{\omega}}{\omega} \right)^{1-\beta} \frac{1}{\Gamma(\mu)} \left({}^{\omega}I^{\mu}_{a_+} \left(\frac{p^{\omega} - a^{\omega}}{\omega} \right)^{\beta-1} \right)(\tau) < 1, $$

where K is in (1.5).

Let us now prove the existence results for NTVP (1.3) utilizing Banach fixed-point theorem.

Theorem 3.2. *Consider (H_1) and (H_2) are hold. Then NTVP (1.3) has unique solution in $\zeta^{\beta}_{1-\beta,\omega}[a,T] \subset \zeta^{\mu,\delta}_{1-\beta,\omega}[a,T]$ provided $A\phi < 1$ where,*

$$ \phi = \frac{|K|}{\Gamma(\mu)} \sum_{\kappa=1}^{m} \eta_{\kappa} \left({}^{\omega}I^{\mu}_{a_+} \left(\frac{p^{\omega} - a^{\omega}}{\omega} \right)^{\beta-1} \right)(\xi_{\kappa}) + \left(\frac{\tau^{\omega} - a^{\omega}}{\omega} \right)^{1-\beta} \frac{1}{\Gamma(\mu)} \left({}^{\omega}I^{\mu}_{a_+} \left(\frac{p^{\omega} - a^{\omega}}{\omega} \right)^{\beta-1} \right)(\tau). \tag{3.7} $$

Proof. Firsly, we prove that the operator $N : \zeta_{1-\beta,\omega}(I) \to \zeta_{1-\beta,\omega}(I)$ defined as:

$$ Nz(\tau) = \frac{K}{\Gamma(\mu)} \left(\frac{\tau^{\omega} - a^{\omega}}{\omega} \right)^{\beta-1} \sum_{\kappa=1}^{m} \eta_{\kappa} \int_a^{\xi_{\kappa}} p^{\omega-1} \left(\frac{{\xi_{\kappa}}^{\omega} - p^{\omega}}{\omega} \right)^{\mu-1} g(p, z(p))dp + \frac{1}{\Gamma(\mu)} \int_a^{\tau} p^{\omega-1} \left(\frac{\tau^{\omega} - p^{\omega}}{\omega} \right)^{\mu-1} g(p, z(p))dp, \tag{3.8} $$

has unique fixed point z^* in $\zeta_{1-\beta,\omega}(I)$.

Let $z, u \in \zeta_{1-\beta,\omega}(I)$ and $\tau \in (a,T]$ then,

$$|Nz(\tau) - Nu(\tau)| \le \frac{K}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa \int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^\omega - p^\omega}{\omega}\right)^{\mu-1} \times |g(p)-h(p)|dp + \frac{1}{\Gamma(\mu)}\int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{\mu-1}|g(p)-h(p)|dp,$$

where $g, h \in \zeta_{1-\beta,\omega}(I)$ such that $g(\tau) = g(\tau, z(\tau))\,, h(\tau) = g(\tau, u(\tau))$.
By (H_1),

$$\begin{aligned}|g(\tau) - h(\tau)| &\le |g(\tau, z(\tau)) - g(\tau, u(\tau))| \\ &\le A|z(\tau) - u(\tau)|.\end{aligned}$$

Hence, for each $\tau \in (a,T]$

$$|Nz(\tau) - Nu(\tau)| \le \frac{KA}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa \int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^\omega - p^\omega}{\omega}\right)^{\mu-1} \times |z(p)-u(p)|dp + \frac{A}{\Gamma(\mu)}\int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{\mu-1}|z(p)-u(p)|dp.$$

Multiply on both sides of above inequation, $\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}$ we get

$$\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}|Nz(\tau) - Nu(\tau)| \le \frac{KA}{\Gamma(\mu)}\sum_{\kappa=1}^{m}\eta_\kappa \int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^\omega - p^\omega}{\omega}\right)^{\mu-1}|z(p)-u(p)|dp + \frac{A}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}\int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{\mu-1}|z(p)-u(p)|dp,$$

$$\begin{aligned}\|Nz(\tau) - Nu(\tau)\| &\le \frac{KA}{\Gamma(\mu)}\|y-u\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_\kappa\left({}^\omega I_{a_+}^\mu\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\xi_\kappa) \\ &\quad + \frac{A}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}\|y-u\|_{\zeta_{1-\beta,\omega}}\left({}^\omega I_{a_+}^\mu\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\tau) \\ &\le A\left[\frac{K}{\Gamma(\mu)}\sum_{\kappa=1}^{m}\eta_\kappa\left({}^\omega I_{a_+}^\mu\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\xi_\kappa) \right. \\ &\quad \left. + \frac{1}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}\left({}^\omega I_{a_+}^\mu\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\tau)\right]\|z-u\|_{\zeta_{1-\beta,\omega}}\end{aligned}$$

$$\|Nz(\tau) - Nu(\tau)\| \le A\phi\|z(\tau) - u(\tau)\|_{\zeta_{1-\beta,\omega}}.$$

By equation (3.8) N is contraction. By implication Banach principle, N has a unique fixed point $z^* \in \zeta_{1-\beta,\omega}(I)$.
Now we show that a fixed point $z* \in \zeta_{1-\beta,\omega}(I)$ is actually in $\zeta_{1-\beta,\omega}(I)$.
Since z^* is unique fixed point of operator N in $\zeta_{1-\beta,\omega}(I)$ then for each $z \in (a,T]$

$$z^*(\tau) = \frac{{}^\omega I_{a_+}^{1-\beta} z(\tau)}{\Gamma(\beta)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1} + \int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{\mu-1}\frac{g(p, z(p))}{\Gamma(\mu)}dp.$$

Applying ${}^\omega D_{a_+}^\beta$ on both sides results in

$$ {}^\omega D_{a_+}^\beta = 0 + \left({}^\omega D_{a_+}^\beta\, {}^\omega I_{a_+}^\mu \phi\right)(\tau) = \left({}^\omega D_{a_+}^{\delta(1-\mu)}\phi\right). \tag{3.9}$$

We can summarize that equations NTVP (1.3) have unique solution in $\zeta_{1-\beta,\omega}(I)$. □

Now we prove existence of solution for NTVP (1.3) using nonlinear alternative of Leray-Schauder. For this we define hypotheses as follows:

(H_3) $\exists$ two continuous functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\eta : I \to \mathbb{R}^+$ which are non decreasing so as

$$|g(\tau, z)| \leq \eta(\tau)\ \psi\big(\|z\|_{\zeta_{1-\beta,\omega}}\big)\ , \qquad \forall\ \tau \in I, z \in \mathbb{R}$$

(H_4) $\exists$ a constant $Q > 0$ such that,

$$\frac{\nu\|\eta\|\psi(Q)}{Q} < 1, \tag{3.10}$$

where

$$\nu = \frac{\Gamma(\beta)}{2\Gamma(\mu)\Gamma(\mu+\beta)}\left(K\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^\omega - a^\omega}{\omega}\right)^{\mu+\beta-1} + \left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\mu}\right).$$

Theorem 3.3. *Assume that $g : I \times \mathbb{R} \to \mathbb{R}$, and (H_3), (H_4) are satisfied. Then NTVP (1.3) has at least one solution on I.*

Proof. Consider $F = \{z \in \zeta_{1-\beta,\omega}(I) : \|z\| < r\}$, where

$$r = \frac{K\left(\frac{t^\omega - a^\omega}{\omega}\right)^{1-\beta} g^* \sum_{\kappa=1}^m \eta_\kappa \left(\frac{\xi_\kappa^\omega - a^\omega}{\omega}\right)^{\mu+\beta-1} + \left(\frac{t^\omega - a^\omega}{\omega}\right)^{1-\beta} g^*}{(-A)\left[K\sum_{\kappa=1}^m \eta_\kappa \left(\frac{\xi_\kappa^\omega - a^\omega}{\omega}\right)^{\mu+\beta-1} + 1\right]} \tag{3.11}$$

and

$$g^* = \sup_{\tau \in (a,T]} |g(t, 0))|.$$

Firstly, we start with operator N as in (3.8), maps F which is again bounded in $\zeta_{1-\beta,\omega}$. For a positive number r, let F be a bounded ball in $\zeta_{1-\beta,\omega}$. Then for $\tau \in I$, we have

$$\begin{aligned}
Nz(\tau) = {} & \frac{K}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^\omega - p^\omega}{\omega}\right)^{\mu-1}|g(p, z(p))|dp \\
& + \frac{1}{\Gamma(\mu)}\int_a^\tau p^{\omega-1}\left(\frac{t^\omega - p^\omega}{\omega}\right)^{\mu-1}|g(p, z(p))|dp, \\
\leq {} & \frac{K}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^\omega - p^\omega}{\omega}\right)^{\mu-1}\eta(p)\ \psi\big(\|z\|\big)dp \\
& + \frac{1}{\Gamma(\mu)}\int_a^\tau p^{\omega-1}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{\mu-1}\eta(p)\ \psi\big(\|z\|\big)dp, \\
\leq {} & \frac{K}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa\int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^\omega - p^\omega}{\omega}\right)^{\mu-1}\left(\frac{p^\omega - a^\omega}{\omega}\right)^{1-\beta+\beta-1} \\
& \times \eta(p)\psi\big(\|z\|\big)dp + \int_a^\tau \frac{p^{\omega-1}}{\Gamma(\mu)}\left(\frac{\tau^\omega - p^\omega}{\omega}\right)^{\mu-1}\left(\frac{p^\omega - a^\omega}{\omega}\right)^{1-\beta+\beta-1}\eta(p)\psi\big(\|z\|\big)dp.
\end{aligned}$$

Using (H_3), for each $\tau \in (a, T]$, we have

$$\begin{aligned}
|\eta(\tau)| &= |g(\tau, z(\tau)) - g(\tau,0) + g(\tau,0)|\, |\eta(\tau)| \\
&\leq |g(\tau, z(\tau)) - g(\tau,0)| + |g(\tau,0)| \\
&\leq A|z(\tau)| + g^* \\
\left|\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}\eta(\tau)\right| &\leq A\left|\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta} z(\tau)\right| + \left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta} g^*, \\
\|\eta(\tau)\|_{\zeta_{1-\beta,\omega}} &\leq Ar + \left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta} g^* \\
&= M.
\end{aligned}$$

Further,

$$\begin{aligned}
|Nz(\tau)| \leq \frac{M\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}}{\Gamma(\mu)}&\Bigg[\left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\tau) \\
&+ K\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_\kappa\left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\xi_\kappa)\Bigg]. \qquad (3.12)
\end{aligned}$$

Upon multiplied by $\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}$ on each side of equation (3.12) we get

$$\begin{aligned}
\left|\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta} Nz(\tau)\right| &\leq \frac{KM}{\Gamma(\mu)}\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_\kappa\left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\xi_\kappa) \\
&\quad + \frac{M}{\Gamma(\mu)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}}\left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^\omega - a^\omega}{\omega}\right)^{\beta-1}\right)(\tau), \\
\|Nz\|_{\zeta_{1-\beta,\omega}} &\leq \frac{KM\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^\omega - a^\omega}{\omega}\right)^{\mu+\beta-1} \\
&\quad + \frac{M\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta}\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{\beta-1} \\
&\leq \frac{KM\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^\omega - a^\omega}{\omega}\right)^{\mu+\beta-1} \\
&\quad + \frac{M\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}} \\
&\leq \frac{KAr\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^\omega - a^\omega}{\omega}\right)^{\mu+\beta-1} \\
&\quad + \frac{K\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta} g^*\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}} \\
&\quad \times\sum_{\kappa=1}^{m}\eta_\kappa\left(\frac{{\xi_\kappa}^\omega - a^\omega}{\omega}\right)^{\mu+\beta-1} + \frac{Ar\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}} \\
&\quad + \frac{\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\left(\frac{\tau^\omega - a^\omega}{\omega}\right)^{1-\beta} g^*\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}} \\
\|Nz\|_{\zeta_{1-\beta,\omega}} &\leq r. \qquad (3.13)
\end{aligned}$$

Thus $Nz \in F$, $N(F)$ is uniformly bounded. Now, we prove that N map bounded set into equicontinuous set of $\zeta_{1-\gamma,\omega}$ that means $N(F)$ is equicontinuous. Let $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$.

For any $z \in F$

$$
\begin{aligned}
|Nz(\tau_1) - Nz(\tau_2)| = & \frac{K}{\Gamma(\mu)}\left[\left(\frac{{\tau_1}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} - \left(\frac{{\tau_2}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1}\right] \\
& \times \sum_{\kappa=1}^{m} \eta_\kappa \int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} |g(p, z(p))| dp \\
& + \frac{1}{\Gamma(\mu)}\int_a^{\tau_1} p^{\omega-1}\left[\left(\frac{{\tau_1}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} - \left(\frac{{\tau_2}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\right] \\
& \times |g(p, z(p))| dp + \frac{1}{\Gamma(\mu)}\int_{\tau_1}^{\tau_2} p^{\omega-1}\left(\frac{{\tau_2}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} |g(p, z(p))| dp, \\
& \leq \frac{K}{\Gamma(\mu)}\left[\left(\frac{{\tau_1}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} - \left(\frac{{\tau_2}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1}\right] \\
& \times \sum_{\kappa=1}^{m} \eta_\kappa \int_a^{\xi_\kappa} p^{\omega-1}\left(\frac{{\xi_\kappa}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} \eta(p)\ \psi(\|z\|) dp \\
& + \frac{1}{\Gamma(\mu)}\int_a^{\tau_1} p^{\omega-1}\left[\left(\frac{{\tau_1}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} - \left(\frac{{\tau_2}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\right] \\
& \times \eta(p)\ \psi(\|z\|) dp + \frac{1}{\Gamma(\mu)}\int_{\tau_1}^{\tau_2} p^{\omega-1}\left(\frac{{\tau_2}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} \eta(p)\ \psi(\|z\|) dp \\
& \leq \frac{K}{\Gamma(\mu)}\left[\left(\frac{{\tau_1}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} - \left(\frac{{\tau_2}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1}\right]\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}} \\
& \times \sum_{\kappa=1}^{m} \eta_\kappa \left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1}\right)(\xi_\kappa) \\
& + \frac{1}{\Gamma(\mu)}\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}}\int_a^{\tau_1} p^{\omega-1}\eta(p)\ \psi(\|z\|) \\
& \times \left[\left(\frac{{\tau_1}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} - \left(\frac{{\tau_2}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\right] dp \\
& + \frac{1}{\Gamma(\mu)}\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}\left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1}\right)(\tau_2) \\
& \leq \frac{K\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\left[\left(\frac{{\tau_1}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1} - \left(\frac{{\tau_2}^{\omega} - a^{\omega}}{\omega}\right)^{\beta-1}\right]\psi(\|z\|)\|\eta\|_{\zeta_{1-\beta,\omega}} \\
& \times \sum_{\kappa=1}^{m} \eta_\kappa \left(\frac{{\xi_\kappa}^{\omega} - a^{\omega}}{\omega}\right)^{\mu+\beta-1} + \frac{1}{\Gamma(\mu)}\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}} \\
& \times \int_a^{\tau_1} p^{\omega-1}\left[\left(\frac{{\tau_1}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1} - \left(\frac{{\tau_2}^{\omega} - p^{\omega}}{\omega}\right)^{\mu-1}\right]\eta(p)\ \psi(\|z\|) dp \\
& + \frac{\Gamma(\beta)}{\Gamma(\mu)\Gamma(\mu+\beta)}\psi(\|z\|)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}\left(\frac{{\tau_2}^{\omega} - {\tau_1}^{\omega}}{\omega}\right)^{\mu+\beta-1}.
\end{aligned}
$$

It is easy to see that $\tau_2 \to \tau_1$ the rhs of above inequality is independent on z and goes to 0. Thus, $N(F)$ is equicontinuous. So followed by Ascoli Arzela's theorem the compactness of N, we conclude the N is completely continuous.

Lastly, we prove that $\exists$ an open set $G \subseteq \zeta_{1-\beta,\omega}$ with $z \neq \lambda Nz$ for $\lambda \in (0,1)$ and $z \in \partial G$. Let $z \in G$ be any solution of $z = \lambda Nz, \quad \lambda \in (0,1)$.

Then

$$
\begin{aligned}
|z(\tau)| &= \lambda |Nz(\tau)| \\
|z(\tau)| &\leq \frac{K\lambda}{\Gamma(\mu)}\left(\frac{\tau^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}\sum_{\kappa=1}^{m}\eta_{\kappa}\int_{a}^{\xi_{\kappa}} p^{\omega-1}\left(\frac{{\xi_{\kappa}}^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}|g(p,z(p))|dp \\
&\quad + \frac{\lambda}{\Gamma(\mu)}\int_{a}^{\tau} p^{\omega-1}\left(\frac{\tau^{\omega}-p^{\omega}}{\omega}\right)^{\mu-1}|g(p,z(p))|dp.
\end{aligned}
$$

Adjusting $\left(\frac{p^{\omega}-a^{\omega}}{\omega}\right)^{1-\beta+\beta-1}$ under integral we get

$$
\begin{aligned}
|z(\tau)| \leq{}& \frac{K}{2\Gamma(\mu)}\psi\big(\|z\|\big)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_{\kappa}\left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}\right)(\xi_{\kappa}) \\
&+ \frac{1}{2\Gamma(\mu)}\psi\big(\|z\|\big)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}\left({}^{\omega}I^{\mu}_{a_+}\left(\frac{p^{\omega}-a^{\omega}}{\omega}\right)^{\beta-1}\right)(\tau) \\
\leq{}& \frac{K\Gamma(\beta)}{2\Gamma(\mu)\Gamma(\mu+\beta)}\psi\big(\|z\|\big)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_{\kappa}\left(\frac{{\xi_{\kappa}}^{\omega}-a^{\omega}}{\omega}\right)^{\mu+\beta-1} \\
&+ \frac{\Gamma(\beta)}{2\Gamma(\mu)\Gamma(\mu+\beta)}\psi\big(\|z\|\big)\|\eta\|_{\zeta_{1-\beta,\omega}}\left(\frac{\tau^{\omega}-a^{\omega}}{\omega}\right)^{\mu+\beta-1}.
\end{aligned}
$$

Multiply on both sides by $\left(\frac{\tau^{\omega}-a^{\omega}}{\omega}\right)^{1-\beta}$, we get

$$
\begin{aligned}
\left|\left(\frac{\tau^{\omega}-a^{\omega}}{\omega}\right)^{1-\beta} z(\tau)\right| \leq{}& \frac{K\Gamma(\beta)}{2\Gamma(\mu)\Gamma(\mu+\beta)}\psi\big(\|z\|\big)\|\eta\|_{\zeta_{1-\beta,\omega}}\sum_{\kappa=1}^{m}\eta_{\kappa}\left(\frac{{\xi_{\kappa}}^{\omega}-a^{\omega}}{\omega}\right)^{\mu+\beta-1} \\
&+ \frac{\Gamma(\beta)}{2\Gamma(\mu)\Gamma(\mu+\beta)}\psi\big(\|z\|\big)\ \ \|\eta\|_{\zeta_{1-\beta,\omega}}\left(\frac{\tau^{\omega}-a^{\omega}}{\omega}\right)^{\mu} \\
\|z\|_{\zeta_{1-\beta,\omega}} \leq{}& \psi\big(\|z\|\big)\ \ \|\eta\|\nu.
\end{aligned}
$$

By rearranging

$$
\frac{\|z\|}{\psi\big(\|z\|\big)\|\eta\|\nu} \leq 1,
$$

where,

$$
\nu = \frac{\Gamma(\beta)}{2\Gamma(\mu)\Gamma(\mu+\beta)}\left(K\sum_{\kappa=1}^{m}\eta_{\kappa}\left(\frac{{\xi_{\kappa}}^{\omega}-a^{\omega}}{\omega}\right)^{\mu+\beta-1}+\left(\frac{\tau^{\omega}-a^{\omega}}{\omega}\right)^{\mu}\right). \tag{3.14}
$$

In view of (H_4), $\exists\, Q$ such that $\|z\| \neq Q$. That is, any solution z of NTVP (1.3) satisfies $\|z\| \neq Q$. Therefore, $G = \{z \in G : \|z\| < Q\}$. Thus, the Leray-Schauder nonlinear alternative gurantees that equation (3.8) has a fixed point on ∂G, which is solution of NTVP (1.3) in $\zeta_{1-\beta,\omega}$. □

4 An applications

Example 4.1. Consider the NTVP

$$
\begin{cases}
\left({}^{\omega}D^{\mu,\delta}_{a_+}z\right)(\tau) = g\big(\tau, z(\tau)\big); & \tau \in (1,2], \\
\left({}^{\omega}I^{1-\beta}_{a_+}z\right)(2+) = 3z\left(\frac{8}{7}\right), & \beta = \mu + \delta(1-\mu).
\end{cases} \tag{4.1}
$$

Denote $\mu = \frac{1}{2}, \delta = \frac{3}{4}$ gives $\beta = \frac{7}{8}$. Let $\omega = \frac{1}{2} = 0.5 > 0$ and set $g(\tau, z) = \sin\left(\frac{1}{3}|z|\right)$. We can see that $g(\tau, z) \in D_{\frac{1}{8},\frac{1}{2}}[1,2]$. Moreover, some computation gives us

$$|K| = \left| \left[\Gamma(0.875) - 2\left(\frac{(8/7)^{0.5} - 1}{0.5} \right)^{-1/8} \right]^{-1} \right| \approx 0.3632 < 1,$$

$$\phi = \frac{K * \Gamma(\frac{7}{8})}{\Gamma(0.5)\Gamma(\frac{7}{8}+1)} \times 3\left(\frac{(8/7)^{0.5} - 1}{0.5} \right)^{0.375} + \frac{1}{\Gamma(1/2)}\left(\frac{2^{1/2} - 1}{0.5} \right)^{0.5} \approx 0.8476 < 1,$$

$$\nu = \frac{\Gamma(0.875)}{2 * \Gamma(0.5)\Gamma(1.375)} \left(K \times 3\left(\frac{(8/7)^{0.5} - 1}{0.5} \right)^{0.375} \right) + \left(\frac{2^{1/2} - 1}{0.5} \right)^{0.5} \approx 0.4872 < 1.$$

All the considerations of Theorems 3.2, 3.3 are satisfied with $|K| \approx 0.3632$, $\phi \approx 0.8476$ and $\nu \approx 0.4872$. Therefore, NTVP (4.1) has at least one solution in $\zeta_{\frac{1}{8},\frac{1}{2}}[1,2]$.

Example 4.2. Consider the NTVP

$$\begin{cases} \left({}^{\omega}D^{\mu,\delta}_{a_+} z\right)(\tau) = g\left(\tau, z(\tau)\right); & \tau \in (1,2], \\ \left({}^{\omega}I^{1-\beta}_{a_+} z\right)(2+) = 8z(\frac{3}{2}). \end{cases} \tag{4.2}$$

Denote $\mu = \frac{3}{5}, \delta = \frac{2}{3}$ and $\omega = \frac{1}{2} = 0.5 > 0$. So $\beta = \frac{5}{6}$ and $\xi = \frac{3}{2}$.
Set $g(\tau, z) = \sin\left(e^{-6}|z|\right)$, for $\tau \in (1,2]$. It is clear to observe $g(\tau, z(\tau)) \in \zeta_{\frac{1}{6},\frac{1}{2}}[1,2]$. Moreover,

$$|K| = \left| \left[\Gamma(0.8334) - \left(8\left(\frac{(3/2)^{0.5} - 1}{0.5} \right)^{-1/6} \right) \right]^{-1} \right| \approx 0.1248 < 1,$$

$$\phi = \frac{K * \Gamma(\frac{5}{6})}{\Gamma(1.5)\Gamma(\frac{5}{6}+1)} \times 8\left(\frac{(3/2)^{0.5} - 1}{0.5} \right)^{1.334} + \frac{1}{\Gamma(3/5)}\left(\frac{2^{1/2} - 1}{0.5} \right)^{0.6} \approx 0.6285 < 1,$$

$$\nu = \frac{\Gamma(0.8334)}{2 * \Gamma(1.5)\Gamma(3/5 + 5/6)} \left(K \times 8\left(\frac{(3/2)^{0.5} - 1}{0.5} \right)^{1.334} \right) + \left(\frac{2^{1/2} - 1}{0.5} \right)^{0.6} \approx 0.6840 < 1.$$

Corresponding to the values of $|K|$, ϕ and ν, the NTVP (4.2) holds assumptions Theorems 3.2, 3.3. Thus, NTVP (4.2) has at least one solution in $\zeta_{\frac{1}{6},\frac{1}{2}}[1,2]$.

5 Conclusions

The NTVP for the class of H-K differential problem is extensively studied. The terminal condition is considered in a nonlocal sense, and the conclusions are driven with fixed point theory. The results obtained here in this study generalised the existing results in the research literature. With some appropriate examples, our theoretical findings are supported.

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