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SCHURER TYPE MODIFICATION OF LUPAŞ-JAIN OPERATORS AND ITS PROPERTIES

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Abstract In this manuscript, Schurer Type Modification of Lupaş-Jain operators are investigated, they are called as Lupaş-Jain-Schurer (LJS-) operators. An estimation of these operators, using moduli of smoothness of the first and second order are discussed. Also, a Voronvskaja-type result is present. Furthermore, discussed these LJS-operators preserve a modulus of continuity. When the attached function is convex and non-decreasing, monotonicity of these sequence of positive linear operators discussed at the end of the paper.

1 Introduction

In 2015, Patel and Mishra [1] modified Jain operators as a variant of the Lupaş operators [2] defined by

$$P_n^{[\mu]}(g,y) = \sum_{k=0}^{\infty} \frac{(ny+k\mu)_k}{2^k k!} 2^{-(ny+k\mu)} g\left(\frac{k}{n}\right), \quad y \geq 0, \quad g : [0,\infty) \to \mathbb{R}, \tag{1.1}$$

where $(ny+k\mu)_0 = 1$, $(ny+k\mu)_1 = ny$ and $(ny+k\mu)_k = ny(ny+k\mu+1)(ny+k\mu+2)\ldots(ny+k\mu+k-1)$, $k \geq 2$, by using analogous Abel and Jensen combinatorial formulas for factorial powers (see [3]). In [4], the authors modified the operators (1.1) into following sense

$$L_n^{[\mu]}(g,y) = \sum_{k=0}^{\infty} \frac{ny(ny+1+k\mu)_{k-1}}{2^k k!} 2^{-(ny+k\mu)} g\left(\frac{k}{n}\right), \tag{1.2}$$

and $L_n^{[\mu]}(g,0) = g(0)$ for real valued bounded functions g on $[0,\infty)$, where $0 \leq \mu < 1$ and μ depending only on n. The authors called these operators (1.2) as Lupaş-Jain operators.
The following operators was introduced by Schurer [5] in year 1965

$$S_{n,q}(g,y) = e^{-(n+q)y} \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right) \frac{(n+q)^k y^k}{k!}, \tag{1.3}$$

where $y \in [0,c]$, $c > 0$, $n \in \mathbb{N}$, $q \geq 0$ and g is real valued and bounded function on $[0,\infty)$. In particular, $q = 0$, the operators (1.3) reduced to the known Szász-Mirakjan operators. Many other generalization Szász-Mirakjan operators are found in [6, 7, 8, 9, 10, 11] with references therein. Schurer's type generalization of many other operators can be found in [12, 13, 14, 15]. Motivated by these works, Schurer's generalization for the Lupaş-Jain operators (1.2) is defined as follow: for a fixed $q = 0, 1, 2, \ldots$,

$$L_{n,q}^{\mu}(g,y) = \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right) \frac{(n+q)y((n+q)y+1+k\mu)_{k-1}}{2^k k!} 2^{-((n+q)y+k\mu)}, y \in [0,\infty) \tag{1.4}$$

and $L_{n,q}^{\mu}(g,0) = g(0)$, for $g \in C_B([0,\infty))$, where $C_B([0,\infty))$ is the set of all continuous and bounded function on $[0,\infty)$, $\mu \in [0,1)$. One can call $L_{n,q}^{\mu}$ as Lupaş-Jain-Schurer operators. For $q = 0$, the operators (1.4) reduce to the operators (1.2). The operators defined by equation (1.4),

maps $C_B([0,\infty))$ into $C_B([0,\infty))$. With the help to the moduli of continuity of first and second order, we have obtain the rate of convergence for the function $g \in C_B([0,\infty))$. A Voronovskaja-type result are established. The preservation properties of these operators $L^{\mu}_{n,q}$ for a general modulus of continuity is established. When the function g is convex and non-decreasing, the monotonicity of the sequence of the operators $L^{\mu}_{n,q}$, with respect to n is discussed.
Now, denoting $e_j(t) = t^j, j \in \mathbb{N} \cup \{0\}$ and $\phi^j_y(t) := (t-y)^j$, $j \in \mathbb{N}$. For the Lupaş-Jain operators, we have

Lemma 1.1 ([4]). *For the operators L^{μ}_n given by (1.2), we have*

(i) $L^{\mu}_n(1,y) = 1;$

(ii) $L^{\mu}_n(t,y) = \dfrac{y}{1-\mu};$

(iii) $L^{\mu}_n(t^2,y) = \dfrac{y^2}{(1-\mu)^2} + \dfrac{2y}{n(1-\mu)^3};$

(iv) $L^{\mu}_n(t^3,y) = \dfrac{y^3}{(1-\mu)^3} + \dfrac{6y^2}{n(1-\mu)^4} + \dfrac{6(1+\mu)y}{n^2(1-\mu)^5};$

(v) $L^{\mu}_n(t^4,y) = \dfrac{y^4}{(1-\mu)^4} + \dfrac{12y^3}{n(1-\mu)^5} + \dfrac{12(2\mu+3)y^2}{n^2(1-\mu)^6} + \dfrac{2(13\mu^2+34\mu+13)y}{n^3(1-\mu)^7}.$

In the following Lemmas 1.1, the moments and central moments of the operators $L^{\mu}_{n,q}$ are established. We omit the calculation as it is straightforward.

Lemma 1.2. *For the Lupaş-Jain-Schurer operators defined by (1.4), We have*

$$L^{\mu}_{n,q}(e_j,y) = L^{\mu}_n\left(e_j, \left(\frac{n+q}{n}\right)y\right), j = 0,1,\dots.$$

Lemma 1.3. *For the Lupaş-Jain-Schurer operators defined by (1.4), We have*

$$L^{\mu}_{n,q}(\phi^1_y,y) = \left(\mu+\frac{q}{n}\right)\frac{y}{1-\mu};$$

$$L^{\mu}_{n,q}(\phi^2_y,y) = \left(\mu+\frac{q}{n}\right)^2\frac{y^2}{(1-\mu)^2} + \frac{2y}{n(1-\mu)^3}\left(1+\frac{q}{n}\right);$$

$$L^{\mu}_{n,q}(\phi^3_y,y) = \left(\frac{q}{n}+\mu\right)^3\frac{y^3}{(1-\mu)^3} + \left(1+\frac{q}{n}\right)\left(\frac{q}{n}+\mu\right)\frac{6y^2}{n(1-\mu)^4} + \frac{6(1+\mu)}{n^2(1-\mu)^5}\left(1+\frac{q}{n}\right)y$$

$$\begin{aligned}L^{\mu}_{n,q}(\phi^4_y,y) = {} & \frac{y^4}{(1-\mu)^4}\left(\frac{q}{n}+\mu\right)^3 + \frac{12y^3}{n(1-\mu)^5}\left(1+\frac{q}{n}\right)\left(\frac{q}{n}+\mu\right)^2; \\ & + \frac{12y^2}{n^2(1-\mu)^6}\left(1+2\mu^2+2\mu+3\frac{q}{n}+2\mu\frac{q}{n}\right)\left(1+\frac{q}{n}\right) \\ & + \frac{2y\left(13\mu^2+34\mu+13\right)}{n^3(1-\mu)^7}\left(1+\frac{q}{n}\right).\end{aligned}$$

2 Rate of Convergence

The aim of this section is the extend the results of Patel and Mishra [1] for the Lupaş-Jain-Schurer operators. To discussed this, let us recall some notations and terminology that will be help us to discussed the rate of convergence. Consider

$$C_B([0,\infty)) = \{g : [0,\infty] \to \mathbb{R} : g \text{ is bounded and continuos on } [0,\infty)\}.$$

Note that $C_B([0,\infty))$ is norm linear space with the norm $\|g\| = \sup_{y\in[0,\infty)}|g(y)|$. for $g \in C_B([0,\infty))$. Consider

$$UC_B([0,\infty)) = \{g : [0,\infty] \to \mathbb{R} : g \text{ is bounded and uniformly continuos on } [0,\infty)\}.$$

Let g be bounded, real valued function defined on $[0,\infty)$ and $\delta>0$. Define

$$\omega_1(g,\delta)=\sup_{|h|\le\delta}\sup_{y,y+h\in[0,\infty)}|g(y+h)-g(y)|$$

and

$$\omega_2(g,\delta)=\sup_{|h|\le\delta}\sup_{y,y+h,y+2h\in[0,\infty)}|g(y+2h)-2g(y+h)+g(y)|.$$

The ω_1 and ω_2 are called first and second moduli of continuity of first and second order respectively. In [12, p. 266, Lemma 5.1.1], the following property of the modulus of smoothness is elaborated.

Remark 2.1. If $g\in UC_B([0,\infty))$, then $\lim_{\delta\to 0^+}\omega_k(g,\delta)=0$ for $k=1,2$.

Define

$$K(g,\delta)=\inf\{h\in C_B^2[0,\infty)\ :\ \|g-h\|+\delta\|h''\|\},\ \text{for } \delta>0.$$

The functional $K(g,\delta)$ is known as Peetre's K-functional. Denote $C_B^2([0,\infty))$ as the set of all function $g\in C_B([0,\infty))$ such that g',g'' are in $C_B([0,\infty))$. The following inequities for the modulus of smoothness and the K-functional of an $g\in C_B([0,\infty))$ are established in [16, p. 177, Theorem 2.4].

$$d_1\omega_2(g,\delta)\le K(g,\delta^2)\le d_2\omega_2(g,\delta) \tag{2.1}$$

for some positive constant d_1 and d_2. In the following theorem quantitative estimation of the sequence operators $\{L_{n,q}^{[\mu_n]}\}_{n\ge1}$ is discussed.

Theorem 2.2. *Let $p\in\mathbb{N}\cup\{0\}$ be fixed, $\mu\in[0,1)$ and $g\in C_B([0,\infty))$. Then, for each $y\in(0,\infty)$, one has*

$$\left|L_{n,q}^{\mu}(g,y)-g(y)\right|\le\omega_1\left(g,\left(\mu+\frac{q}{n}\right)\frac{y}{1-\mu}\right)+C\omega_2\left(g,\delta_{n,q}^{\mu}(y)\right), \tag{2.2}$$

where $C>0$ is a constant and

$$\delta_{n,q}^{\mu}(y):=\frac{1}{2}\sqrt{\left(\mu+\frac{q}{n}\right)^2\frac{y^2}{(1-\mu)^2}+\frac{y}{n(1-\mu)^3}\left(1+\frac{q}{n}\right)}. \tag{2.3}$$

Proof. For $f\in C_B([0,\infty))$ and $n\in\mathbb{N}$, defined the following auxiliary operators

$$\bar{L}_{n,q}^{\mu}(g,y):=L_{n,q}^{\mu}(g,y)+g(y)-g\left(\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}\right), \tag{2.4}$$

We note that, $\bar{L}_{n,q}^{\mu}$ are linear and positive. Also, they preserves the linear functions. Let $h\in C_b^2[0,\infty)$. For any $y\in[0,\infty)$, using Taylor's formula one has

$$h(t)=h(y)+h'(y)(t-y)+\int_t^y(t-u)h''(u)du \tag{2.5}$$

for $t\in[0,\infty)$. Applying the operators $\bar{L}_{n,q}^{\mu}$ both side, we obtain

$$\bar{L}_{n,q}^{\mu}(h,y)-h(y)=h'(y)\bar{L}_{n,q}^{\mu}(t-y,y)+\bar{L}_{n,q}^{\mu}\left(\int_t^y(t-u)h''(u)du,y\right) \tag{2.6}$$

Using the definition of $\bar{L}_{n,q}^{\mu}$ and $g(t)=\int_t^y(t-u)h''(u)du$, the above equality gives

$$\bar{L}_{n,q}^{\mu}(h,y)-h(y)=L_{n,q}^{\mu}\left(\int_t^y(t-u)h''(u)du,y\right)-\int_y^{\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}}\left[\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}-u\right]h''(u)du.$$

Using the fact

$$\left|\int_y^{\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}}\left[\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}-u\right]g''(u)du\right|\le\frac{1}{2}\left(L_{n,q}^{\mu}(t-y,y)\right)^2\|h''\|,$$

By Lemma 1.3, we obtain

$$\begin{aligned}
\left|\bar{L}^{\mu}_{n,q}(h,y)-h(y)\right| &\le L^{\mu}_{n,q}\left(\left|\int_t^y (t-u)h''(u)du\right|,y\right)\\
&\quad+\left|\int_y^{\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}}\left[\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}-u\right]h''(u)du\right|\\
&\le \frac{\|h''\|}{2}\left[L^{\mu}_{n,q}(\phi_y^2,y)+\left(L^{\mu}_{n,q}(\phi_y^1,y)\right)^2\right]\\
&=\|h''\|\left[\left(\mu+\frac{q}{n}\right)^2\frac{y^2}{(1-\mu)^2}+\frac{y}{n(1-\mu)^3}\left(1+\frac{q}{n}\right)\right]. \qquad (2.7)
\end{aligned}$$

From $\bar{L}^{\mu}_{n,q}$ and using Lemma 1.2, For any $g\in C_B([0,\infty))$, we achieve

$$\left|\bar{L}^{\mu}_{n,q}(g,y)\right|\le|L^{\mu}_{n,q}(g,y)|+2\|g\|\le 3\|g\|. \qquad (2.8)$$

Using (2.4), (2.7) and (2.8), for $g,h\in C_B([0,\infty))$ one has

$$\begin{aligned}
\left|L^{\mu}_{n,q}(g,y)-g(y)\right| &\le \left|\bar{L}^{\mu}_{n,q}(g-h,y)-(g-h)(y)\right|+\left|\bar{L}^{\mu}_{n,q}(h,y)-h(y)\right|\\
&\quad+\left|g(y)-g\left(\left(1+\frac{q}{n}\right)\frac{y}{1-\mu}\right)\right|\\
&\le \omega_1\left(g,\left(\mu+\frac{q}{n}\right)\frac{y}{1-\mu}\right)\\
&\quad+4\left\{\|g-h\|+\frac{1}{4}\left[\left(\mu+\frac{q}{n}\right)^2\frac{y^2}{(1-\mu)^2}+\frac{y}{n(1-\mu)^3}\left(1+\frac{q}{n}\right)\right]\|h''\|\right\}.
\end{aligned}$$

Now, taking infimum over all $h\in C_B^2([0,\infty))$ and applying (2.1), we obtain

$$\begin{aligned}
\left|L^{\mu}_{n,q}(g,y)-g(y)\right| &\le \omega_1\left(g,\left(\mu+\frac{q}{n}\right)\frac{y}{1-\mu}\right)+K\left(g,\left(\delta^{\mu}_{n,q}(y)\right)^2\right)\\
&\le \omega_1\left(g,\left(\mu+\frac{q}{n}\right)\frac{y}{1-\mu}\right)+C\omega_2\left(g,\delta^{\mu}_{n,q}(y)\right),
\end{aligned}$$

where $\delta^{\mu}_{n,q}(y)$ is given by (2.3). □

The particular result [17] can be achieve using the above Theorem by putting $q=0$.

3 A Voronovskaja Approximation Result

In [1], Patel and Mishra have discussed the version of L^{μ}_{n} and established following Voronovskaja-type result for operators as

$$\lim_{n\to\infty} n\left(P_n^{[\mu]}(g,y)-g(y)\right)=yg''(y), \qquad y>0, g\in C^2([0,\infty)),$$

where $C^2([0,\infty))$ is denoted by the space of all continuous functions having continuous second order derivative and $0\le\mu_n<1$ is a sequence such that $\lim_{n\to\infty}\mu_n=0$ and $\lim_{n\to\infty}n\mu_n=0$.
In this section is to establish a Voronovskaja approximation result for the LJS operators $L^{\mu}_{n,q}$, $n\in\mathbb{N}$, which is generalized the above result.

Theorem 3.1. *Let $p\in\mathbb{N}\cup\{0\}$ be fixed and $\{\mu_n\}$ be a sequence satisfies $\mu_n\in[0,1)$ with $\lim_{n\to\infty}n\mu_n=0$ and $\lim_{n\to\infty}\mu_n=0$. If $g\in C_B([0,\infty))$, also g has the second order derivative at some $y\in(0,\infty)$, then*

$$\lim_{n\to\infty} n\left\{L^{\mu_n}_{n,q}(g,y)-g(y)\right\}=pyg'(y)+yg''(y).$$

Proof. Applying Taylor's formula on g, we have

$$g(t) = g(y) + g'(y)\phi_y^1(t) + \frac{1}{2}g''(y)\phi_y^2(t) + \eta(t-y)\phi_y^2(t) \tag{3.1}$$

at fixed point $y \in [0,\infty)$, where $\eta(t-y)$ is bounded for all $t \in [0,\infty)$ and $\lim_{t\to y}\eta(t-y) = 0$. Applying the operators $L_{n,q}^{\mu_n}$ to (3.1) gives

$$n\left[L_{n,q}^{\mu_n}(g,y) - g(y)\right] = g'(y)nL_{n,q}^{\mu_n}(\phi_y^1(t),y) + \frac{1}{2}g''(y)nL_{n,q}^{\mu_n}\left(\phi_y^2(t),y\right) + nL_{n,q}^{\mu_n}\left(\eta(t-y)\phi_y^2(t),y\right).$$

Using the conditions $\lim_{n\to\infty}\mu_n = 0$, $\lim_{n\to\infty} n\mu_n = 0$ and Lemma 1.3, we obtain

$$\lim_{n\to\infty} nL_{n,q}^{\mu_n}(\phi_y^1(t),y) = py$$

and

$$\lim_{n\to\infty} nL_{n,q}^{\mu_n}(\phi_y^2(t),y) = 2y.$$

Hence,

$$\lim_{n\to\infty} n\left[L_{n,q}^{\mu_n}(g,y) - g(y)\right] = pyg'(y) + yg''(y) + \lim_{n\to\infty} nL_{n,q}^{\mu_n}\left(\eta(t-y)\phi_y^2(t),y\right).$$

It is enough to prove that $\lim_{n\to\infty} nL_{n,q}^{\mu}\left(\eta(t-y)(t-y)^2,y\right) = 0$. By defining $\eta(0) = 0$ and using $\lim_{t\to y}\eta(t-y) = 0$, we say that η is continuous at y. Therefore, for each $\epsilon > 0$ such that $|\eta(t-y)| < \epsilon$ for all t such that $|t-y| < \delta$.
As $\eta(t-y)$ is bounded on $[0,\infty)$, there exist $M > 0$ such that $|\eta(t-y)| \le M$ for all t. Hence, $|\eta(t-y)| \le M\frac{(t-y)^2}{\delta^2}$, whenever $|t-y| \ge \delta$. Hence, $|\eta(t-y)| \le \epsilon + M\frac{(t-y)^2}{\delta^2}$ for all t. The monotonicity and linearity of $L_{n,q}^{\mu}$ given that

$$L_{n,q}^{\mu_n}\left(\eta(t-y)\phi_y^2(t),y\right) \le \epsilon L_{n,q}^{\mu_n}\left(\phi_y^2(t),y\right) + \frac{M}{\delta^2}L_{n,q}^{\mu_n}\left(\phi_y^4(t),y\right).$$

Using Lemma 1.3, with $\mu = \mu_n$,

$$\lim_{n\to\infty} nL_{n,q}^{\mu_n}\left(\eta(t-y)\phi_y^2(t),y\right) = 0,$$

which completes the proof. □

4 Preserves of Modulus of Continuity

A continuos function $\omega : [0,\infty) \to \mathbb{R}$ satisfied the following conditions

(i) $\omega(x+y) \le \omega(x) + \omega(y)$ for $x, y, x+y \in [0,\infty)$, i.e. ω is semi-additive.

(ii) $\omega(x) \ge \omega(y)$ for $x \ge y > 0$, i.e. ω is non-decreasing.

(iii) $\lim_{x\to 0+}\omega(x) = \omega(0) = 0$, [18, pp. 106]

is called a modulus of continuity. For the Bernstein polynomial preserves the properties of modulus of continuity on $[0,1]$ was discussed by Li [19]. This motivated us to discussed the same results for LJS operator. To establish the proof, we need the Jensen and Abel combinatorial formulas for factorial powers. The following formula is obtain by from the work of Stancu and Occorsio [3, pp.175-176] for the increment -1, respectively

$$(u+v)(u+v+1+m\mu)_{m-1} = \sum_{k=0}^{m}\binom{m}{k} u(u+1+k\mu)_{k-1}v(v+1+(m-k)\mu)_{m-k-1} \tag{4.1}$$

and

$$(u+v+m\mu)_m = \sum_{k=0}^{m}\binom{m}{k}(u+k\mu)_k v(v+1+(m-k)\mu)_{m-k-1}. \tag{4.2}$$

Theorem 4.1. *Let ω be a modulus of continuous function. Then, $L^{\mu}_{n,q}(\omega)$ is a modulus of continuity function, for all n.*

Proof. Take $x, y \in [0, \infty)$ with $x \leq y$. From the definition of $L^{\mu}_{n,q}$, we write

$$L^{\mu}_{n,q}(\omega, y) = \sum_{k=0}^{\infty} \frac{(n+q)y((n+q)y+1+k\mu)_{k-1}}{2^k k!} 2^{-((n+q)y+k\mu)} \omega\left(\frac{k}{n}\right).$$

Replacing $(n+q)x$ and $(n+q)(y-x)$ and u and v, respectively in (4.1), we have

$$(n+q)y((n+q)y+1+m\mu)_{m-1}$$
$$= \sum_{i=0}^{k} \binom{k}{i} (n+q)x((n+q)x+1+i\mu)_{i-1}(n+q)(y-x)((n+q)(y-x)+1+(k-i)\mu)_{k-i-1}, \tag{4.3}$$

which implies

$$L^{\mu}_{n,q}(\omega, y) = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \omega\left(\frac{k}{n}\right)\binom{k}{i} \frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^k k!} 2^{-((n+q)y+k\mu)}$$
$$\times (n+q)(y-x)((n+q)(y-x)+1+(k-i)\mu)_{k-i-1}.$$

By swapping the order of above summations, we have

$$L^{\mu}_{n,q}(\omega, y) = \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \omega\left(\frac{k}{n}\right) \frac{1}{i!(k-i)!} \frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^k} 2^{-((n+q)y+k\mu)}$$
$$\times (n+q)(y-x)((n+q)(y-x)+1+(k-i)\mu)_{k-i-1}. \tag{4.4}$$

Putting $k-i=l$, (4.4) lower down to

$$L^{\mu}_{n,q}(\omega, y) = \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \omega\left(\frac{i+1}{n}\right) \frac{1}{i!l!} \frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^{i+l}} 2^{-((n+q)y+(i+l)\mu)}$$
$$\times (n+q)(y-x)((n+q)(y-x)+1+l\mu)_{l-1}. \tag{4.5}$$

Also, $L^{\mu}_{n,q}(\omega, x)$ can be express as

$$L^{\mu}_{n,q}(\omega, x) = \sum_{i=0}^{\infty} \omega\left(\frac{i}{n}\right) \frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^i i!} 2^{-((n+q)x+i\mu)}$$
$$= \sum_{i=0}^{infty} \omega\left(\frac{i}{n}\right) \frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^i i!} 2^{-((n+q)y+i\mu)} 2^{(n+q)(y-x)}. \tag{4.6}$$

Since,

$$2^{(n+q)(y-x)} = \sum_{l=0}^{\infty} (n+q)(y-x)(n(y-x)+1+l\mu)_{l-1} \frac{2^{-l\mu}}{2^l l!}$$

then, one may write

$$L^{\mu}_{n,q}(\omega, x) = \sum_{i=0}^{infty}\sum_{l=0}^{\infty} \omega\left(\frac{i}{n}\right) \frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^{i+l} i! l!} 2^{-((n+q)y+(i+l)\mu)}$$
$$\times (n+q)(y-x)((n+q)n(y-x)+1+l\mu)_{l-1} \tag{4.7}$$

Subtracting (4.7) from (4.5), we get

$$L^{\mu}_{n,q}(\omega, y) - L^{\mu}_{n,q}(\omega, x) = \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \left[\omega\left(\frac{i+l}{n}\right) - \omega\left(\frac{i}{n}\right)\right] \frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^{i+l} i! l!}$$
$$\times 2^{-((n+q)y+(i+l)\mu)} (n+q)(y-x)((n+q)(y-x)+1+l\mu)_{l-1} \tag{4.8}$$

and using the given condition, ω is a modulus of continuity function, we have

$$\begin{aligned}
L^{\mu}_{n,q}(\omega,y)-L^{\mu}_{n,q}(\omega,x) &\le \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)\frac{(n+q)x((n+q)x+1+i\mu)_{i-1}}{2^{i+l}i!l!}2^{-((n+q)y+(i+l)\mu)}\\
&\times (n+q)(y-x)((n+q)(y-x)+1+l\mu)_{l-1}\\
&=\sum_{i=0}^{\infty}(n+q)x((n+q)x+1+i\mu)_{i-1}\frac{2^{-i\mu}}{2^i i!}\\
&\times\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)\frac{2^{-((n+q)y+l\mu)}}{2^l l!}(n+q)(y-x)((n+q)(y-x)+1+l\mu)_{l-1}\\
&=\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)\frac{2^{-((n+q)(y-x)+l\mu)}}{2^l l!}(n+q)(y-x)((n+q)(y-x)+1+l\mu)_{l-1}\\
&=L^{\mu}_{n,q}(\omega,y-x). \qquad (4.9)
\end{aligned}$$

This proves that $L^{\mu}_{n,q}(\omega)$ satisfying the subjectivity property. Using (4.8), we say that $L^{\mu}_{n,q}(\omega,y)\ge L^{\mu}_{n,q}(\omega,x)$ when $y\ge x$, namely, $L^{\mu}_{n,q}(\omega)$ is non-decreasing because $omega$ is non-decreasing. Also, $\lim_{x\to 0}L^{\mu}_{n,q}(\omega,x)=L^{\mu}_{n,q}(\omega,0)=\omega(0)=0$, which is follows from the definition of $L^{\mu}_{n,q}$. Hence, $L^{\mu}_{n,q}(\omega)$ is a function of modulus of continuity. □

5 The Monotonicity of the sequence of LJS Operators

A continuous function g is said to be convex in $D\subset\mathbb{R}$, if

$$g\left(\sum_{i=1}^{\infty}\alpha_i t_i\right)\le\sum_{i=1}^{n}\alpha_i g\left(t_i\right)$$

for every $t_1,t_2,\dots t_n\in D$ and for every non-negative numbers $\alpha_1,\alpha_2,\dots\alpha_n$ such that $\alpha_1+\alpha_2+\cdots+\alpha_n=1$.

Cheney and Sharma [20] have establish the monotonicity of Szazs-Mirakjan operators of convex function. Erençin *et al.* [21] have prove the same result for the Lupaş operators. In [4], authors present the monotonicity of the Lupaş-Jain. Now, we discussed the monotonicity of the operators $L^{\mu}_{n,q}$ for n, when g is a convex function.

Theorem 5.1. *Let $g:[0,\infty)\to\mathbb{R}$ be a convex function. Then $L^{\mu}_{n,q}(g)$ is non-decreasing in n.*

Proof. Clearly, the result is true for $y=0$. For $y>0$, we have

$$2^y=\sum_{k=0}^{\infty}\frac{y(y+1+k\mu)_{k-1}}{2^k k!}2^{-k\mu}$$

by [4, p. 527] with $\alpha=y$. Using above formula, we get

$$\begin{aligned}
&L^{\mu}_{n,q}(g,y)-L^{\mu}_{n+1,q}(g,y)\\
&=2^y\sum_{k=0}^{\infty}g\left(\frac{k}{n}\right)\frac{(n+q)y((n+q)y+1+k\mu)_{k-1}}{2^k k!}2^{-((n+q+1)y+k\mu)}\\
&-\sum_{k=0}^{\infty}g\left(\frac{k}{n+1}\right)\frac{(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}}{2^k k!}2^{-((n+1+p)y+k\mu)}\\
&=\sum_{l=0}^{\infty}\frac{y(y+1+l\mu)_{l-1}}{2^l l!}2^{-l\mu}\sum_{k=0}^{\infty}g\left(\frac{k}{n}\right)\frac{(n+q)y((n+q)y+1+k\mu)_{k-1}}{2^k k!}2^{-((n+q+1)y+k\mu)}\\
&-\sum_{k=0}^{\infty}g\left(\frac{k}{n+1}\right)\frac{(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}}{2^k k!}2^{-((n+1+p)y+k\mu)}.
\end{aligned}$$

Further,

$$
\begin{aligned}
&L^{\mu}_{n,q}(g,y)-L^{\mu}_{n+1,q}(g,y)\\
&=\sum_{l=0}^{\infty}\frac{y(y+1+l\mu)_{l-1}}{2^l l!}2^{-l\mu}\\
&\times\sum_{k=l}^{\infty}g\left(\frac{k-l}{n}\right)\frac{(n+q)y((n+q)y+1+(k-l)\mu)_{k-l-1}}{2^{k-l}(k-l)!}2^{-((n+q+1)y+(k-l)\mu)}\\
&-\sum_{k=0}^{\infty}g\left(\frac{k}{n+1}\right)\frac{(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}}{2^k k!}2^{-((n+1+p)y+k\mu)}.
\end{aligned}
$$

On swapping order of the above summations, we have

$$
\begin{aligned}
&L^{\mu}_{n,q}(g,y)-L^{\mu}_{n+1,q}(g,y)\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{y(y+1+l\mu)_{l-1}}{l!}g\left(\frac{k-l}{n}\right)\frac{(n+q)y((n+q)y+1+(k-l)\mu)_{k-l-1}}{2^k(k-l)!}2^{-((n+q+1)y+k\mu)}\\
&-\sum_{k=0}^{\infty}g\left(\frac{k}{n+1}\right)\frac{(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}}{2^k k!}2^{-((n+1+p)y+k\mu)}\\
&=\sum_{k=0}^{\infty}\left\{\sum_{l=0}^{k}\frac{y(y+1+(k-l)\mu)_{k-l-1}}{l!}g\left(\frac{l}{n}\right)\frac{(n+q)y((n+q)y+1+l\mu)_{l-1}}{2^k(k-l)!}\right.\\
&\left.-g\left(\frac{k}{n+1}\right)\frac{(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}}{2^k k!}\right\}2^{-((n+1+p)y+k\mu)}. \qquad (5.1)
\end{aligned}
$$

Now, denote

$$
\alpha_l=\binom{k}{l}\frac{y(y+1+(k-l)\mu)_{k-l-1}(n+q)y((n+q)y+1+l\mu)_{l-1}}{(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}}>0
$$

and

$$
t_l:=\frac{l}{n+q}.
$$

Taking $u=(n+q)y$, $v=y$ and $m=k$ in (4.1), one has

$$
\begin{aligned}
&(n+q+1)y\left((n+q+1)y+1+k\mu\right)_{k-1}\\
&=\sum_{l=0}^{k}\binom{k}{l}(n+q)y((n+q)y+1+l\mu)_{l-1}y(y+1+(k-l)\mu)_{k-l-1},
\end{aligned}
$$

which gives

$$
\sum_{l=0}^{k}\alpha_l=1.
$$

Taking $u=(n+q)y+\mu+1$, $v=y$ and $m=k-1$ in (4.2), it has

$$
\begin{aligned}
&((n+q)y+\mu+1+y+(k-1)\mu)_{k-1}=((n+q+1)y+1+k\mu)_{k-1}\\
&=\sum_{l=0}^{k-1}\binom{k-1}{l}((n+q)y+\mu+1+l\mu)_l y(y+1+(k-1-l)\mu)_{k-l-2}.
\end{aligned}
$$

Using the above fact, we can write

$$\begin{aligned}
\sum_{l=0}^{k}\alpha_l t_l &= \frac{\sum_{l=1}^{k}\binom{k}{l}y(y+1+(k-l)\mu)_{k-l-1}(n+q)y((n+q)y+1+l\mu)_{l-1}\frac{l}{n+q}}{(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}} \\
&= \frac{k\sum_{l=0}^{k-1}\binom{k-1}{l}y(y+1+(k-1-l)\mu)_{k-l-2}(n+q)y((n+q)y+1+\mu+l\mu)_l}{(n+q)(n+q+1)y((n+q+1)y+1+k\mu)_{k-1}} \\
&= \frac{k}{n+q+1}\frac{\sum_{l=0}^{k-1}\binom{k-1}{l}y(y+1+(k-1-l)\mu)_{k-l-2}((n+q)y+1+\mu+l\mu)_l}{((n+q+1)y+1+k\mu)_{k-1}} \\
&= \frac{k}{n+q+1}.
\end{aligned}$$

Therefore, using the convexity of g, (5.1) provide that

$$L^{\mu}_{n,q}(g,y) \geq L^{\mu}_{n+1,q}(g,y)$$

for all $n \in \mathbb{N}$, which completes the proof. □

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