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P-OPERATORS ON HILBERT SPACES

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Abstract : A real square matrix A is called a P-matrix if all its principal minors are positive. Using the sign non-reversal property of matrices, the notion of P-matrix has been recently extended by Kannan and Sivakumar to infinite-dimensional Banach spaces relative to a given Schauder basis. Motivated by their work, we discuss P-operators on separable real Hilbert spaces. We also investigate P-operators relative to various orthonormal bases.

1 Introduction

An $n \times n$ real matrix A is said to be a P-matrix [4] if all its principal minors are positive. The study of P-matrices originated in the context of some of the notable classes of matrices, such as positive matrices, M-matrices, and totally positive matrices. But the first systematic study of P-matrices appeared in the work of Fiedler and Ptákk [4]. Since then, researchers have been captivated by this class of matrices. They play an important role in a wide range of applications, including the linear complementarity problem, global univalence of maps, linear differential inclusion problems, interval matrices, and computational complexity [1, 2, 3, 5, 7, 8]. The linear complementarity problem (LCP) is stated as follows : Given an $n \times n$ real matrix A and a vector $q \in \mathbb{R}^n$, the LCP is written as $LCP(A, q)$, and it is defined as to find a vector $x \in \mathbb{R}^n$ such that $x \geq 0$, $Ax + q \geq 0$ and $x^T(Ax + q) = 0$, where the notation $x \geq 0$ denotes each coordinate of the vector x is non-negative. It is shown in [3] that given a real square matrix A, the linear complementarity problem $LCP(A, q)$ has a unique solution for each vector $q \in \mathbb{R}^n$ if and only if A is a P-matrix.

We say that an $n \times n$ matrix A reverses the sign of a vector $x \in \mathbb{R}^n$ if $x_i(Ax)_i \leq 0$ for all $i = 1, 2, 3, \ldots, n$, where x_i denotes the i^{th} coordinate of the vector x. Fiedler and Ptákk [4] have shown that A is a P-matrix if and only if A does not reverse the sign of any non-zero vector. Inspired by this characterization of P-matrices, Kannan and Sivakumar [9] extended the notion of P-operator to infinite-dimensional Banach spaces having a Schauder basis. In this paper, we discuss the notion of P-operator to separable real Hilbert spaces and some results in this setting.

In what follows, we will use separable real Hilbert space $\mathcal{H}$ and the term operator on $\mathcal{H}$ to mean a linear operator from $\mathcal{H}$ into itself. We denote $\mathcal{B}(\mathcal{H})$ for the space of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we denote the adjoint of A by A^*.

2 P-Operators on Hilbert Spaces

Let us begin with the definition of P-operator in Banach spaces introduced by Kannan and Sivakumar [9]. A sequence $\{z_n\}_n$ in a real Banach space X is said to be a Schauder basis for X if for each $x \in X$, there exists a unique sequence of scalars $\{\alpha_n(x)\}_n$ such that $x = \sum_n \alpha_n(x) z_n$. In such a case, we denote $x_n = \alpha_n(x)$, for any natural number n. Throughout the set of natural numbers is the index set, and we write simply $\{\alpha_n(x)\}_n$ instead of $\{\alpha_n(x)\}_{n=1}^{\infty}$.

Definition 2.1. [9] Let X be a Banach space with a Schauder basis. A bounded linear operator $T : X \to X$ is said to be a P-operator relative to the given Schauder basis if for $x \in X$, the inequalities $x_n(Tx)_n \leq 0$ for all n imply that $x = 0$.

It is well-known that a countable orthonormal basis $\mathcal{B} = \{e_n\}_n$ exists for every separable Hilbert space $\mathcal{H}$ such that for any $x \in \mathcal{H}$ we have $x = \sum_n \langle x, e_n \rangle e_n$. If an orthonormal basis is known, say $\{e_n\}_n$, then any orthonormal basis of $\mathcal{H}$ is of the form $\{Ue_n\}_n$ for some unitary operator U on $\mathcal{H}$. We define the P-operator on separable real Hilbert spaces as follows.

Definition 2.2. Let $\mathcal{B} = \{e_n\}_n$ be an orthonormal basis of $\mathcal{H}$. A bounded linear operator T on $\mathcal{H}$ is said to be a P-operator relative to the given orthonormal basis $\mathcal{B}$ if for $x \in \mathcal{H}$, the inequalities

$$\langle x, e_n\rangle\langle Tx, e_n\rangle \leq 0$$

for all n imply that $x = 0$.

Example 2.3. Let ℓ_2 denote the square summable sequence space of real numbers. Let $\mathcal{B} = \{e_n\}_n$ be the standard orthonormal basis of ℓ_2, where e_n denotes the vector whose n^{th} entry is one, and all other entries are zero. Define $T : \ell_2 \to \ell_2$ by

$$T(x_1, x_2, x_3, \ldots) = (\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots),$$

for any $(x_1, x_2, x_3, \ldots) \in \ell_2$ with $\alpha_n > 0$ for all n and $\sup_n |\alpha_n| < \infty$. Then T is a bounded linear operator, and it is a P-operator relative to $\mathcal{B}$.

Example 2.4. The right shift operator T_R and the left shift operator T_L on ℓ_2 are defined by

$$T_R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$$

and

$$T_L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

respectively. The operators T_R and T_L are not P-operators relative to the standard orthonormal basis $\mathcal{B} = \{e_n\}_n$ of ℓ_2. Indeed, the non-zero element $x = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots) \in \ell_2$ satisfies the inequalities $\langle x, e_n\rangle\langle T_R(x), e_n\rangle \leq 0$ and $\langle x, e_n\rangle\langle T_L(x), e_n\rangle \leq 0$, for all n.

Example 2.5. The operators $I + T_R$ and $I + T_L$ on ℓ_2 are P-operators relative to the standard orthonormal basis $\mathcal{B} = \{e_n\}_n$ of ℓ_2, where I is the identity operator on ℓ_2. Note that $I + T_R$ is a bounded linear operator. Suppose for $x = (x_1, x_2, x_3, \ldots) \in \ell_2$, the inequalities $\langle x, e_n\rangle\langle (I + T_R)x, e_n\rangle \leq 0$ for all n. This leads to the inequalities $x_1^2 \leq 0$, $x_{n-1}x_n + x_n^2 \leq 0$, for all $n \geq 2$. From these inequalities, we get that $x_n = 0$, for all n, hence $x = 0$.

Next, to see that $I + T_L$ is a P-operator relative to $\mathcal{B}$, it is noted that $I + T_L$ is a bounded linear operator. Suppose for $x = (x_1, x_2, x_3, \ldots) \neq 0 \in \ell_2$ the inequalities

$$\langle x, e_n\rangle\langle (I + T_L)x, e_n\rangle \leq 0$$

hold for all n. This implies that $x_n^2 + x_n x_{n+1} \leq 0$ for all n. Suppose $x_i = 0$ for some index i, then we get that $x_{i-1}^2 + x_{i-1}x_i \leq 0$, hence $x_n = 0$ for all $n \leq i$. Thus x is of the form $x = (0, 0, \ldots, 0, 0, x_{i+1}, \ldots)$ with $x_n \neq 0$ for all $n \geq i + 1$ satisfying $x_n^2 + x_n x_{n+1} \leq 0$. Thus $|x_n^2| \leq |x_n|\,|x_{n+1}|$ for all $n \geq i + 1$.

Now as $x_n \neq 0$ for all $n \geq i + 1$, we have $|x_n| > 0$ for all $n \geq i + 1$. Thus by dividing the inequalities by $|x_n|$, we get $|x_n| \leq |x_{n+1}|$ for all $n \geq i + 1$. This shows that the absolute values of the components of x are increasing, and hence n^{th} term of x will not converge to 0, hence $x \notin \ell_2$. Thus if $\langle x, e_n\rangle\langle (I + T_L)x, e_n\rangle \leq 0$ for all n, then x must be equal to 0.

We next give a result that says that an operator can be a P-operator relative to several orthonormal bases.

Theorem 2.6. *Let T be a bounded linear operator on $\mathcal{H}$ satisfying $TU = UT$ for a unitary operator U on $\mathcal{H}$. Then T is a P-operator relative to an orthonormal basis $\mathcal{B} = \{e_n\}_n$ of $\mathcal{H}$ if and only if T is a P-operator relative to the orthonormal basis $\mathcal{B}' = \{Ue_n\}_n$ of $\mathcal{H}$.*

Proof. Suppose T is a P-operator relative to the orthonormal basis $\mathcal{B} = \{e_n\}_n$ of $\mathcal{H}$ satisfying $TU = UT$. Suppose $\langle x, Ue_n\rangle\langle Tx, Ue_n\rangle \leq 0$ for all n. Then $\langle U^*x, e_n\rangle\langle U^*Tx, e_n\rangle \leq 0$ for all n, hence $\langle U^*x, e_n\rangle\langle TU^*x, e_n\rangle \leq 0$ for all n, because $TU = UT$. As T is a P-operator relative to the orthonormal basis $\mathcal{B}$, we get $U^*x = 0$, hence $x = 0$. Therefore T is a P-operator relative to the orthonormal basis $\mathcal{B}' = \{Ue_n\}_n$.

On the other hand, assume that T is a P-operator relative to the orthonormal basis $\mathcal{B}' = \{Ue_n\}_n$ of $\mathcal{H}$ satisfying $TU = UT$. Suppose $\langle x, e_n\rangle\langle Tx, e_n\rangle \le 0$ for all n. As U is a unitary operator, we get $\langle x, U^*Ue_n\rangle\langle Tx, U^*Ue_n\rangle \le 0$ for all n, so $\langle Ux, Ue_n\rangle\langle UTx, Ue_n\rangle \le 0$ for all n. Hence $\langle Ux, Ue_n\rangle\langle TUx, Ue_n\rangle \le 0$ for all n. Since T is a P-operator relative to $\mathcal{B}' = \{Ue_n\}_n$, we get $Ux = 0$, hence $x = 0$. Therefore T is a P-operator relative to the orthonormal basis $\mathcal{B} = \{e_n\}_n$. □

The condition $TU = UT$ in Theorem 2.6 cannot be dropped which is illustrated in the example given below. The example also tells that an operator T can be a P-operator relative to one orthonormal basis, whereas the same operator relative to another orthonormal basis may not be a P-operator.

Example 2.7. Define $T : \ell_2 \to \ell_2$ by

$$T(x_1, x_2, x_3, \dots) = (x_1, 2x_1 + x_2, 2x_2 + x_3, \dots)$$

for $x = (x_1, x_2, x_3, \dots) \in \ell_2$. Then T is a bounded linear operator and it is a P-operator relative to the standard orthonormal basis $\mathcal{B} = \{e_n\}_n$ of ℓ_2.

Now consider the unitary operator U on ℓ_2 given by

$$U(x_1, x_2, x_3, \dots) = \Big(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} + \frac{x_4}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} - \frac{x_4}{\sqrt{2}}, \dots\Big),$$

then $U^* = U$ and $UT \neq TU$. The operator T is not a P-operator relative to the orthonormal basis $\mathcal{B}' = \{Ue_n\}$ of $\mathcal{H}$, because the non-zero element $x = (1, -1, 0, 0, \dots) \in \ell_2$ satisfies the inequalities $\langle x, Ue_n\rangle\langle Tx, Ue_n\rangle \le 0$, for all n.

Remark 2.8. Theorem 2.6 tells us that the condition $TU = UT$ is sufficient for the operator T to be a P-operator relative to the orthonormal bases $\mathcal{B} = \{e_n\}_n$ and $\mathcal{B}' = \{Ue_n\}_n$. But it is not a necessary condition. That is, an operator T can be P-operator relative to two orthonormal bases $\mathcal{B} = \{e_n\}_n$ and $\mathcal{B}' = \{Ue_n\}_n$, but it may not satisfy the relation $TU = UT$. The following example shows this fact.

Example 2.9. Define $T : \ell_2 \to \ell_2$ by

$$T(x_1, x_2, x_3, \dots) = (x_1 - x_2, x_1 + x_2, x_3 - x_4, x_3 + x_4, \dots)$$

for $x = (x_1, x_2, x_3, \dots) \in \ell_2$. Then T is a P-operator relative to the standard orthonormal basis $\mathcal{B} = \{e_n\}_n$ of $\mathcal{H}$. To see this, here the operator T is bounded linear. Suppose $\langle x, e_n\rangle\langle Tx, e_n\rangle \le 0$ for all n. Then $x_n(x_n - x_{n+1}) \le 0$ for odd n and $x_n(x_n + x_{n-1}) \le 0$ for even n. Solving these inequalities together will lead to $x = 0$. Hence T is a P-operator relative to $\mathcal{B}$.

Now consider the unitary operator U on $\mathcal{H}$ given by

$$U(x_1, x_2, x_3, \dots) = \Big(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} + \frac{x_4}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} - \frac{x_4}{\sqrt{2}}, \dots\Big),$$

then $U^* = U$ and $\mathcal{B}' = \{Ue_n\}_n$ is the another orthonormal basis of $\mathcal{H}$. Then T is also a P-operator relative to $\mathcal{B}'$. To see this, suppose $\langle x, Ue_n\rangle\langle Tx, Ue_n\rangle \le 0$ for all n. Then $x_n(x_n - x_{n-1}) \le 0$ for odd n and $x_n(x_n + x_{n+1}) \le 0$ for even n. Solving these inequalities together will lead to $x = 0$. Hence T is a P-operator relative to $\mathcal{B}'$. Note that

$$UT(x) = (\sqrt{2}x_1, -\sqrt{2}x_2, \sqrt{2}x_3, \dots)$$

and

$$TU(x) = (x_1, x_2, x_3, \dots)$$

for any $x = (x_1, x_2, x_3, \dots) \in \ell_2$, thus $TU \neq UT$.

Theorem 2.10. *Let $\mathcal{B} = \{e_n\}_n$ be an orthonormal basis of $\mathcal{H}$. Then the following statements hold good :*

(a) T is a P-operator on $\mathcal{H}$ relative to $\mathcal{B}$ if and only if the operator UTU^* is a P-operator relative to $\mathcal{B}' = \{Ue_n\}_n$, for any unitary operator U.

(b) T is a P-operator on $\mathcal{H}$ relative to $\mathcal{B}' = \{Ue_n\}_n$ of $\mathcal{H}$ where U is any unitary operator on $\mathcal{H}$ if and only if the operator U^*TU is a P-operator on $\mathcal{H}$ relative to $\mathcal{B}$.

Proof. (a) Assume that T is a P-operator relative to $\mathcal{B}$. Then we have, if $\langle x, e_n\rangle\langle Tx, e_n\rangle \leq 0$ for all n imply that $x = 0$. Suppose $\langle x, Ue_n\rangle\langle UTU^*x, Ue_n\rangle \leq 0$ for all n. Then

$$\langle U^*x, e_n\rangle\langle TU^*x, e_n\rangle \leq 0$$

for all n. As T is a P-operator relative to $\mathcal{B}$, we get here $U^*x = 0$ and hence $x = 0$. Therefore UTU^* is a P-operator relative to $\mathcal{B}' = \{Ue_n\}_n$.

Conversely, assume that UTU^* is a P-operator relative to the orthonormal basis $\mathcal{B}' = \{Ue_n\}_n$. Suppose $\langle x, e_n\rangle\langle Tx, e_n\rangle \leq 0$ for all n. Then $\langle Ux, Ue_n\rangle\langle UTU^*Ux, Ue_n\rangle \leq 0$ for all n. As UTU^* is a P-operator relative to $\mathcal{B}' = \{Ue_n\}_n$, we get that $x = 0$. Therefore T is a P-operator relative to $\mathcal{B}$.

(b) Assume that T is a P-operator relative to $\mathcal{B}' = \{Ue_n\}_n$. Suppose $\langle x, e_n\rangle\langle U^*TUx, e_n\rangle \leq 0$ for all n. Then $\langle Ux, Ue_n\rangle\langle TUx, Ue_n\rangle \leq 0$ for all n. As T is a P-operator relative to $\mathcal{B}'$, we get that $Ux = 0$ and hence $x = 0$. Therefore U^*TU is a P-operator relative to $\mathcal{B}$.

Conversely, assume that U^*TU is a P-operator relative to $\mathcal{B}$. Suppose that

$$\langle x, Ue_n\rangle\langle Tx, Ue_n\rangle \leq 0$$

for all n. Then $\langle U^*x, e_n\rangle\langle U^*TUU^*x, e_n\rangle \leq 0$ for all n. As U^*TU is a P-operator relative to $\mathcal{B}$, we get that $x = 0$. Therefore T is a P-operator relative to $\mathcal{B}' = \{Ue_n\}_n$. □

Remark 2.11. A bounded linear operator T on $\mathcal{H}$ is called invertible if there is a bounded linear operator S on $\mathcal{H}$ so that ST and TS are the identity operators. We say that S is the inverse of T in this case and it is denoted by T^{-1}. It is observed in [9] that every P-matrix is invertible and its inverse is also a P-matrix. But P-operator does not guarantee its invertibility in infinite-dimensional spaces which is shown in the following example. Moreover, if we have a P-operator which is invertible, then its inverse is also a P-operator.

Example 2.12. Consider the linear operator $T : \ell_2 \to \ell_2$ defined by

$$T(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots),$$

for $x = (x_1, x_2, x_3, \ldots) \in \ell_2$. Then T is a P-operator relative to the standard orthonormal basis of ℓ_2 but T is not invertible.

Theorem 2.13. *Let T be an invertible P-operator on $\mathcal{H}$ relative to an orthonormal basis $\mathcal{B} = \{e_n\}_n$. Then the inverse of T is also a P-operator relative to $\mathcal{B}$.*

Proof. Since T is a P-operator on $\mathcal{H}$ relative to $\mathcal{B}$, we have, if the inequalities $\langle x, e_n\rangle\langle Tx, e_n\rangle \leq 0$ for all n imply $x = 0$. Suppose $\langle y, e_n\rangle\langle T^{-1}y, e_n\rangle \leq 0$ for all n. Hence $\langle Tx, e_n\rangle\langle x, e_n\rangle \leq 0$ for all n, where $x = T^{-1}y$. As T is a P-operator, we get $x = 0$, hence $y = 0$. Thus T^{-1} is a P-operator relative to the orthonormal basis $\mathcal{B}$. □

Theorem 2.14. *Let T be a positive definite operator on $\mathcal{H}$. Then T is a P-operator on $\mathcal{H}$ relative to any orthonormal basis $\mathcal{B} = \{e_n\}_n$ of $\mathcal{H}$.*

Proof. Assume that T is a positive definite operator on $\mathcal{H}$. Then for every $0 \neq x \in \mathcal{H}$, $\langle Tx, x\rangle > 0$. Let x be a non-zero element of $\mathcal{H}$. Then $x = \sum_n x_n e_n$ and $Tx = \sum_n (Tx)_n e_n$. Then $\langle Tx, x\rangle = \sum_n (Tx)_n x_n > 0$. Therefore there exists some m, for which $\langle x, e_m\rangle\langle Tx, e_m\rangle > 0$. Hence if $\langle x, e_n\rangle\langle Tx, e_n\rangle \leq 0$ for all n, imply that $x = 0$. Hence T is a P-operator. □

The converse of the above theorem need not be true which is shown in the following example.

Example 2.15. Let $T : \ell_2 \to \ell_2$ be defined by

$$T(x_1, x_2, x_3, \ldots) = (x_1 - 7x_2, x_2, x_3, \ldots),$$

for $x = (x_1, x_2, x_3, \ldots) \in \ell_2$. Then T is a P-operator on ℓ_2 relative to the standard basis $\mathcal{B}$ of ℓ_2 as $\langle x, e_n\rangle\langle Tx, e_n\rangle \leq 0$ for all n imply that $x = 0$. But T is not a positive definite operator because $\langle Tx, x\rangle = x_1^2 - 7x_1x_2 + x_2^2$ is negative for $x = (1, 1, 0, \ldots) \neq 0$.

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