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ERROR OF APPROXIMATION OF FUNCTIONS BY $C^{\gamma}.T$-MEANS OF ITS FOURIER-LAGUERRE SERIES

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Abstract In this present work, we determine the error of approximation of functions f belonging to $L[0,\infty)$-class by $C^{\gamma}.T(\gamma \geq 1)$-means of its Fourier-Laguerre series at a point $x = 0$. Our results generalize previously known results of Krasniqi [On the degree of approximation of a function by $(C,1)(E,q)$ means of its Fourier-Laguerre series, *Int. J. Anal. Appl.* **1(1)**, 33-39 (2013)], Sonker [Approximation of functions by $(C,2)(E,q)$ means of its Fourier–Laguerre series, *In Proc. ICMS-2014 ISBN: 978-93-5107-261-4*, 125– 128 (2014)] and Sonker [Approximation of functions by $(C^1.T)$means of its Fourier-Laguerre series, *In Proc. ICMS-2014 ISBN: 978-1-61804-267-5* **1(1)**, 122-125 (2014)]. We also discuss some particular cases of $C^{\gamma}.T$-means.

1 Introduction

The Fourier-Laguerre expansion of function $f \in L[0,\infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n \, L_n^{(\alpha)}(x), \tag{1.1}$$

where

$$a_n = \frac{n! \; \alpha!}{(n+\alpha)! \; \Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} \, e^{-x} \, f(x) \, L_n^{(\alpha)}(x) dx \;\text{ and} \tag{1.2}$$

$L_n^{(\alpha)}(x)$ is n^{th} Laguerre polynomial of order $\alpha > -1$, is defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \, \omega^n = \frac{e^{\frac{\omega \, x}{\omega - 1}}}{(1-\omega)^{1+\alpha}}.$$

It is pretended that integral 1.2 exists.
The $(n+1)^{\text{th}}$ partial sum of the Fourier-Laguerre series of 1.1 is defined by

$$s_n(f;x) = \sum_{k=0}^{n} a_k \, L_k^{(\alpha)}(x), \; n \in \mathbb{N}_0. \tag{1.3}$$

Define

$$[t]_n(f;x) = \sum_{k=0}^{n} a_{n,k} \, s_k(f;x), \; n \in \mathbb{N}_0,$$

where $T \equiv (a_{n,k} \geq 0$ for every $n,k)$ is a lower triangular matrix such that $a_{n,-1} = 0, A_{n,k} = \sum_{k=r}^{n} a_{n,k}$ and $A_{n,0} = 1, n \in \mathbb{N}_0$. The Fourier-Laguerre series is called T-summable to s, if $[t]_n(f;x) \to s$ as $n \to \infty$.

If $a_{n,k} = \begin{cases} \frac{n! \, \gamma!}{(n+\gamma)!} \binom{n+\gamma-k-1}{\gamma-1}, & 0 \leq k \leq n, \\ 0\,, & k > n, \end{cases}$ then the matrix T converts to Cesàro matrix of order $\gamma \geq 1$ and denoted by C^γ. The Fourier-Laguerre series is called C^γ-summable to s, if $[C^\gamma]_n(f;x) \to s$ as $n \to \infty$.

The product of C^γ-summable with T-summable defines $C^\gamma.T$-summable. Thus $C^\gamma.T$-summability of sequence $\{s_n(f;x)\}$ denoted by

$$[C^\gamma.T]_n(f;x) = \frac{n!\ \gamma!}{(n+\gamma)!} \sum_{v=0}^{n} \binom{n+\gamma-v-1}{\gamma-1} \sum_{k=0}^{v} a_{v,k} s_k(f;x). \tag{1.4}$$

If $[C^\gamma.T]_n(f;x) \to s$ as $n \to \infty$, then the Fourier-Laguerre series is called $C^\gamma.T$-summable to s. The regularity of T and C^γ methods implies the regularity of the $C^\gamma.T$ method. The following cases are important and particular cases of $C^\gamma.T$ method:

(i) If $a_{v,k} = \frac{p_{v-k}}{P_v}$ where $P_v = \sum_{k=0}^{v} p_k \neq 0$, then $C^\gamma.T$ reduce to $C^\gamma.N_p$ or $(C,\gamma)(N,p_v)$.

(ii) If $a_{v,k} = \frac{1}{(v-k+1)\,log(v+1)}$, then $C^\gamma.T$ reduce to $C^\gamma.H$ or $(C,\gamma)(H,\frac{1}{v+1})$.

(iii) If $a_{v,k} = \frac{p_k}{P_v}$, then $C^\gamma.T$ reduce to $C^\gamma.\overline{N}_p$ or $(C,\gamma)(\overline{N},p_v)$.

(iv) If $a_{v,k} = \frac{1}{(1+q)^v}\binom{v}{k} q^{v-k}$, then $C^\gamma.T$ reduce to $C^\gamma.E^q$ or $(C,\gamma)(E,q)$.

(v) If $a_{v,k} = \frac{p_{v-k}\, q_k}{R_v}$ where $R_v = \sum_{k=0}^{v} p_k\, q_{v-k}$, then $C^\gamma.T$ reduce to $C^\gamma.N_{pq}$ or $(C,\gamma)(N,p,q)$.

(vi) If $a_{v,k} = \frac{1}{2^v}\binom{v}{k}$, then $C^\gamma.T$ reduce to $C^\gamma.E^1$ or $(C,\gamma)(E,1)$.

If we take $\gamma = 1$ in the above cases, then we get

(vii) If $a_{v,k} = \frac{p_{v-k}}{P_v}$ where $P_v = \sum_{k=0}^{v} p_k \neq 0$, then $C^1.T$ reduce to $C^1.N_p$ or $(C,1)(N,p_v)$.

(viii) If $a_{v,k} = \frac{1}{(v-k+1)\,log(v+1)}$, then $C^1.T$ reduce to $C^1.H$ or $(C,1)(H,\frac{1}{v+1})$.

(ix) If $a_{v,k} = \frac{p_k}{P_v}$, then $C^1.T$ reduce to $C^1.\overline{N}_p$ or $(C,1)(\overline{N},p_v)$.

(x) If $a_{v,k} = \frac{1}{(1+q)^v}\binom{v}{k} q^{v-k}$, then $C^1.T$ reduce to $C^1.E^q$ or $(C,1)(E,q)$.

(xi) If $a_{v,k} = \frac{p_{v-k}\, q_k}{R_v}$ where $R_v = \sum_{k=0}^{v} p_k\, q_{v-k}$, then $C^1.T$ reduce to $C^1.N_{pq}$ or $(C,1)(N,p,q)$.

(xii) If $a_{v,k} = \frac{1}{2^v}\binom{v}{k}$, then $C^1.T$ reduce to $C^1.E^1$ or $(C,1)(E,1)$.

where p_v and q_v are non-negative, monotonic and non-increasing sequence of real constants.

Remark 1.1. We consider the series $1 - 2n\sum_{i=1}^{\infty}(-2n+1)^{i-1}$ and matrix $a_{i,k} = \frac{1}{n^i}\binom{i}{k}(n-1)^{i-k}$ for $n \in \mathbb{N}$, then the i^{th} partial sum of the series is given by $s_i = (-2n+1)^i$. It can be seen that the series is not T-summable and also not C^γ-summable (for $\gamma = 1$), but it is $C^\gamma.T$-summable (for $\gamma = 1$). We can observe that product summabilities are more effective than the single summability.

We also use the following notations:

$$\phi(x) = \frac{f(x) - f(0)}{\Gamma(\alpha+1)} \text{ and } L_n^{(\alpha)}(0) = \frac{(n+\alpha)!}{\alpha!\ n!}.$$

2 Known Results

Many investigators have analyzed the problem of approximation of a function using single or product means of its Fourier-Laguerre series at a point $x = 0$. Some authors like Gupta [3], Singh [12, 13], Beohar and Jadiya [1, 2], Nigam and Sharma [9], Lal and Nigam [6], Singh and Saini [11, 14] and Sahani et al. [10] have used the single summability method to obtain error of approximation. Gupta [3], Singh [13], and Beohar and Jadiya [1] have got the result using Cesàro mean of order $k > \alpha + 1/2$. In 1980, Beohar and Jadiya [2] extended the results of Gupta [3], Singh [13], and Beohar and Jadiya [1] using Cesàro mean of order $k > \alpha > -1$. Further, Nigam and Sharma [9], and Lal and Nigam [6] have obtained interesting results using $(E,1)$ and (N,p,q) means. Also, Singh and Saini [11] have obtained the same result using the Hausdorff mean, and using this result, they generalized previously known results. The

Laguerre functions form an orthogonal basis for $L_2[0,\infty)$-space, which successively defines the Fourier-Laguerre series. It has also been shown that Laguerre's polynomial theory directly solves the problem of determining Fourier-Laguerre approximations for a large class of delay systems. Moreover, these findings are necessary for studying the regular order of identification as a standard method for identifying infinite-dimensional systems [7]. Recently, Singh and Saini [14] extended this study to generalized Laguerre polynomials to obtain an approximation of functions $f \in L[0,\infty)$ using the Cesàro mean of its Fourier-Laguerre series for $x > 0$. In 2020, Sahani et al. [10] have obtained error of approximation using the Nörlund summability method. As mentioned in Remark 1.1, product summability is more effective than single summability. Thus product summability provides an approximation for a vast class of functions than the single summability. Keeping this vital point in mind, many investigators such as Krasniqi [5], Sonker [15, 16], Khatri and Mishra [4], and Mittal and Singh [8] have obtained error of approximation using different types of product summability methods. In 2013, Krasniqi [5] obtained the the error of approximation using product mean $(C,1)(E,q), q \geq 1$- means of its Fourier-Laguerre series. Sonker [15, 16] derived same results using $(C,2)(E,q)$ and $C^1.T$- means of its Fourier-Laguerre series, respectively. Krasniqi [5, pp. 35] and Sonker [15, 16] have used $\alpha \in (-1,-1/2)$ in their results. Khatri and Mishra [4] have obtained error of approximation using Harmonic-Euler means. On the other hand, Mittal and Singh [8] have obtained an error of approximation using Matrix-Euler operators.

Remark 2.1. In 2015, Saini and Singh [11, pp. 210, Remark 1] noted that Krasniqi [5, pp. 37] had optimized $\sum_{k=0}^{v}\binom{v}{k}q^k k^{(2\alpha+1)/4}$ by its supreme value $(q+1)^v v^{(2\alpha+1)/4}$ but it is true, if $\alpha > -1/2$. The same error can also be seen in [15].

3 Main Results

The above-mentioned particular cases of $C^\gamma.T$-means, the importance of the product summability method discussed in Remark 1.1 and the observation mentioned in Remark 2.1 motivate us to generalize the above results. In this present work, we analyze the problem of the error of approximation of functions f using the product mean $C^\gamma.T$. More precisely, we prove:

Theorem 3.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix satisfy the following conditions:*

(i) $a_{n,k}$ *be a non-negative and non-decreasing with respect to k, for $0 \leq k \leq n$,*

(ii) $\sum_{v=t}^{n} A_{v,v-t} = O(n+1),\ n \in \mathbb{N}_0.$

Then error of approximation of functions $f \in L[0,\infty)$-class by $C^\gamma.T$-means of its Fourier-Laguerre series at a point $x = 0$ by is given by

$$|[C^\gamma.T]_n(0) - f(0)| = o(\xi(n)), \tag{3.1}$$

provided $\phi(t)$ satisfies following conditions:

$$\Phi(t) = \int_0^t e^{-y}|y^\alpha\,\phi(y)|dy = o\left(t^{\alpha+1}\,\xi\left(\frac{1}{t}\right)\right),\ t \to 0, \tag{3.2}$$

$$\int_\eta^n \frac{e^{-y/2}\,|\phi(y)|}{y^{(3-2\alpha)/4}}dy = o\left(\frac{\xi(n)}{n^{(2\alpha+1)/4}}\right), \tag{3.3}$$

$$\int_n^\infty \frac{e^{-y/2}\,|\phi(y)|}{y^{(1-3\alpha)/3}}dy = o(\xi(n)),\ n \to \infty, \tag{3.4}$$

where η is a fixed positive number, $\alpha > -1/2$ and $\xi(t)$ is a positive monotonically increasing function such that $\xi(t) \to \infty$ as $t \to \infty$.

Here few lemmas are given, which are useful to prove our theorems:

Lemma 3.2. *Let ϵ be a fixed positive constant and α be an arbitrary real number. Then*

$$L_n^{(\alpha)}(x) = \begin{cases} O(n^{\alpha}), & 0 \le x \le 1/n, \\ O(x^{-(2\alpha+1)/4}\, n^{(2\alpha-1)/4}), & 1/n \le x \le \epsilon, \end{cases} \quad \text{as } n \to \infty.$$

The proof is given in [17, pp. 177, Theorem 7.6.4] .

Lemma 3.3. *Let ρ and α be arbitrary real numbers,* $0 < \eta < 4$ *and* $\omega > 0$. *Then*

$$max\, e^{(-x/2)} x^{\rho} |L_n^{(\alpha)}(x)| = O(n^{Q}),$$

where

$$Q = \begin{cases} max\left(\rho - \frac{1}{2}, \frac{\alpha}{2} - \frac{1}{4}\right), & \omega \le x \le (4-\eta)n, \\ max\left(\rho - \frac{1}{3}, \frac{\alpha}{2} - \frac{1}{4}\right), & x > n. \end{cases}$$

The proof is given in [17, pp. 241, Theorem 8.91.7] .

Proof of Theorem 3.1. We have

$$\begin{aligned} s_n(0) &= \sum_{k=0}^{n} a_k\, L_k^{(\alpha)}(0) \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} y^{\alpha}\, e^{-y}\, f(y)\, L_n^{(\alpha+1)}(y)\, dy \end{aligned} \tag{3.5}$$

Applying T-summability on equation 3.5,

$$\begin{aligned} [T]_n(0) &= \sum_{k=0}^{n} a_{n,k}\, s_k(0) \\ &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{n} a_{n,k} \int_0^{\infty} y^{\alpha}\, e^{-y}\, f(y)\, L_k^{(\alpha+1)}(y)\, dy. \end{aligned} \tag{3.6}$$

Applying (C,γ)-summability on equation 3.6,

$$\begin{aligned} &[C^{\gamma}.T]_n(0) \\ &= \frac{\gamma!\; n!}{(n+\gamma)!\; \Gamma(\alpha+1)} \sum_{v=0}^{n} \binom{n+\gamma-v-1}{\gamma-1} \sum_{k=0}^{v} a_{v,v-k} \int_0^{\infty} y^{\alpha}\, e^{-y}\, f(y)\, L_k^{(\alpha+1)}(y)\, dy, \end{aligned} \tag{3.7}$$

we have

$$\begin{aligned} [C^{\gamma}.T]_n(0) - f(0) &= \frac{\gamma!\; n!}{(n+\gamma)!} \sum_{v=0}^{n} \binom{n+\gamma-v-1}{\gamma-1} \sum_{k=0}^{v} a_{v,v-k} \times \\ &\qquad \int_0^{\infty} y^{\alpha}\, e^{-y}\, \phi(y)\, L_k^{(\alpha+1)}(y)\, dy \\ &= \frac{\gamma!\; n!}{(n+\gamma)!} \sum_{v=0}^{n} \binom{n+\gamma-v-1}{\gamma-1} \sum_{k=0}^{v} a_{v,v-k} \times \\ &\qquad \left[\int_0^{1/n} + \int_{1/n}^{\eta} + \int_{\eta}^{n} + \int_{n}^{\infty}\right] y^{\alpha}\, e^{-y}\, \phi(y)\, L_k^{(\alpha+1)}(y)\, dy \\ &= \sum_{m=1}^{4} J_m \end{aligned} \tag{3.8}$$

Applying property of the orthogonality, condition 3.2 and Lemma 3.2, we have

$$\begin{aligned}
|J_1| &= \left|\frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_0^{1/n} y^{\alpha}\,e^{-y}\,\phi(y)\,L_k^{(\alpha+1)}(y)\,dy\right| \\
&\leq \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_0^{1/n} e^{-y}\,|y^{\alpha}\,\phi(y)|\,|L_k^{(\alpha+1)}(y)|\,dy \\
&= \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\,O\left(k^{\alpha+1}\right)\int_0^{1/n} e^{-y}\,|y^{\alpha}\,\phi(y)|\,dy \\
&= \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\,O\left(n^{\alpha+1}\right)o\left(\frac{\xi(n)}{n^{\alpha+1}}\right) \\
&= o(\xi(n))\frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k} \\
&= o(\xi(n)). \qquad (3.9)
\end{aligned}$$

Applying property of the orthogonality, condition 3.3 and Lemma 3.2, we have

$$\begin{aligned}
|J_2| &= \left|\frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_{1/n}^{\eta} y^{\alpha}\,e^{-y}\,\phi(y)\,L_k^{(\alpha+1)}(y)\,dy\right| \\
&\leq \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_{1/n}^{\eta} e^{-y}\,|y^{\alpha}\,\phi(y)|\,|L_k^{(\alpha+1)}(y)|\,dy \\
&= \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\,O\left(k^{(2\alpha+1)/4}\right)\times \\
&\qquad\qquad \int_{1/n}^{\eta} y^{-(2\alpha+3)/4}\,e^{-y}\,|y^{\alpha}\,\phi(y)|\,dy \\
&= \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\,O\left(k^{(2\alpha+1)/4}\right)o\left(\frac{\xi(n)}{n^{(2\alpha+1)/4}}\right) \\
&= o(\xi(n))\frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k} \\
&= o(\xi(n)). \qquad (3.10)
\end{aligned}$$

as stated in [9, pp. 6] $\int_{1/n}^{\eta} y^{-(2\alpha+3)/4}\,e^{-y}\,|y^{\alpha}\,\phi(y)|\,dy = o\left(\frac{\xi(n)}{n^{(2\alpha+1)/4}}\right)$.

Applying property of the orthogonality, condition 3.3 and Lemma 3.3 (taking $\eta = 3$ and $\alpha + 1$ for α), we have

$$\begin{aligned}
|J_3| &= \left|\frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_{\eta}^{n} y^{\alpha}\,e^{-y}\,\phi(y)\,L_k^{(\alpha+1)}(y)\,dy\right| \\
&\leq \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_{\eta}^{n} \frac{e^{-y/2}\,|\phi(y)|}{y^{(3-2\alpha)/4}}\,\frac{y^{(2\alpha+3)/4}|L_k^{(\alpha+1)}(y)|}{e^{y/2}}\,dy \\
&= \frac{\gamma!\,n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}O\left(k^{(2\alpha+1)/4}\int_{\eta}^{n}\frac{e^{-y/2}\,|\phi(y)|}{y^{(3-2\alpha)/4}}dy\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\gamma!\;n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}O(k^{(2\alpha+1)/4})o\left(n^{-(2\alpha+1)/4}\xi(n)\right)\\
&= o\left(\xi(n)\right)\frac{\gamma!\;n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\\
&= o(\xi(n)). \qquad (3.11)
\end{aligned}
$$

Applying property of the orthogonality, Lemma 3.3 and condition 3.4, we have

$$
\begin{aligned}
|J_4| &= \left|\frac{\gamma!\;n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_{n}^{\infty}y^{\alpha}\,e^{-y}\,\phi(y)\,L_k^{(\alpha+1)}(y)\,dy\right|\\
&\le \frac{\gamma!\;n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\int_{n}^{\infty}\frac{y^{(3\alpha-5)/6}|\phi(y)|}{e^{y/2}}\,\frac{y^{(3\alpha+5)/6}|L_k^{(\alpha+1)}(y)|}{e^{y/2}}dy\\
&= \frac{\gamma!\;n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}O\left(k^{(\alpha+1)/2}\int_{n}^{\infty}\frac{e^{-y/2}y^{(3\alpha-1)/3}|\phi(y)|}{y^{(\alpha+1)/2}}dy\right)\\
&= \frac{\gamma!\;n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}O(k^{(\alpha+1)/2})o\left(n^{-(\alpha+1)/2}\xi(n)\right)\\
&= o\left(\xi(n)\right)\frac{\gamma!\;n!}{(n+\gamma)!}\sum_{v=0}^{n}\binom{n+\gamma-v-1}{\gamma-1}\sum_{k=0}^{v}a_{v,v-k}\\
&= o(\xi(n)). \qquad (3.12)
\end{aligned}
$$

Combining 3.8 – 3.12, we have

$$|[C^{\gamma}.T]_n(0)-f(0)| = o(\xi(n)).$$

This completes the proof of Thoerem 3.1. □

4 Corollaries

Here few corollaries are given, which are derived from our Theorem 3.1.

Corollary 4.1. *If we take* $a_{v,k}=\frac{p_{v-k}}{P_v}$, *where* $P_v=\sum_{k=0}^{v}p_k\neq 0$ *in equation 1.4, then* $C^{\gamma}.T$ *reduce to* $C^{\gamma}.N_p$ *or* $(C,\gamma)(N,p_v)$, *then for* $f\in L[0,\infty)$, *we have*

$$|[C^{\gamma}.N_p]_n(0)-f(0)| = o(\xi(n)).$$

Corollary 4.2. *If we take* $a_{v,k}=\frac{1}{(v-k+1)\,log(v+1)}$ *in equation 1.4, then* $C^{\gamma}.T$ *reduce to* $C^{\gamma}.H$ *or* $(C,\gamma)(H,\frac{1}{v+1})$, *then for* $f\in L[0,\infty)$, *we have*

$$|[C^{\gamma}.H]_n(0)-f(0)| = o(\xi(n)).$$

Corollary 4.3. *If we take* $a_{v,k}=\frac{p_k}{P_v}$ *in equation 1.4, then* $C^{\gamma}.T$ *reduce to* $C^{\gamma}.\overline{N}_p$ *or* $(C,\gamma)(\overline{N},p_v)$, *then for* $f\in L[0,\infty)$, *we have*

$$\left|[C^{\gamma}.\overline{N}_p]_n(0)-f(0)\right| = o(\xi(n)).$$

Corollary 4.4. *If we take* $a_{v,k}=\frac{1}{(1+q)^v}\binom{v}{k}q^{v-k}$ *in equation 1.4, then* $C^{\gamma}.T$ *reduce to* $C^{\gamma}.E^q$ *or* $(C,\gamma)(E,q)$, *then for* $f\in L[0,\infty)$, *we have*

$$|[C^{\gamma}.E^q]_n(0)-f(0)| = o(\xi(n)).$$

Corollary 4.5. *If we take* $a_{v,k}=\frac{p_{v-k}\,q_k}{R_v}$, *where* $R_v=\sum_{k=0}^{v}p_k\,q_{v-k}$ *in equation 1.4, then* $C^{\gamma}.T$ *reduce to* $C^{\gamma}.N_{pq}$ *or* $(C,\gamma)(N,p,q)$, *then for* $f\in L[0,\infty)$, *we have*

$$|[C^{\gamma}.N_{pq}]_n(0)-f(0)| = o(\xi(n)).$$

Corollary 4.6. *If we take* $a_{v,k} = \frac{1}{2^v}\binom{v}{k}$ *in equation 1.4, then* $C^\gamma.T$ *reduce to* $C^\gamma.E^1$ *or* $(C,\gamma)(E,1)$, *then for* $f \in L[0,\infty)$, *we have*

$$\left|[C^\gamma.E^1]_n(0) - f(0)\right| = o(\xi(n)).$$

Remark 4.7. If we take $\gamma = 1$ in the above cases, then we get $C^1.N_p, C^1.H, C^1.\overline{N}_p, (C,1)(E,q)$, $(C,1)(N,p,q)$ and $(C,1)(E,1)$ are also particular cases of the $C^\gamma.T$ method.

5 Conclusion

The results of this paper are aimed to construct the problem of approximation of function f using $C^\gamma.T$-means of its Fourier-Laguerre series in a simpler manner. The followings are the particular cases of the results of this paper :

Remark 5.1. If we take $\gamma = 1$ and replace matrix means T by (E,q) in Theorem 3.1, then $C^\gamma.T$ reduce to $(C,1)(E,q)$, then result of Krasniqi [5] become a particular case of our Theorem 3.1.

Remark 5.2. If we take $\gamma = 2$ and replace matrix means T by (E,q) in Theorem 3.1, then $C^\gamma.T$ reduce to $(C,2)(E,q)$, then result of Sonker [15] become a particular case of our Theorem 3.1.

Remark 5.3. If we take $\gamma = 1$ in Theorem 3.1, then $C^\gamma.T$ reduce to $C^1.T$, then result of Sonker [16] become a particular case of our Theorem 3.1.

6 Conflict of Interest

The authors declare that they have no conflict of interest.

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