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PARTITION AND LOCAL METRIC DIMENSION OF AN EXTENDED ANNIHILATING-IDEAL GRAPH

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Abstract In this paper, we compute the partition dimension and local metric dimension of the extended annihilating-ideal graph $\mathbb{EAG}(R)$ associated to a commutative ring R which is denoted by $dim_P(\mathbb{EAG}(R))$ and $dim_l(\mathbb{EAG}(R))$ respectively. In addition, we characterize $dim_l(\mathbb{EAG}(R))$ for direct product of rings and the ring of integers $\mathbb{Z}_n$.

1 Introduction

All over this paper R denotes a commutative ring with identity $1 \neq 0$ and $\mathbb{I}(R)$ is the collection of all ideals of R. An ideal I is called an annihilating-ideal of R if $IJ = (0)$ for some ideal $J \neq (0)$ of R and $\mathbb{A}(R)$ is the collection of all annihilating-ideals of R. Typically, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Z}^+$ and $\mathbb{R}$ denote the integers, integers modulo n, positive integers and the real numbers respectively. For ring theoretic definitions, refer to [3].

In [9], Nithya and Elavarasi initiated and examined the extended annihilating-ideal graph $\mathbb{EAG}(R)$ related to R, whose vertices are $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$ and for distinct vertices I and J are adjacent if and only if $I^nJ^m = (0)$ with $I^n \neq (0)$ and $J^m \neq (0)$, for some $n, m \in \mathbb{Z}^+$. The authors discussed in detail the diameter and girth of $\mathbb{EAG}(R)$ and investigated the coincidence of $\mathbb{EAG}(R)$ and $\mathbb{AG}(R)$. They noted that $\mathbb{EAG}(R)$ is a null graph if and only if R is an integral domain. Also in [10], the authors studied the metric dimension, upper dimension and the resolving number of $\mathbb{EAG}(R)$ denoted by $dim_M(\mathbb{EAG}(R))$, $dim^+(\mathbb{EAG}(R))$ and $res(\mathbb{EAG}(R))$ respectively and illustrated these parameters with examples. One can refer [2] and [8], for studying various graphs from ring theoretic structures and the metric dimension of the annihilating-ideal graph of a finite commutative ring respectively.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Recall that $S \subseteq V(G)$, the induced subgraph $\langle S\rangle$ is the graph with vertex set S and two vertices are adjacent if and only if they are adjacent in G. The distance between two vertices x and y of G, $d(x, y)$ is the length of the shortest path from x to y. A complete graph is a graph where every pair of distinct vertices are adjacent and K_n denotes the complete graph on n vertices. If $V(G)$ can be split into two disjoint sets V_1 and V_2 such that every edge joins a vertex in V_1 to one in V_2, then G is a bipartite graph. A complete bipartite graph is a bipartite graph in which every vertex of one set is adjacent to every vertex of the other set and $K_{m,n}$ is the complete bipartite graph on m and n vertices and $K_{1,n}$ is a star graph. The order of the largest complete subgraph (clique) in G is known as the clique number $\omega(G)$ of G. The set of all vertices of G adjacent to the vertex v is known as the neighborhood $N(v)$ of v and $N[v] = N(v) \cup \{v\}$. For $|V(G)| \geq 2$, if $d(u, x) = d(v, x)$, for all $x \in V(G)\backslash\{u, v\}$ and $u \neq v$, then u and v are twins. If $uv \notin E(G)$ and $N(u) = N(v)$, then they are referred to as false twins. If $uv \in E(G)$ and $N[u] = N[v]$, then they are known as true twins. It can be verified that the twins produce an equivalence relation on $V(G)$ and two distinct vertices u and v are twins if they are either false twin vertices or true twin vertices. See [7], for terminology and notations in graph theory not described here .

In Sections 2 and 3, we discuss the partition dimension and local metric dimension of $\mathbb{EAG}(R)$ respectively.

2 Partition dimension of $\mathbb{EAG}(R)$

The concept of partition dimension of a connected graph was studied in [5, 6]. For $S \subseteq V(G)$ and a vertex $v \in G$, the distance between v and S is defined as $d(v, S) = \min\{d(v, x) | x \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$ and a vertex $v \in G$, the representation of v with respect to Π is defined as the k-vector $D(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$. If the k-vectors $D(v|\Pi), v \in V(G)$, are distinct, then Π is called a resolving partition. The minimum k for which there is a resolving k-partition of $V(G)$ is the partition dimension $dim_P(G)$ of G. In this Section, we ascertain the exact value of partition dimension of $\mathbb{EAG}(R)$. The following theorem shows the comparison between the metric dimension and the partition dimension of G as seen in [5].

Theorem 2.1. [5, Theorem 1.1] If G is a nontrivial connected graph, then $dim_P(G) \leq dim_M(G)+1$.

Note that if G is a connected graph of order $n \geq 4$ that is neither a path nor a complete graph, then $3 \leq dim_P(G) \leq n-1$.

Theorem 2.2. *If $R \cong R_1 \times R_2 \times ... \times R_n$ where $R_i's$ are fields for every $i = 1$ to n, then*

(i) $dim_p(\mathbb{EAG}(R)) = n$ *for* $n = 2, 3, 4$.

(ii) $dim_p(\mathbb{EAG}(R)) \leq n+1$ *for* $n \geq 5$.

Proof. (i) For $n = 2$, clearly $\mathbb{EAG}(R) \cong K_2$ so that $dim_P(\mathbb{EAG}(R)) = 2$. For $n = 3$, as said in the above note and Theorem 2.1, $3 \leq dim_P(\mathbb{EAG}(R)) \leq dim_M(\mathbb{EAG}(R)) + 1$. Theorem 2.5 (i) in [10] shows that $dim_M(\mathbb{EAG}(R)) = 2$ and hence $dim_P(\mathbb{EAG}(R)) = 3$.

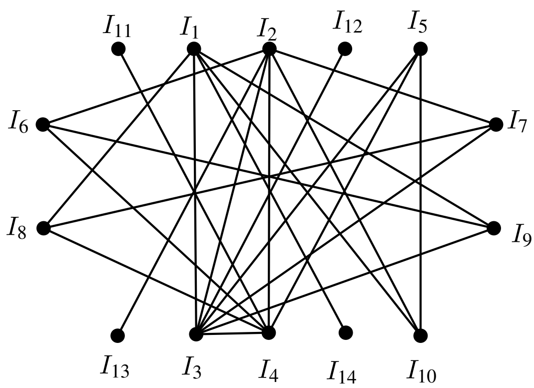

Figure 2.1

For $n = 4$, again Theorem 2.5 (i) in [10] implies that $dim_M(\mathbb{EAG}(R)) = 3$ and so $3 \leq dim_p(\mathbb{EAG}(R)) \leq 4$. Let $I_1 = R_1 \times (0) \times (0) \times (0), I_2 = (0) \times R_2 \times (0) \times (0), I_3 = (0) \times (0) \times R_3 \times (0), I_4 = (0) \times (0) \times (0) \times R_4, I_5 = R_1 \times R_2 \times (0) \times (0), I_6 = R_1 \times (0) \times R_3 \times (0), I_7 = R_1 \times (0) \times (0) \times R_4, I_8 = (0) \times R_2 \times R_3 \times (0), I_9 = (0) \times R_2 \times (0) \times R_4, I_{10} = (0) \times (0) \times R_3 \times R_4, I_{11} = R_1 \times R_2 \times R_3 \times (0), I_{12} = R_1 \times R_2 \times (0) \times R_4, I_{13} = R_1 \times (0) \times R_3 \times R_4$ and $I_{14} = (0) \times R_2 \times R_3 \times R_4$. Consider 3-partition $\Pi = \{S_1, S_2, S_3\}$ of $\mathbb{A}(R)^*$, where $S_1 = \{I_1, I_2, I_5, I_8, I_{13}\}$, $S_2 = \{I_3, I_6, I_9, I_{12}, I_{14}\}$ and $S_3 = \{I_4, I_7, I_{10}, I_{11},\}$. Then $D(I_1|\Pi) = D(I_2|\Pi) = D(I_5|\Pi)$ implies Π is not a resolving 3-partition. From Figure 2.1, one can verify that resolving 3-partition does not exist in $\mathbb{EAG}(R)$ for other cases. Hence $dim_P(\mathbb{EAG}(R)) = 4$.
(ii) Follows from Theorem 2.1 and Theorem 2.5 (ii) and (iii) in [10]. □

Recall that R is called a principal ideal ring (PIR), if every ideal is a principal ideal in R. An integral domain in which every ideal is principal is called a principal ideal domain (PID). A local artinian PIR is called a special principal ring (SPR) and it has only finitely many ideals, each of which is a power of the maximal ideal.

Theorem 2.3. *If R is a SPR, then* $dim_P(\mathbb{EAG}(R)) = |\mathbb{A}(R)^*|$.

Proof. By Theorem 2.4 in [9] and Proposition 2.3 in [5], the result holds. □

The following theorem computes $dim_p(\mathbb{EAG}(R))$ for direct product of certain rings.

Theorem 2.4. *If $R \cong R_1 \times R_2$, then the following cases occur.*

(i) *If R_1 is a field and R_2 is a ring with unique nonzero proper ideal, then $dim_P(\mathbb{EAG}(R)) = 3$.*

(ii) *If R_1 and R_2 are rings with unique nonzero proper ideal, then $dim_P(\mathbb{EAG}(R)) = 4$.*

(iii) *If R_1 is a field and R_2 is a SPR with more than one nonzero proper ideals, then $dim_P(\mathbb{EAG}(R)) = |\mathbb{I}(R_2)|$.*

Proof. (i) As $\mathbb{EAG}(R) \cong K_{2,2}$, then by Theorem 2.4 in [5], $dim_P(\mathbb{EAG}(R)) = 3$.
(ii) Assume that R_1 and R_2 are rings with unique nonzero proper ideal, say I_1 and I_2 respectively. Then the above note and Theorem 2.1 show that $3 \leq dim_P(\mathbb{EAG}(R)) \leq dim_M(\mathbb{EAG}(R)) + 1$. As noted in the proof of the Theorem 2.6 in [10], $dim_M(\mathbb{EAG}(R)) = 3$ implies that $3 \leq dim_P(\mathbb{EAG}(R)) \leq 4$. Clearly $d(R_1 \times (0), J) = d(R_1 \times I_2, J)$, for all $J \in \mathbb{A}(R)^* \backslash \{R_1 \times (0), R_1 \times I_2\}$ and $d((0) \times R_2, J) = d(I_1 \times R_2, J)$, for all $J \in \mathbb{A}(R)^* \backslash \{(0) \times R_2, I_1 \times R_2\}$. Suppose that 3-partition $\Pi = \{S_1, S_2, S_3\}$ of $\mathbb{A}(R)^*$. Then by Lemma 2.2 in [5], $R_1 \times (0)$ and $R_1 \times I_2$ contained in distinct elements of Π. Similarly, $(0) \times R_2$ and $I_1 \times R_2$ contained in distinct elements of Π. Let $S_1 = \{(0) \times R_2, R_1 \times (0)\}$, $S_2 = \{I_1 \times R_2, R_1 \times I_2\}$ and the remaining vertices $J_1 = I_1 \times (0), J_2 = (0) \times I_2, J_3 = I_1 \times I_2$ contained in anyone of the elements of Π. Consider $S_3 = \{J_1, J_2, J_3\}$. Then $D(J_1|\Pi) = D(J_2|\Pi)$ implies Π is not a resolving 3-partition. From Figure 2.2, one can view that Π is not a resolving 3-partition for other cases. Hence $dim_P(\mathbb{EAG}(R)) = 4$.

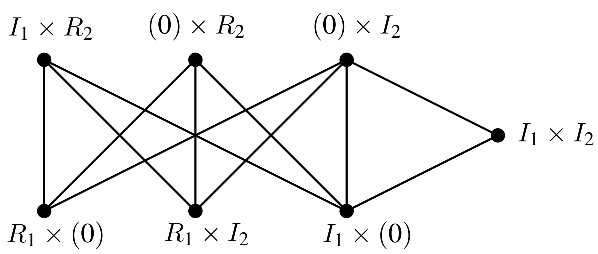

Figure 2.2

(iii) Let M_2 be the maximal ideal in R_2 such that ${M_2}^m = (0)$. The nonzero annihilating-ideals of R are $R_1 \times (0), (0) \times R_2$, $V_1 = \{(0) \times {M_2}^j\}$ and $V_2 = \{R_1 \times {M_2}^j\}$, for $1 \leq j < m$. The induced subgraphs $\langle V_1 \rangle$ is complete and $\langle V_2 \rangle$ is totally disconnected. Also any one edge ends at V_i means that edge adjacent to all the vertices in V_i.

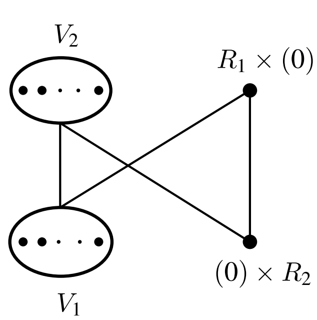

Figure 2.3

From Figure 2.3, $d((0) \times M_2, J) = d((0) \times {M_2}^2, J) = ... = d((0) \times {M_2}^{m-1}, J)$, for all $J \in \mathbb{A}(R)^* \backslash V_1$ and $d(R_1 \times (0), J) = d(R_1 \times M_2, J) = d(R_1 \times {M_2}^2, J) = ... = d(R_1 \times {M_2}^{m-1}, J)$, for all $J \in \mathbb{A}(R)^* \backslash (\{R_1 \times (0)\} \cup V_2)$. Now let $m+1$-partition $\Pi = \{S_1, S_2, ..., S_{m+1}\}$ of $\mathbb{A}(R)^*$. Then again by Lemma 2.2 in [5], consider $S_1 = \{(0) \times M_2, R_1 \times M_2\}$, $S_2 = \{(0) \times {M_2}^2, R_1 \times {M_2}^2\}, ..., S_{m-1} = \{(0) \times {M_2}^{m-1}, R_1 \times {M_2}^{m-1}\}, S_m = \{R_1 \times (0)\}, S_{m+1} = \{(0) \times R_2\}$. Clearly, Π is a resolving $(m+1)$-partition and so $|\mathbb{I}(R_2)^*| = m \leq dim_P(\mathbb{EAG}(R)) \leq m + 1 = |\mathbb{I}(R_2)|$.

Suppose that m-partition $\Pi = \{S_1, S_2, ..., S_m\}$ of $\mathbb{A}(R)^*$, where $S_1, S_2, ..., S_m$ are constructed as above and the remaining vertex $(0) \times R_2$ contained in any one of the elements of Π. If $(0) \times R_2 \in S_i$, then $D((0) \times R_2|\Pi) = D((0) \times {M_2}^i|\Pi)$, for $i = 1$ to $m - 1$. If $(0) \times R_2 \in S_m$, then vertices in each S_i have same partition metric representations about Π, for every $i = 1$ to m. Thus Π is not a resolving m-partition. Finally, resolving m-partition does not exist for all cases. Thus $dim_p(\mathbb{EAG}(R)) = m + 1 = |\mathbb{I}(R_2)|$. □

The following examples point up the previous theorem.

Example 2.5. (a) If $R \cong \frac{\mathbb{Z}_2[X]}{(X^2+X+1)} \times \frac{\mathbb{R}[X]}{(X^2)}$ where $\frac{\mathbb{Z}_2[X]}{(X^2+X+1)}$ is a field and (X) is a unique nonzero proper ideal in $\frac{\mathbb{R}[X]}{(X^2)}$, then clearly $\mathbb{EAG}(R) \cong K_{2,2}$. Consider 3-partition $\Pi = \{S_1, S_2, S_3\}$ of $\mathbb{A}(R)^*$, where $S_1 = \{(0) \times (X), \frac{\mathbb{Z}_2[X]}{(X^2+X+1)} \times (0)\}$, $S_2 = \{(0) \times \frac{\mathbb{R}[X]}{(X^2)}\}$ and $S_3 = \{\frac{\mathbb{Z}_2[X]}{(X^2+X+1)} \times (X)\}$. From this, Π is a resolving 3-partition and hence $dim_P(\mathbb{EAG}(R)) = 3$.
(b) Let $R \cong \frac{\mathbb{Z}_2[X]}{(X^2)} \times \frac{\mathbb{Z}[i]}{(1+i)^2}$. In this case, (X) and $(1 + i)$ are the unique nonzero proper ideal in $\frac{\mathbb{Z}_2[X]}{(X^2)}$ and $\frac{\mathbb{Z}[i]}{(1+i)^2}$ respectively. Consider 4-partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $\mathbb{A}(R)^*$, where $S_1 = \{\frac{\mathbb{Z}_2[X]}{(X^2)} \times (0), (0) \times (1+i)\}$, $S_2 = \{\frac{\mathbb{Z}_2[X]}{(X^2)} \times (1+i), (X) \times (1+i)\}$, $S_3 = \{(0) \times \frac{\mathbb{Z}[i]}{(1+i)^2}, (X) \times (0)\}$ and $S_4 = \{(X) \times \frac{\mathbb{Z}[i]}{(1+i)^2}\}$. This forms a resolving 4-partition and so $dim_P(\mathbb{EAG}(R)) = 4$.
(c) Let $R \cong \mathbb{Z}_2 \times \frac{\mathbb{R}[X]}{(X^3)}$ where $\frac{\mathbb{R}[X]}{(X^3)}$ is a SPR with the maximal ideal (X) such that $(X^3) = (0)$ and $\mathbb{A}(R)^* = \{\mathbb{Z}_2 \times (0), (0) \times \frac{\mathbb{R}[X]}{(X^3)}\} \cup V_1 \cup V_2$ where $V_1 = \{(0) \times (X), (0) \times (X^2)\}$ and $V_2 = \{\mathbb{Z}_2 \times (X), \mathbb{Z}_2 \times (X^2)\}$. Consider 4-partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $\mathbb{A}(R)^*$, where $S_1 = \{(0) \times (X), \mathbb{Z}_2 \times (X)\}$, $S_2 = \{(0) \times (X^2), \mathbb{Z}_2 \times (X^2)\}$, $S_3 = \{(0) \times \frac{\mathbb{R}[X]}{(X^3)}\}$ and $S_4 = \{\mathbb{Z}_2 \times (0)\}$. This implies Π is a resolving 4-partition and hence $dim_P(\mathbb{EAG}(R)) = 4$.

Theorem 2.6. *If $R \cong R_1 \times R_2 \times R_3$, then the following holds.*

(i) *If R_1, R_2 are fields and R_3 is a SPR and not a field, then $dim_P(\mathbb{EAG}(R)) = |\mathbb{A}(R_3)| + 2$.*

(ii) *If R_1 is a field, R_2 and R_3 are rings with unique nonzero proper ideal, then $dim_P(\mathbb{EAG}(R))$ $= 6$.*

(iii) *If R_1, R_2 and R_3 are rings with unique nonzero proper ideal, then $dim_P(\mathbb{EAG}(R)) = 7$.*

Proof. (i) Let M_3 be the maximal ideal in R_3 such that ${M_3}^r = (0)$. Consider $V_1 = \{(0) \times (0) \times {M_3}^k\}$, $V_2 = \{(0) \times R_2 \times {M_3}^k\}$, $V_3 = \{R_1 \times (0) \times {M_3}^k\}$ and $V_4 = \{R_1 \times R_2 \times {M_3}^k\}$, for $1 \leq k < r$. In Figure 2.4, the induced subgraphs $\langle V_1 \rangle$ is complete and $\langle V_2 \rangle, \langle V_3 \rangle$ and $\langle V_4 \rangle$ are totally disconnected. Also $d((0) \times (0) \times {M_3}^{k_1}, J) = d((0) \times (0) \times {M_3}^{k_2}, J)$, for all $J \in \mathbb{A}(R)^* \backslash V_1$, $d((0) \times R_2 \times {M_3}^{k_1}, J) = d((0) \times R_2 \times {M_3}^{k_2}, J) = d((0) \times R_2 \times (0), J)$, for all $J \in \mathbb{A}(R)^* \backslash (V_2 \cup \{(0) \times R_2 \times (0)\})$, $d(R_1 \times (0) \times {M_3}^{k_1}, J) = d(R_1 \times (0) \times {M_3}^{k_2}, J) = d(R_1 \times (0) \times (0), J)$, for all $J \in \mathbb{A}(R)^* \backslash (V_3 \cup \{R_1 \times (0) \times (0)\})$ and $d(R_1 \times R_2 \times {M_3}^{k_1}, J) = d(R_1 \times R_2 \times {M_3}^{k_2}, J) = d(R_1 \times R_2 \times (0), J)$, for all $J \in \mathbb{A}(R)^* \backslash (V_4 \cup \{R_1 \times R_2 \times (0)\})$ and $1 \leq k_1 < k_2 < r$. Let Π be a partition of $\mathbb{A}(R)^*$. Then by Lemma 2.2 in [5], $|\mathbb{A}(R_3)| = r \leq dim_P(\mathbb{EAG}(R))$. Choose r-partition $\Pi = \{S_1, S_2, ..., S_r\}$ of $\mathbb{A}(R)^*$ and $S_1 = \{(0) \times (0) \times M_3, (0) \times R_2 \times M_3, R_1 \times (0) \times M_3, R_1 \times R_2 \times M_3\}$, $S_2 = \{(0) \times (0) \times {M_3}^2, (0) \times R_2 \times {M_3}^2, R_1 \times (0) \times {M_3}^2, R_1 \times R_2 \times {M_3}^2\}, ..., S_{r-1} = \{(0) \times (0) \times {M_3}^{r-1}, (0) \times R_2 \times {M_3}^{r-1}, R_1 \times (0) \times {M_3}^{r-1}, R_1 \times R_2 \times {M_3}^{r-1}\}$, $S_r = \{(0) \times R_2 \times (0), R_1 \times R_2 \times (0), R_1 \times (0) \times (0)\}$ and the remaining vertices $J_1 = (0) \times R_2 \times R_3$, $J_2 = R_1 \times (0) \times R_3$ and $J_3 = (0) \times (0) \times R_3$ contained in any one of S_i, for $i = 1$ to r. Then $D((0) \times (0) \times M_3|\Pi) = D((0) \times R_2 \times M_3|\Pi) = D(R_1 \times (0) \times M_3|\Pi) = (0, 1, 1, ..., 1)$ and so Π is not a resolving r-partition. From Figure 2.4, resolving r-partition does not exist for all cases. Hence $|\mathbb{A}(R_3)| + 1 = r + 1 \leq dim_P(\mathbb{EAG}(R))$.

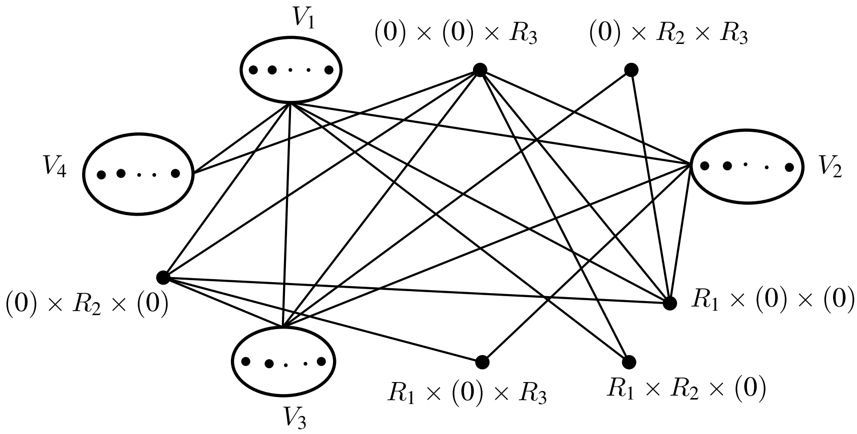

Figure 2.4

Consider $r+1$-partition $\Pi = \{S_1, S_2, ..., S_{r+1}\}$ of $\mathbb{A}(R)^*$ and $S_1, S_2, ..., S_r$ are constructed as above. Consider either $S_{r+1} = \{J_1, J_2, J_3\}$ or any two vertices of J_1, J_2, J_3 contained in S_{r+1}. Then $D((0) \times R_2 \times {M_3}^k|\Pi) = D(R_1 \times (0) \times {M_3}^k|\Pi)$, for $1 \leq k < r$ implies Π is not a resolving $(r+1)$-partition. Suppose that any one vertex of J_1, J_2, J_3 contained in S_{r+1} and remaining two vertices in any one of S_i, for all $i = 1$ to r. Consider $J_1 \in S_{r+1}$, then $D((0) \times (0) \times M_3|\Pi) = D((0) \times R_2 \times M_3|\Pi) = (0, 1, 1, ..., 1, 2)$. Suppose $J_2 \in S_{r+1}$, then $D((0) \times (0) \times M_3|\Pi) = D(R_1 \times (0) \times M_3|\Pi) = (0, 1, 1, ..., 1, 2)$. If $J_3 \in S_{r+1}$, then $D((0) \times R_2 \times M_3|\Pi) = D(R_1 \times (0) \times M_3|\Pi) = (0, 1, 1, .., 1, 1)$. This shows that Π is not a resolving $(r+1)$-partition. Hence in all cases, resolving $(r+1)$-partition does not exist and so $dim_p(\mathbb{EAG}(R)) \geq r+2 = |\mathbb{A}(R_3)| + 2$.
Consider $r+2$-partition $\Pi = \{S_1, S_2, ..., S_{r+2}\}$ of $\mathbb{A}(R)^*$, where $S_1, S_2, ..., S_{r-1}$ are constructed as above and $S_r = \{(0) \times R_2 \times (0), R_1 \times R_2 \times (0), R_1 \times (0) \times (0), J_3\}, S_{r+1} = \{J_2\}, S_{r+2} = \{J_1\}$. It is clear that the vertices in $\mathbb{A}(R)^*$ have different partition metric representations about Π and so Π is a resolving $(r+2)$-partition. Hence $dim_P(\mathbb{EAG}(R)) = r+2 = |\mathbb{A}(R_3)| + 2$.
(ii) Let I_2 and I_3 be unique nonzero proper ideal in R_2 and R_3 respectively. Clearly $d(R_1 \times (0) \times (0), J) = d(R_1 \times (0) \times I_3, J) = d(R_1 \times I_2 \times (0), J) = d(R_1 \times I_2 \times I_3, J)$, for all $J \in \mathbb{A}(R)^* \backslash \{R_1 \times (0) \times (0), R_1 \times (0) \times I_3, R_1 \times I_2 \times (0), R_1 \times I_2 \times I_3\}$, $d(R_1 \times (0) \times R_3, J) = d(R_1 \times I_2 \times R_3, J)$, for all $J \in \mathbb{A}(R)^* \backslash \{R_1 \times (0) \times R_3, R_1 \times I_2 \times R_3\}$, $d(R_1 \times R_2 \times (0), J) = d(R_1 \times R_2 \times I_3, J)$, for all $J \in \mathbb{A}(R)^* \backslash \{R_1 \times R_2 \times (0), R_1 \times R_2 \times I_3\}$, $d((0) \times (0) \times R_3, J) = d((0) \times I_2 \times R_3, J)$, for all $J \in \mathbb{A}(R)^* \backslash \{(0) \times (0) \times R_3, (0) \times I_2 \times R_3\}$, $d((0) \times R_2 \times (0), J) = d((0) \times R_2 \times I_3, J)$, for all $J \in \mathbb{A}(R)^* \backslash \{(0) \times R_2 \times (0), (0) \times R_2 \times I_3\}$. Consider 6-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ of $\mathbb{A}(R)^*$. By Lemma 2.2 in [5], $S_1 = \{R_1 \times (0) \times (0), R_1 \times (0) \times R_3, (0) \times (0) \times R_3, R_1 \times R_2 \times (0), (0) \times R_2 \times (0)\}$, $S_2 = \{R_1 \times (0) \times I_3, R_1 \times I_2 \times R_3, (0) \times I_2 \times R_3, R_1 \times R_2 \times I_3, (0) \times R_2 \times I_3\}$, $S_3 = \{R_1 \times I_2 \times (0), (0) \times R_2 \times R_3, (0) \times (0) \times I_3\}$, $S_4 = \{R_1 \times I_2 \times I_3\}$, $S_5 = \{(0) \times I_2 \times I_3\}$ and $S_6 = \{(0) \times I_2 \times (0)\}$. Then Π is a resolving 6-partition and so $4 \leq dim_P(\mathbb{EAG}(R)) \leq 6$. Suppose that 4-partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $\mathbb{A}(R)^*$ and $S_1 = \{R_1 \times (0) \times (0), R_1 \times (0) \times R_3, (0) \times (0) \times R_3, R_1 \times R_2 \times (0), (0) \times R_2 \times (0)\}$, $S_2 = \{R_1 \times (0) \times I_3, R_1 \times I_2 \times R_3, (0) \times I_2 \times R_3, R_1 \times R_2 \times I_3, (0) \times R_2 \times I_3\}$, $S_3 = \{R_1 \times I_2 \times (0)\}$, $S_4 = \{R_1 \times I_2 \times I_3\}$ and the remaining vertices $J_1 = (0) \times R_2 \times R_3, J_2 = (0) \times (0) \times I_3, J_3 = (0) \times I_2 \times I_3, J_4 = (0) \times I_2 \times (0)$ contained in any set of $S_i{'}s$, for all $i = 1$ to 4, implies that $D((0) \times (0) \times R_3|\Pi) = D((0) \times R_2 \times (0)|\Pi) = (0, 1, 1, 1)$ and so Π is not a resolving 4-partition. Similarly in all cases, $dim_P(\mathbb{EAG}(R)) \geq 5$.
Consider 5-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$ of $\mathbb{A}(R)^*$ and S_1, S_2, S_3, S_4 are constructed as above. Consider any set of k vertices of J_1, J_2, J_3, J_4 contained in S_5, for $k = 2$ to 4, then $D(J_t|\Pi) = D(J_m|\Pi) = (1, 1, 1, 1, 0)$ where $J_t, J_m \in S_5$, $t, m = 1$ to 4 and $t \neq m$. Hence Π is not a resolving 5-partition. Suppose that any one vertex of J_1, J_2, J_3, J_4 contained in S_5 and remaining three vertices in any one of S_i, for all $i = 1$ to 4. Consider $J_1 \in S_5$, then $D((0) \times (0) \times R_3|\Pi) = D((0) \times R_2 \times (0)|\Pi) = (0, 1, 1, 1, 2)$. Thus Π is not a resolving 5-partition. Similarly, resolving 5-partition does not exist for all cases so that $dim_p(\mathbb{EAG}(R)) = 6$.
(iii) Let I_1, I_2 and I_3 be unique nonzero proper ideal in R_1, R_2 and R_3 respectively. It is clear that $d(R_1 \times (0) \times (0), J) = d(R_1 \times (0) \times I_3, J) = d(R_1 \times I_2 \times (0), J) = d(R_1 \times I_2 \times I_3, J)$, for all $J \in A(R)^* \backslash \{R_1 \times (0) \times (0), R_1 \times (0) \times I_3, R_1 \times I_2 \times (0), R_1 \times I_2 \times I_3\}$, $d((0) \times (0) \times R_3, J) = d((0) \times$

$I_2\times R_3, J) = d(I_1\times I_2\times R_3, J) = d(I_1\times(0)\times R_3, J)$, for all $J \in \mathbb{A}(R)^*\backslash\{(0)\times(0)\times R_3, (0)\times I_2\times R_3, I_1\times I_2\times R_3, I_1\times(0)\times R_3\}$, $d((0)\times R_2\times(0), J) = d((0)\times R_2\times I_3, J) = d(I_1\times R_2\times(0), J) = d(I_1\times R_2\times I_3, J)$, for all $J \in \mathbb{A}(R)^*\backslash\{(0)\times R_2\times(0), (0)\times R_2\times I_3, I_1\times R_2\times(0), I_1\times R_2\times I_3\}$, $d((0)\times R_2\times R_3, J) = d(I_1\times R_2\times R_3, J)$, for all $J \in \mathbb{A}(R)^*\backslash\{(0)\times R_2\times R_3, I_1\times R_2\times R_3\}$, $d(R_1\times R_2\times(0), J) = d(R_1\times R_2\times I_3, J)$, for all $J \in \mathbb{A}(R)^*\backslash\{R_1\times R_2\times(0), R_1\times R_2\times I_3\}$, $d(R_1\times(0)\times R_3, J) = d(R_1\times I_2\times R_3, J)$, for all $J \in \mathbb{A}(R)^*\backslash\{R_1\times(0)\times R_3, R_1\times I_2\times R_3\}$. Suppose that 7-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ of $\mathbb{A}(R)^*$. Then again by Lemma 2.2 in [5], consider $S_1 = \{R_1\times(0)\times(0), (0)\times(0)\times R_3, (0)\times R_2\times(0), R_1\times R_2\times(0), R_1\times(0)\times R_3, (0)\times R_2\times R_3, (0)\times I_2\times I_3\}$, $S_2 = \{R_1\times(0)\times I_3, (0)\times I_2\times R_3, (0)\times R_2\times I_3, R_1\times R_2\times I_3, R_1\times I_2\times R_3, I_1\times R_2\times R_3\}$, $S_3 = \{R_1\times I_2\times(0), I_1\times I_2\times R_3, I_1\times R_2\times(0), I_1\times(0)\times I_3\}$, $S_4 = \{R_1\times I_2\times I_3, I_1\times(0)\times R_3, I_1\times R_2\times I_3, I_1\times I_2\times(0)\}$, $S_5 = \{I_1\times(0)\times(0), I_1\times I_2\times I_3\}$, $S_6 = \{(0)\times I_2\times(0)\}$ and $S_7 = \{(0)\times(0)\times I_3\}$. Then the vertex in every S_i, for all $i = 1$ to 7 has distinct partition metric representations about Π. Consequently, Π is a resolving 7-partition and so $4 \le dim_P(\mathbb{EAG}(R)) \le 7$. Suppose that 4-partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $\mathbb{A}(R)^*$ and $S_1 = \{R_1\times(0)\times(0), (0)\times(0)\times R_3, (0)\times R_2\times(0), R_1\times R_2\times(0), R_1\times(0)\times R_3, (0)\times R_2\times R_3\}$, $S_2 = \{R_1\times(0)\times I_3, (0)\times I_2\times R_3, (0)\times R_2\times I_3, R_1\times R_2\times I_3, R_1\times I_2\times R_3, I_1\times R_2\times R_3\}$, $S_3 = \{R_1\times I_2\times(0), I_1\times I_2\times R_3, I_1\times R_2\times(0)\}$, $S_4 = \{R_1\times I_2\times I_3, I_1\times(0)\times R_3, I_1\times R_2\times I_3\}$ and the remaining vertices $J_1 = (0)\times I_2\times I_3, J_2 = I_1\times(0)\times I_3, J_3 = I_1\times I_2\times(0), J_4 = I_1\times(0)\times(0), J_5 = I_1\times I_2\times I_3, J_6 = (0)\times I_2\times(0)$ and $J_7 = (0)\times(0)\times I_3$ contained in any one of S_i, for $i = 1$ to 4. This implies $D((0)\times(0)\times R_3|\Pi) = D((0)\times R_2\times(0)|\Pi) = D((0)\times R_2\times R_3|\Pi) = (0, 1, 1, 1)$ and so Π is not a resolving 4-partition. As similar argument for all other cases, $dim_P(\mathbb{EAG}(R)) \ge 5$.
Consider 5-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$ of $\mathbb{A}(R)^*$ and S_1, S_2, S_3, S_4 are constructed as above. Any set of k vertices of $J_1, J_2, J_3, J_4, J_5, J_6, J_7$ contained in S_5, for $k = 1$ to 7 does not form a resolving 5-partition about Π. Since $D(I|\Pi) = (0, 1, 1, 1, d(I, S_5))$, for all $I \in S_1\backslash\{J_5\}$ and $d(I, S_5) = 1$ or 2 implies that any two vertices in S_1 have same partition metric representations about Π. Argument is similar if the vertices of S_i are replaced, for $i = 1$ to 5. Hence $dim_P(\mathbb{EAG}(R)) \ge 6$.
Suppose that 6-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ of $\mathbb{A}(R)^*$ and S_1, S_2, S_3, S_4 are constructed as above and if $J_1 \in S_1, J_2 \in S_3, J_3 \in S_4, S_5 = \{J_4, J_5\}$ and $S_6 = \{J_6, J_7\}$, then $D((0)\times(0)\times R_3|\Pi) = D((0)\times R_2\times(0)|\Pi) = D(J_1|\Pi)$. Hence Π is not a resolving 6-partition. Similarly, placing J_t in any S_i, for all $t = 1$ to 7 and $i = 1$ to 6 implies that Π is not a resolving 6-partition. Also in all cases, $dim_P(\mathbb{EAG}(R)) \ge 7$, so that $dim_P(\mathbb{EAG}(R))) = 7$. □

We conclude this section by providing certain examples which demonstrates the previous theorem.

Example 2.7. (a) Let $R \cong \frac{\mathbb{Z}_5[X]}{(X^2+2)} \times \frac{\mathbb{Z}_5[X]}{(X^2+2)} \times \frac{\mathbb{R}[X]}{(X^3)}$. Here (X) is the maximal ideal in $\frac{\mathbb{R}[X]}{(X^3)}$. Consider 5-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$ of $\mathbb{A}(R)^*$, where $S_1 = \{(0)\times(0)\times(X), (0)\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(X), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0)\times(X), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(X)\}$, $S_2 = \{(0)\times(0)\times(X^2), (0)\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(X^2), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0)\times(X^2), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(X^2)\}$, $S_3 = \{(0)\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0)\times(0), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0), (0)\times(0)\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\}$, $S_4 = \{\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0)\times\frac{\mathbb{R}[X]}{(X^3)}\}$ and $S_5 = \{(0)\times\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times\frac{\mathbb{R}[X]}{(X^3)}\}$. This forms a resolving 5-partition so that $dim_P(\mathbb{EAG}(R)) = 5$.
(b) Let $R \cong \frac{\mathbb{Z}_5[X]}{(X^2+2)} \times \mathbb{Z}_4 \times \mathbb{Z}_9$. In this, (2) and (3) are the unique nonzero proper ideal in $\mathbb{Z}_4$ and $\mathbb{Z}_9$ respectively. Consider 6-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ of $\mathbb{A}(R)^*$, where $S_1 = \{\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0)\times(0), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0)\times\mathbb{Z}_9, (0)\times(0)\times\mathbb{Z}_9, \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times\mathbb{Z}_4\times(0), (0)\times\mathbb{Z}_4\times(0)\}$, $S_2 = \{\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(0)\times(3), \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(2)\times\mathbb{Z}_9, (0)\times(2)\times\mathbb{Z}_9, \frac{\mathbb{Z}_5[X]}{(X^2+2)}\times\mathbb{Z}_4\times(3), (0)\times\mathbb{Z}_4\times(3)\}$, $S_3 = \{\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(2)\times(0), (0)\times\mathbb{Z}_4\times\mathbb{Z}_9, (0)\times(0)\times(3)\}$, $S_4 = \{\frac{\mathbb{Z}_5[X]}{(X^2+2)}\times(2)\times(3)\}$, $S_5 = \{(0)\times(2)\times(3)\}$ and $S_6 = \{(0)\times(2)\times(0)\}$. From this, Π is a resolving 6-partition and so $dim_p(\mathbb{EAG}(R)) = 6$.
(c) Let $R \cong \frac{\mathbb{R}[X]}{(X^2)} \times \frac{\mathbb{R}[X]}{(X^2)} \times \frac{\mathbb{R}[X]}{(X^2)}$. Here (X) is a unique nonzero proper ideal in $\frac{\mathbb{R}[X]}{(X^2)}$. Suppose that 7-partition $\Pi = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ of $\mathbb{A}(R)^*$, where $S_1 = \{\frac{\mathbb{R}[X]}{(X^2)}\times(0)\times(0), (0)\times(0)\times\frac{\mathbb{R}[X]}{(X^2)}, (0)\times\frac{\mathbb{R}[X]}{(X^2)}\times(0), \frac{\mathbb{R}[X]}{(X^2)}\times\frac{\mathbb{R}[X]}{(X^2)}\times(0), \frac{\mathbb{R}[X]}{(X^2)}\times(0)\times\frac{\mathbb{R}[X]}{(X^2)}, (0)\times\frac{\mathbb{R}[X]}{(X^2)}\times\frac{\mathbb{R}[X]}{(X^2)}, (0)\times(X)\times(X)\}$,

$S_2 = \{\frac{\mathbb{R}[X]}{(X^2)} \times (0) \times (X), (0) \times (X) \times \frac{\mathbb{R}[X]}{(X^2)}, (0) \times \frac{\mathbb{R}[X]}{(X^2)} \times (X), \frac{\mathbb{R}[X]}{(X^2)} \times \frac{\mathbb{R}[X]}{(X^2)} \times (X), \frac{\mathbb{R}[X]}{(X^2)} \times (X) \times \frac{\mathbb{R}[X]}{(X^2)}, (X) \times \frac{\mathbb{R}[X]}{(X^2)} \times \frac{\mathbb{R}[X]}{(X^2)}\}$, $S_3 = \{\frac{\mathbb{R}[X]}{(X^2)} \times (X) \times (0), (X) \times (X) \times \frac{\mathbb{R}[X]}{(X^2)}, (X) \times \frac{\mathbb{R}[X]}{(X^2)} \times (0), (X) \times (0) \times (X)\}$, $S_4 = \{\frac{\mathbb{R}[X]}{(X^2)} \times (X) \times (X), (X) \times (0) \times \frac{\mathbb{R}[X]}{(X^2)}, (X) \times \frac{\mathbb{R}[X]}{(X^2)} \times (X), (X) \times (X) \times (0)\}$, $S_5 = \{(X) \times (0) \times (0), (X) \times (X) \times (X)\}$, $S_6 = \{(0) \times (X) \times (0)\}$ and $S_7 = \{(0) \times (0) \times (X)\}$. This forms a resolving 7-partition. Hence $dim_p(\mathbb{EAG}(R)) = 7$.

3 Local metric dimension of $\mathbb{EAG}(R)$

The local metric dimension of a graph was introduced by Okamoto et al. [11]. For an ordered subset $W = \{v_1, v_2, ..., v_k\}$ of $V(G)$ and a vertex $v \in G$, the representation of v with respect to W is defined as the k-vector $D(v|W) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$. If $D(u|W) \neq D(v|W)$ for every pair u, v of adjacent vertices of G, then the set W is a local metric set of G. The minimum cardinality of a local metric set W is the local metric basis for G and the number of elements in the local metric basis is the local metric dimension of G and it is denoted by $dim_l(G)$. Note that if G is a nontrivial connected graph of order n, then $1 \leq dim_l(G) \leq dim_M(G) \leq n - 1$. In this Section, we explore the local metric dimension of $\mathbb{EAG}(R)$. The following theorem computes $dim_l(\mathbb{EAG}(R))$ for direct product of fields.

Theorem 3.1. *If $R \cong R_1 \times R_2 \times ... \times R_n$ where $R_i's$ are fields for every $i = 1$ to n and $n \geq 2$, then*

(i) $dim_l(\mathbb{EAG}(R)) = n - 1$ where $2 \leq n \leq 5$.

(ii) $dim_l(\mathbb{EAG}(R)) \leq n$ where $n \geq 6$.

Proof. (i) For $n = 2$. Then clearly $dim_l(\mathbb{EAG}(R)) = 1$. Let $n = 3$. As $dim_l(\mathbb{EAG}(R)) \leq dim_M(\mathbb{EAG}(R))$, then by Theorem 2.5 (i) in [10], $dim_l(\mathbb{EAG}(R)) \leq 2$. Since $\omega(\mathbb{EAG}(R)) = 3$, then by Theorem 3.1 in [11]., $dim_l(\mathbb{EAG}(R)) \geq \lceil log_2 3\rceil$. As $\lceil log_2 3\rceil = 2$, $dim_l(\mathbb{EAG}(R)) \geq 2$. Hence $dim_l(\mathbb{EAG}(R)) = 2$. For $n = 4$, by Theorem 2.5 (i) in [10], $dim_l(\mathbb{EAG}(R)) \leq 3$. From Figure 2.1, $\omega(\mathbb{EAG}(R)) = 4$. Then again by Theorem 3.1 in [11], $dim_l(\mathbb{EAG}(R)) \geq \lceil log_2 4\rceil = 2$. Obviously, any collection of two vertices in $\mathbb{EAG}(R)$ does not form a local metric set so that $dim_l(\mathbb{EAG}(R)) \geq 3$. Hence $dim_l(\mathbb{EAG}(R)) = 3$.
For $n = 5$, the nonzero annihilating-ideals of R are $V_1 = \{I_1, I_2, I_3, I_4, I_5\}$, $V_2 = \{J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J_9, J_{10}\}$, $V_3 = \{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}\}$ and $V_4 = \{L_1, L_2, L_3, L_4, L_5\}$, where $I_1 = R_1 \times (0) \times (0) \times (0) \times (0), I_2 = (0) \times R_2 \times (0) \times (0) \times (0), I_3 = (0) \times (0) \times R_3 \times (0) \times (0), I_4 = (0) \times (0) \times (0) \times R_4 \times (0), I_5 = (0) \times (0) \times (0) \times (0) \times R_5, J_1 = R_1 \times R_2 \times (0) \times (0) \times (0), J_2 = R_1 \times (0) \times R_3 \times (0) \times (0), J_3 = R_1 \times (0) \times (0) \times R_4 \times (0), J_4 = R_1 \times (0) \times (0) \times (0) \times R_5, J_5 = (0) \times R_2 \times R_3 \times (0) \times (0), J_6 = (0) \times R_2 \times (0) \times R_4 \times (0), J_7 = (0) \times R_2 \times (0) \times (0) \times R_5, J_8 = (0) \times (0) \times R_3 \times R_4 \times (0), J_9 = (0) \times (0) \times R_3 \times (0) \times R_5, J_{10} = (0) \times (0) \times (0) \times R_4 \times R_5, N_1 = R_1 \times R_2 \times R_3 \times (0) \times (0), N_2 = R_1 \times R_2 \times (0) \times R_4 \times (0), N_3 = R_1 \times R_2 \times (0) \times (0) \times R_5, N_4 = R_1 \times (0) \times R_3 \times R_4 \times (0), N_5 = R_1 \times (0) \times R_3 \times (0) \times R_5, N_6 = R_1 \times (0) \times (0) \times R_4 \times R_5, N_7 = (0) \times R_2 \times R_3 \times R_4 \times (0), N_8 = (0) \times R_2 \times R_3 \times (0) \times R_5, N_9 = (0) \times R_2 \times (0) \times R_4 \times R_5, N_{10} = (0) \times (0) \times R_3 \times R_4 \times R_5, L_1 = R_1 \times R_2 \times R_3 \times R_4 \times (0), L_2 = R_1 \times R_2 \times R_3 \times (0) \times R_5, L_3 = R_1 \times R_2 \times (0) \times R_4 \times R_5, L_4 = R_1 \times (0) \times R_3 \times R_4 \times R_5$ and $L_5 = (0) \times R_2 \times R_3 \times R_4 \times R_5$. Clearly, $\langle V_1 \rangle$ forms a complete graph K_5. Also $\omega(\mathbb{EAG}(R)) = 5$ then again Theorem 3.1 in [11] implies that $dim_l(\mathbb{EAG}(R)) \geq 3$. If W is a collection of any three vertices of V_k, for $k = 1, 2$, then any two adjacent vertices of $V_k \backslash W$ have same local metric representations about W. Let W be a collection of any three vertices of V_3. Then any two adjacent vertices of V_2 have same local metric representations about W. Let W be any three vertices of V_4. Then any two vertices of V_1 have same local metric representations about W. Let W be any three vertices of the form either $\{I_i, J_j, N_s\}$ or $\{I_i, J_j, L_t\}$ or $\{I_i, N_s, L_t\}$ or $\{J_j, N_s, L_t\}$, for all $i, t = 1$ to 5, $j, s = 1$ to 10. Then any two adjacent vertices of $V_1 \backslash W$ or $V_2 \backslash W$ have same local metric representations about W. Hence for all cases, every collection of three vertices of $\mathbb{EAG}(R)$ does not form a local metric set so that $dim_l(\mathbb{EAG}(R)) \geq 4$. If $W = \{I_1, I_2, I_3, I_4\}$, then every pair of adjacent vertices in $\mathbb{EAG}(R)$ have different local metric representations about W. Hence $dim_l(\mathbb{EAG}(R)) = 4$.
(ii) The result follows from $dim_l(\mathbb{EAG}(R)) \leq dim_M(\mathbb{EAG}(R))$ and by Theorem 2.5 (iii) in [10]. □

Theorem 3.2. *If R is a SPR, then $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R)^*| - 1$.*

Proof. By Theorem 2.4 in [9], $|\mathbb{A}(R)^*| - 1$ vertices of $\mathbb{EAG}(R)$ form a local metric basis so that $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R)^*| - 1$. □

Consider two vertices being true twins produce an equivalence relation on $V(G)$. If the resulting true twin equivalence classes are $U_1, U_2, ..., U_l$, then every local metric set of G must contain at least $|U_i| - 1$ vertices from U_i, for all $1 \leq i \leq l$.
The subsequent theorem characterizes the local metric dimension of $\mathbb{EAG}(R)$ for direct product of rings.

Theorem 3.3. *If R is a PIR and $R \cong R_1 \times R_2$, then*

- (i) *R_1 is an integral domain and R_2 is either an integral domain or a ring with unique nonzero proper ideal if and only if $dim_l(\mathbb{EAG}(R)) = 1$.*
- (ii) *R_1 and R_2 are rings with unique nonzero proper ideal if and only if $dim_l(\mathbb{EAG}(R)) = 2$.*
- (iii) *R_1 is an integral domain and R_2 is a ring with more than one nonzero proper ideals if and only if $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_2)^*|$.*
- (iv) *R_1 is not an integral domain and R_2 is a ring with more than one nonzero proper ideals if and only if $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)| - 2$.*

Proof. As R is a PIR, then $R \cong \prod_{i=1}^{n} R_i$ where $R_i's$ are either PIDs or SPRs.
(i) Assume that R_1 and R_2 are integral domains and not fields. Then $P \cap Q = (0)$, for some nonzero prime ideals $P = R_1 \times (0)$ and $Q = (0) \times R_2$ and they are not minimal ideals. Since R_1 and R_2 are reduced, then by Theorems 2.3 in [9] and 2.4 in [1], $\mathbb{EAG}(R)$ is a complete bipartite graph. Thus by Lemma 2.8 in [11], $dim_l(\mathbb{EAG}(R)) = 1$. Now consider R_1 is a field and R_2 is an integral domain. Since R_1 and R_2 are reduced, so by Theorem 2.3 in [9] and Corollary 2.3 in [4], $\mathbb{EAG}(R)$ is a star graph so that $dim_l(\mathbb{EAG}(R)) = 1$. Consider R_1 is an integral domain and R_2 is a ring with unique nonzero proper ideal. Then Theorem 2.5 in [9] and Lemma 2.8 in [11] show that $dim_l(\mathbb{EAG}(R)) = 1$.
Conversely, assume that $dim_l(\mathbb{EAG}(R)) = 1$. Suppose that R_1 is an integral domain and R_2 is a ring with more than one nonzero proper ideals. Consider I is a nonzero proper ideal in R_1 and M_2 is the maximal ideal in R_2 such that ${M_2}^m = (0)$. Then $\mathbb{A}(R)^* = \{R_1 \times (0)\} \cup \{(0) \times R_2\} \cup V_1 \cup V_2 \cup V_3 \cup V_4$ where $V_1 = \{(0) \times {M_2}^j\}$, $V_2 = \{R_1 \times {M_2}^j\}$, $V_3 = \{I \times (0) : I \in R_1\}$ and $V_4 = \{I \times {M_2}^j : I \in R_1\}$, for $1 \leq j < m$. Here the induced subgraphs $\langle V_1 \rangle$ is complete and $\langle V_2 \rangle$, $\langle V_3 \rangle$ and $\langle V_4 \rangle$ are totally disconnected.

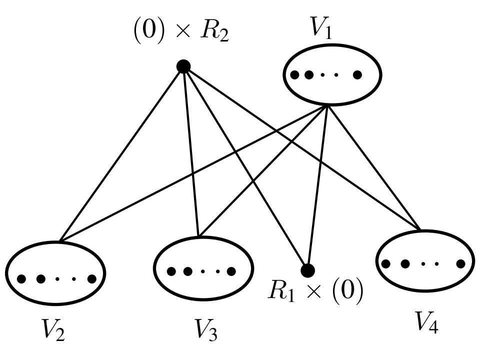

Figure 3.1

From Figure 3.1, $\mathbb{EAG}(R)$ has only one true twin equivalence class, say $U_1 = V_1$. Then every local metric set of $\mathbb{EAG}(R)$ must contain at least $|U_1| - 1 = m - 2$ vertices from U_1. Then $m - 2 \leq dim_l(\mathbb{EAG}(R)$. Consider $W \subseteq V_1$ and $|W| = m - 2$, then the adjacent vertices in $\mathbb{A}(R)^* \backslash W$ have same local metric representations with respect to W. Therefore $dim_l(\mathbb{EAG}(R)) \neq m - 2$. Consider $W = V_1$ and $|W| = m - 1$, then the adjacent vertices in $\mathbb{EAG}(R)$ have different local metric representations about W. Then by definition, $dim_l(\mathbb{EAG}(R)) = m - 1 = |\mathbb{A}(R_2)^*|$.
Suppose that R_1 and R_2 are not integral domains. Consider R_1 and R_2 have unique nonzero

proper ideal say, M_1 and M_2 respectively. As $\omega(\mathbb{EAG}(R)) = 3$, then by Theorem 3.1 in [11], $dim_l(\mathbb{EAG}(R)) \geq \lceil log_2 3 \rceil = 2$. It is clear that $W = \{M_1 \times R_2, M_1 \times M_2\}$ is a local metric basis for $\mathbb{EAG}(R)$. Hence $dim_l(\mathbb{EAG}(R)) = 2$.
Now consider R_2 has more than one nonzero proper ideals. Let M_1 and M_2 be the maximal ideals in R_1 and R_2 respectively such that ${M_1}^n = (0)$ and ${M_2}^m = (0)$. Then the nonzero annihilating-ideals of R are $R_1 \times (0)$, $(0) \times R_2$, $V_1 = \{{M_1}^i \times (0)\}$, $V_{21} = \{{M_1}^i \times {M_2}^j : ({M_1}^i)^l \neq (0)$ and $({M_2}^j)^l = (0)$, for some $l \in \mathbb{Z}^+\}$, $V_{22} = \{{M_1}^i \times {M_2}^j : ({M_1}^i)^l = ({M_2}^j)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{23} = \{{M_1}^i \times {M_2}^j : ({M_1}^i)^l = (0)$ and $({M_2}^j)^l \neq (0)$ for some $l \in \mathbb{Z}^+\}$, $V_3 = \{(0) \times {M_2}^j\}$, $V_4 = \{{M_1}^i \times R_2\}$ and $V_5 = \{R_1 \times {M_2}^j\}$, for $1 \leq i < n, 1 \leq j < m$.

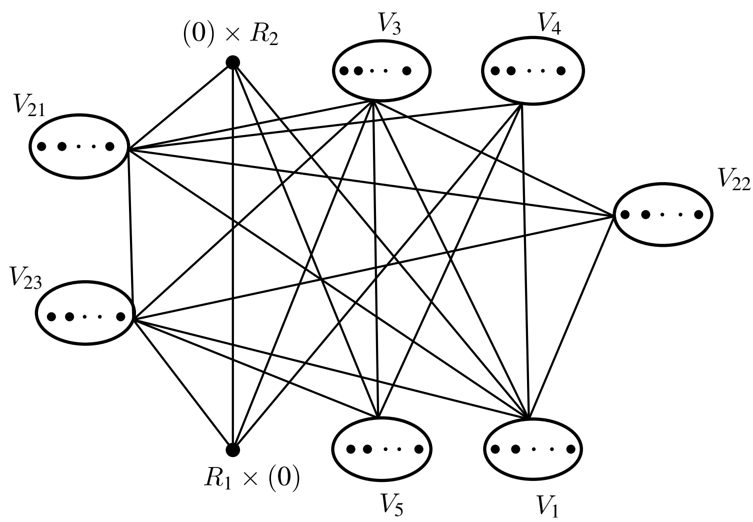

Figure 3.2

In Figure 3.2, the induced subgraphs $\langle V_1 \rangle, \langle V_{21} \rangle, \langle V_{22} \rangle, \langle V_{23} \rangle, \langle V_3 \rangle$ are complete graphs and $\langle V_4 \rangle$, $\langle V_5 \rangle$ are totally disconnected. Here the true twin equivalence classes in $\mathbb{EAG}(R)$ are $U_1 = V_1 \cup V_{21}$, $U_2 = V_{22}$ and $U_3 = V_3 \cup V_{23}$. Then at least $|U_i| - 1$ vertices from U_i, for all $i = 1, 2, 3$ must contained in the local metric set of $\mathbb{EAG}(R)$. Let $W = \bigcup_{i=1}^{3}(U_i \backslash \{J_i\}) \cup \{R_1 \times (0), (0) \times R_2\}$ where $J_i \in U_i$ and so $|W| = |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)| - 2$. Then all the adjacent vertices in $\mathbb{EAG}(R)$ have different local metric representations about W. Hence W is a local metric set for $\mathbb{EAG}(R)$ and so $dim_l(\mathbb{EAG}(R)) \leq |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)| - 2$. Suppose that $W = \bigcup_{i=1}^{3}(U_i \backslash \{J_i\}) \cup \{R_1 \times (0)\}$, $J_i \in U_i$ and so the cardinality is $|\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)| - 3$. Then W is not a local metric set for $\mathbb{EAG}(R)$ since $D(J_1|W) = D(J_2|W)$, for the adjacent vertices J_1 and J_2 of $\mathbb{EAG}(R)$. Similarly for all cases, $dim_l(\mathbb{EAG}(R)) \neq |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)| - 3$. Hence $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)| - 2$. From all cases, R_1 is an integral domain and R_2 is either an integral domain or a ring with unique nonzero proper ideal.
(ii), (iii) and (iv) follow from the proof of (i). □

Next we provide certain examples for the previous theorem.

Example 3.4. (a) If $R \cong \mathbb{Z} \times \mathbb{Z}_2$, then clearly $\mathbb{EAG}(R)$ is a star graph so that $dim_l(\mathbb{EAG}(R)) = 1$.
(b) Let $R \cong \mathbb{Z} \times \frac{\mathbb{Z}_2[X]}{(X^2)}$. Obviously, $\mathbb{EAG}(R)$ forms a complete bipartite graph so that $dim_l(\mathbb{EAG}(R)) = 1$.
(c) Let $R \cong \frac{\mathbb{R}[X]}{(X^2)} \times \frac{\mathbb{R}[X]}{(X^2)}$. In this, (X) is a unique nonzero proper ideal in $\frac{\mathbb{R}[X]}{(X^2)}$. Then the local metric basis for $\mathbb{EAG}(R)$ is $W = \{(X) \times \frac{\mathbb{R}[X]}{(X^2)}, (X) \times (X)\}$. Hence $dim_l(\mathbb{EAG}(R)) = 2$.
(d) Let $R \cong \mathbb{Z}[i] \times \frac{\mathbb{R}[X]}{(X^3)}$. Here (X) is the maximal ideal in $\frac{\mathbb{R}[X]}{(X^3)}$ such that $(X^3) = (0)$. Then $W = \{(0) \times (X), (0) \times (X^2)\}$ is a local metric basis for $\mathbb{EAG}(R)$. This shows that $dim_l(\mathbb{EAG}(R)) = 2$.
(e) Let $R \cong \frac{\mathbb{Z}[i]}{(1+i)^3} \times \frac{\mathbb{Z}[i]}{(1+i)^3}$. Here $(1+i)$ is the maximal ideal in $\frac{\mathbb{Z}[i]}{(1+i)^3}$ such that $(1+i)^3 = (0)$.

Then the local metric basis for $\mathbb{EAG}(R)$ is $W = \{(0) \times (1+i), (1+i) \times (0), (1+i) \times (1+i), (1+i) \times (1+i)^2, (1+i)^2 \times (1+i), (0) \times \frac{\mathbb{Z}[i]}{(1+i)^3}, \frac{\mathbb{Z}[i]}{(1+i)^3} \times (0)\}$. Hence $dim_l(\mathbb{EAG}(R)) = 7$.

Theorem 3.5. *If R is a PIR and $R \cong R_1 \times R_2 \times R_3$, then*

(i) *Either R_1, R_2 and R_3 are integral domains or R_1 is an integral domain, R_2 and R_3 are rings with unique nonzero proper ideal if and only if $dim_l(\mathbb{EAG}(R)) = 2$.*

(ii) *R_1, R_2 are integral domains and R_3 is not an integral domain if and only if $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_3)|$.*

(iii) *R_1 is an integral domain, R_2 is not an integral domain and R_3 is a ring with more than one nonzero proper ideals if and only if $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 2$.*

(iv) *R_1, R_2 and R_3 are rings with unique nonzero proper ideal if and only if $dim_l(\mathbb{EAG}(R)) = 3$.*

(v) *R_1, R_2 are not integral domains and R_3 is a ring with more than one nonzero proper ideals if and only if $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 5$.*

Proof. As R is a PIR, then $R \cong \prod_{i=1}^{n} R_i$ where $R_i's$ are either PIDs or SPRs.
(i) Assume that R_1, R_2 and R_3 are integral domains. Let I_1, I_2 and I_3 be nonzero proper ideals in R_1, R_2 and R_3 respectively. Consider $V_1 = \{R_1 \times (0) \times I_3 : I_3 \in R_3\}$, $V_2 = \{R_1 \times I_2 \times (0) : I_2 \in R_2\}$, $V_3 = \{I_1 \times (0) \times (0) : I_1 \in R_1\}$, $V_4 = \{I_1 \times (0) \times R_3 : I_1 \in R_1\}$, $V_5 = \{I_1 \times (0) \times I_3 : I_1 \in R_1, I_3 \in R_3\}$, $V_6 = \{I_1 \times I_2 \times (0) : I_1 \in R_1, I_2 \in R_2\}$, $V_7 = \{I_1 \times R_2 \times (0) : I_1 \in R_1\}$, $V_8 = \{(0) \times (0) \times I_3 : I_3 \in R_3\}$, $V_9 = \{(0) \times I_2 \times (0) : I_2 \in R_2\}$, $V_{10} = \{(0) \times I_2 \times I_3 : I_2 \in R_2, I_3 \in R_3\}$, $V_{11} = \{(0) \times I_2 \times R_3 : I_2 \in R_2\}$ and $V_{12} = \{(0) \times R_2 \times I_3 : I_3 \in R_3\}$. Here the induced subgraphs $\langle V_1\rangle, \langle V_2\rangle, \langle V_3\rangle, \langle V_4\rangle, \langle V_5\rangle, \langle V_6\rangle, \langle V_7\rangle, \langle V_8\rangle, \langle V_9\rangle, \langle V_{10}\rangle, \langle V_{11}\rangle$ and $\langle V_{12}\rangle$ are totally disconnected. From Figure 3.3, the twin equivalence classes in $\mathbb{EAG}(R)$ are $U_1 = \{R_1 \times (0) \times (0)\} \cup V_3$, $U_2 = \{R_1 \times (0) \times R_3\} \cup V_1 \cup V_4 \cup V_5$, $U_3 = \{R_1 \times R_2 \times (0)\} \cup V_2 \cup V_6 \cup V_7$, $U_4 = \{(0) \times (0) \times R_3\} \cup V_8$, $U_5 = \{(0) \times R_2 \times (0)\} \cup V_9$ and $U_6 = \{(0) \times R_2 \times R_3\} \cup V_{10} \cup V_{11} \cup V_{12}$. Let $W = \{R_1 \times (0) \times R_3, R_1 \times (0) \times (0)\}$. Then the adjacent vertices in $\mathbb{EAG}(R)$ have different local metric representations about W. Thus $dim_l(\mathbb{EAG}(R)) \leq 2$. From Figure 3.3, one can check that any set of one vertex of $\mathbb{A}(R)^*$ does not form a local metric set and hence $dim_l(\mathbb{EAG}(R)) = 2$.

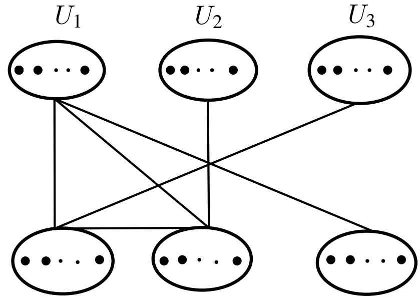

Figure 3.3

Assume that R_1 is an integral domain, R_2 and R_3 are rings with unique nonzero proper ideal. Suppose that I_1 is a nonzero proper ideal in R_1 and M_2, M_3 are unique nonzero proper ideal in R_2 and R_3 respectively. Consider $V_1 = \{I_1 \times (0) \times M_3{}^j : I_1 \in R_1\}$, $V_2 = \{I_1 \times M_2 \times M_3{}^j : I_1 \in R_1\}$, $V_3 = \{I_1 \times M_2{}^j \times R_3 : I_1 \in R_1\}$ and $V_4 = \{I_1 \times R_2 \times M_3{}^j : I_1 \in R_1\}$, for $j = 1, 2$. Here the induced subgraphs $\langle V_1\rangle$, $\langle V_2\rangle$, $\langle V_3\rangle$ and $\langle V_4\rangle$ are totally disconnected. From Figure 3.4, the twin equivalence classes in $\mathbb{EAG}(R)$ are $U_1 = \{R_1 \times (0) \times M_3{}^j, R_1 \times M_2 \times M_3{}^j\} \cup V_1 \cup V_2$, $U_2 = \{R_1 \times M_2{}^j \times R_3\} \cup V_3$, $U_3 = \{R_1 \times R_2 \times M_3{}^j\} \cup V_4$, $U_4 = \{(0) \times (0) \times M_3\}$, $U_5 = \{(0) \times R_2 \times R_3\}$, $U_6 = \{(0) \times R_2 \times M_3{}^j\}$, $U_7 = \{(0) \times M_2{}^j \times R_3\}$, $U_8 = \{(0) \times M_2 \times (0)\}$ and $U_9 = \{(0) \times M_2 \times M_3\}$, for $j = 1, 2$. Let $W = \{(0) \times (0) \times R_3, (0) \times R_2 \times (0)\}$ with cardinality 2. Then the adjacent vertices in $\mathbb{EAG}(R)$ have different local metric representations about W. Thus $dim_l(\mathbb{EAG}(R)) \leq 2$. Suppose that $W = \{(0) \times (0) \times R_3\}$. Then $D(U_1|W) = D(U_8|W)$, $D(U_1|W) = D(U_6|W)$ and $D(U_4|W) = D(U_9|W)$, so that W is not a local metric set of

$\mathbb{EAG}(R)$. Similarly, Figure 3.4 explicitly shows that $dim_l(\mathbb{EAG}(R)) \neq 1$ for all cases. Hence $dim_l(\mathbb{EAG}(R)) = 2$.

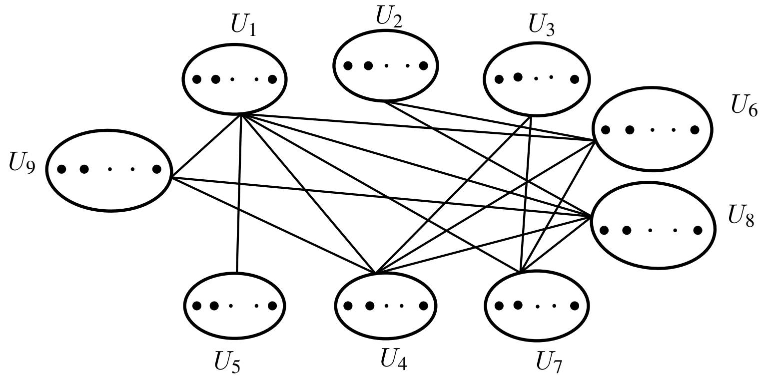

Figure 3.4

Conversely, assume that $dim_l(\mathbb{EAG}(R)) = 2$. Suppose that R_1, R_2 are integral domains and R_3 is not an integral domain. Here I_1, I_2 are nonzero proper ideals in R_1, R_2 respectively and M_3 is the maximal ideal in R_3 such that $M_3^r = (0)$. Consider $V_1 = \{(0) \times (0) \times M_3^k\}$, $V_2 = \{(0) \times I_2 \times M_3^k : I_2 \in R_2\}$, $V_3 = \{(0) \times I_2 \times R_3 : I_2 \in R_2\}$, $V_4 = \{(0) \times I_2 \times (0) : I_2 \in R_2\}$, $V_5 = \{(0) \times R_2 \times M_3^k\}$, $V_6 = \{I_1 \times (0) \times (0) : I_1 \in R_1\}$, $V_7 = \{I_1 \times (0) \times R_3 : I_1 \in R_1\}$, $V_8 = \{I_1 \times (0) \times M_3^k : I_1 \in R_1\}$, $V_9 = \{I_1 \times I_2 \times (0) : I_1 \in R_1, I_2 \in R_2\}$, $V_{10} = \{I_1 \times I_2 \times M_3^k : I_1 \in R_1, I_2 \in R_2\}$, $V_{11} = \{I_1 \times R_2 \times (0) : I_1 \in R_1\}$, $V_{12} = \{I_1 \times R_2 \times M_3^k : I_1 \in R_1\}$, $V_{13} = \{R_1 \times (0) \times M_3^k\}$, $V_{14} = \{R_1 \times I_2 \times (0) : I_2 \in R_2\}$, $V_{15} = \{R_1 \times I_2 \times M_3^k : I_2 \in R_2\}$, $V_{16} = \{R_1 \times R_2 \times M_3^k\}$, for $1 \leq k < r$.

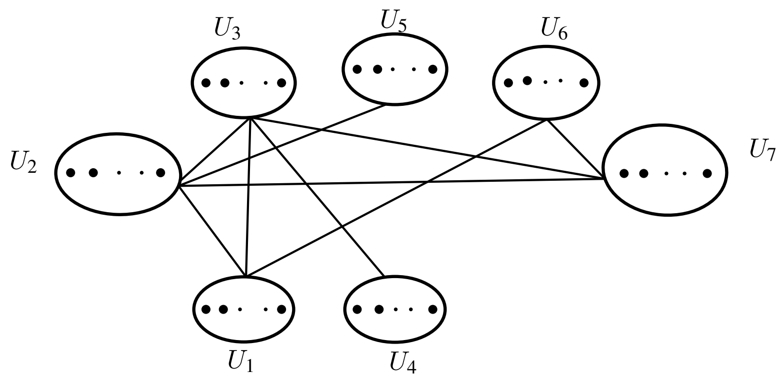

Figure 3.5

Here the induced subgraphs $\langle V_1 \rangle$ is complete and $\langle V_2 \rangle$, $\langle V_3 \rangle$, $\langle V_4 \rangle$, $\langle V_5 \rangle$, $\langle V_6 \rangle$, $\langle V_7 \rangle$, $\langle V_8 \rangle$, $\langle V_9 \rangle$, $\langle V_{10} \rangle$, $\langle V_{11} \rangle$, $\langle V_{12} \rangle$, $\langle V_{13} \rangle$, $\langle V_{14} \rangle$, $\langle V_{15} \rangle$ and $\langle V_{16} \rangle$ are totally disconnected. From Figure 3.5, the twin equivalence classes in $\mathbb{EAG}(R)$ are $U_1 = V_1$, $U_2 = \{(0) \times R_2 \times (0)\} \cup V_2 \cup V_4 \cup V_5$, $U_3 = \{R_1 \times (0) \times (0)\} \cup V_6 \cup V_8 \cup V_{13}$, $U_4 = \{(0) \times R_2 \times R_3\} \cup V_3$, $U_5 = \{R_1 \times (0) \times R_3\} \cup V_7$, $U_6 = \{R_1 \times R_2 \times (0)\} \cup V_9 \cup V_{10} \cup V_{11} \cup V_{12} \cup V_{14} \cup V_{15} \cup V_{16}$, $U_7 = \{(0) \times (0) \times R_3\}$ and the true twin equivalence class in $\mathbb{EAG}(R)$ is U_1 so that at least $|U_1| - 1$ vertices from U_1 must contained in the local metric set. Let $W = U_1 \cup \{(0) \times R_2 \times (0)\}$ and so $|W| = |\mathbb{A}(R_3)|$. Then the adjacent vertices in $\mathbb{EAG}(R)$ have different local metric representations about W. So $dim_l(\mathbb{EAG}(R)) \leq |\mathbb{A}(R_3)|$. Suppose that $W = U_1$ and $|W| = |\mathbb{A}(R_3)| - 1$. Then $D(U_2|W) = D(U_3|W)$ so that W is not a local metric set for $\mathbb{EAG}(R)$. Similarly for all cases, $dim_l(\mathbb{EAG}(R)) \neq |\mathbb{A}(R_3)| - 1$. Hence $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_3)|$.

Suppose that R_1 is an integral domain, R_2 is not an integral domain and R_3 is a ring with more than one nonzero proper ideals. In this, I_1 is a nonzero proper ideal in R_1 and M_2, M_3 are the maximal ideals in R_2, R_3 respectively such that ${M_2}^m = (0)$ and ${M_3}^r = (0)$. Consider $V_1 = \{(0) \times (0) \times {M_3}^k\}$, $V_2 = \{(0) \times {M_2}^j \times (0)\}$, $V_{3a} = \{(0) \times {M_2}^j \times {M_3}^k : ({M_2}^j)^l = (0), ({M_3}^k)^l \neq$

(0) for some $l \in \mathbb{Z}^+\}$, $V_{3b} = \{(0) \times M_2{}^j \times M_3{}^k : (M_2{}^j)^l = (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{3c} = \{(0) \times M_2{}^j \times M_3{}^k : (M_2{}^j)^l \neq (0), (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_4 = \{(0) \times M_2{}^j \times R_3\}$, $V_5 = \{(0) \times R_2 \times M_3{}^k\}$, $V_6 = \{I_1 \times (0) \times M_3{}^k : I_1 \in R_1\}$, $V_7 = \{I_1 \times M_2{}^j \times (0) : I_1 \in R_1\}$, $V_8 = \{I_1 \times M_2{}^j \times M_3{}^k : I_1 \in R_1\}$, $V_9 = \{I_1 \times M_2{}^j \times R_3 : I_1 \in R_1\}$, $V_{10} = \{I_1 \times R_2 \times M_3{}^k : I_1 \in R_1\}$, $V_{11} = \{R_1 \times (0) \times M_3{}^k\}$, $V_{12} = \{R_1 \times M_2{}^j \times (0)\}$, $V_{13} = \{R_1 \times M_2{}^j \times M_3{}^k\}$, $V_{14} = \{R_1 \times M_2{}^j \times R_3\}$, $V_{15} = \{R_1 \times R_2 \times M_3{}^k\}$, $V_{16} = \{I_1 \times (0) \times (0) : I_1 \in R_1\}$, $V_{17} = \{I_1 \times (0) \times R_3 : I_1 \in R_1\}$ and $V_{18} = \{I_1 \times R_2 \times (0) : I_1 \in R_1\}$, for $1 \leq j < m, 1 \leq k < r$. Here the induced subgraphs $\langle V_1\rangle$, $\langle V_2\rangle$, $\langle V_{3a}\rangle$, $\langle V_{3b}\rangle$, $\langle V_{3c}\rangle$ are complete and $\langle V_4\rangle$, $\langle V_5\rangle$, $\langle V_6\rangle$, $\langle V_7\rangle$, $\langle V_8\rangle$, $\langle V_9\rangle$, $\langle V_{10}\rangle$, $\langle V_{11}\rangle$, $\langle V_{12}\rangle$, $\langle V_{13}\rangle$, $\langle V_{14}\rangle$, $\langle V_{15}\rangle$, $\langle V_{16}\rangle$, $\langle V_{17}\rangle$ and $\langle V_{18}\rangle$ are totally disconnected. Figure 3.6 shows that the twin equivalence classes in $\mathbb{EAG}(R)$ are $U_1 = V_1 \cup V_{3a}$, $U_2 = V_2 \cup V_{3c}$, $U_3 = V_{3b}$, $U_4 = \{(0) \times (0) \times R_3\} \cup V_4$, $U_5 = \{(0) \times R_2 \times (0)\} \cup V_5$, $U_6 = \{(0) \times R_2 \times R_3\}$, $U_7 = \{R_1 \times (0) \times (0)\} \cup V_6 \cup V_7 \cup V_8 \cup V_{11} \cup V_{12} \cup V_{13} \cup V_{16}$, $U_8 = \{R_1 \times (0) \times R_3\} \cup V_9 \cup V_{14} \cup V_{17}$ and $U_9 = \{R_1 \times R_2 \times (0)\} \cup V_{10} \cup V_{15} \cup V_{18}$ and the true twin equivalence classes in $\mathbb{EAG}(R)$ are U_1, U_2 and U_3 so that at least $|U_i| - 1$ vertices from U_i must contained in the local metric set of $\mathbb{EAG}(R)$, for every $i = 1, 2, 3$.

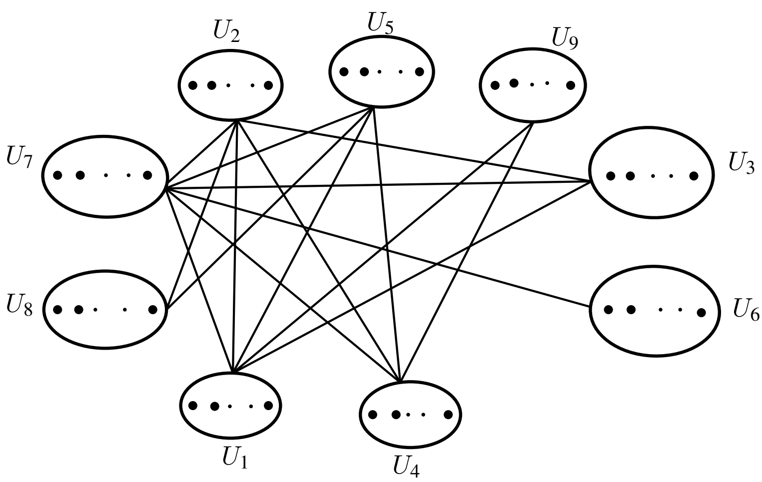

Figure 3.6

Let $W = (\bigcup_{i=1}^{3} U_i \backslash \{J_i\}) \cup \{(0) \times (0) \times R_3, (0) \times R_2 \times (0)\}$ where $J_i \in U_i$ so that $|W| = |\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 2$. Then the adjacent vertices in $\mathbb{EAG}(R)$ have different local metric representations about W. Thus W is a local metric set so that $dim_l(\mathbb{EAG}(R)) \leq |\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 2$. Suppose that $W = (\bigcup_{i=1}^{3} U_i \backslash \{J_i\}) \cup \{(0) \times (0) \times R_3\}$ where $J_i \in U_i$ and the cardinality is $|\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 3$. Then $D(U_7|W) = D(J_2|W)$ and $D(J_1|W) = D(J_3|W)$ so that W is not a local metric set for $\mathbb{EAG}(R)$. From Figure 3.6, one can check that $dim_l(\mathbb{EAG}(R)) \neq |\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 3$ for all cases. Hence $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 2$.

Suppose that R_1, R_2 and R_3 are not integral domains. Consider R_1, R_2 and R_3 have unique nonzero proper ideal say, M_1, M_2 and M_3 respectively. Here the twin equivalence classes in $\mathbb{EAG}(R)$ are $U_1 = \{R_1 \times (0) \times M_3{}^j, R_1 \times M_2 \times M_3{}^j\}$, $U_2 = \{(0) \times M_2{}^j \times R_3, M_1 \times M_2{}^j \times R_3\}$, $U_3 = \{(0) \times R_2 \times M_3{}^j, M_1 \times R_2 \times M_3{}^j\}$, $U_4 = \{M_1{}^j \times R_2 \times R_3\}$, $U_5 = \{R_1 \times R_2 \times M_3{}^j\}$, $U_6 = \{R_1 \times M_2{}^j \times R_3\}$, $U_7 = \{(0) \times (0) \times M_3\}$, $U_8 = \{(0) \times M_2 \times (0)\}$, $U_9 = \{M_1 \times (0) \times (0)\}$, $U_{10} = \{M_1 \times (0) \times M_3\}$, $U_{11} = \{M_1 \times M_2 \times (0)\}$, $U_{12} = \{(0) \times M_2 \times M_3\}$ and $U_{13} = \{M_1 \times M_2 \times M_3\}$, for $j = 1, 2$. Let $W = \{R_1 \times (0) \times (0), (0) \times R_2 \times (0), (0) \times (0) \times R_3\}$ with cardinality 3. Then the adjacent vertices in $\mathbb{EAG}(R)$ have distinct local metric representations about W. Consequently, W is a local metric set of $\mathbb{EAG}(R)$ so that $dim_l(\mathbb{EAG}(R)) \leq 3$. Suppose that $W = \{R_1 \times (0) \times (0), (0) \times R_2 \times (0)\}$ with cardinality 2. Then $D(U_9|W) = D(U_{10}|W)$, $D(U_8|W) = D(U_{12}|W)$, $D(U_{11}|W) = D(U_{13}|W)$ and so W is not a local metric set of $\mathbb{EAG}(R)$. Similarly, $dim_l(\mathbb{EAG}(R)) \neq 2$ for all cases. Hence $dim_l(\mathbb{EAG}(R)) = 3$.

Finally consider R_3 to be a ring with more than one nonzero proper ideals. Here M_1, M_2 and M_3 are the maximal ideals in R_1, R_2 and R_3 respectively such that $M_1{}^n = (0)$, $M_2{}^m = (0)$ and $M_3{}^r = (0)$. Consider $V_{1a} = \{M_1{}^i \times (0) \times M_3{}^k : (M_1{}^i)^l = (0), (M_3{}^k)^l \neq (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{1b} = \{M_1{}^i \times (0) \times M_3{}^k : (M_1{}^i)^l = (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{1c} = \{M_1{}^i \times (0) \times M_3{}^k :$

$(M_1{}^i)^l \neq (0), (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{2a} = \{M_1{}^i \times M_2{}^j \times (0) : (M_1{}^i)^l = (0), (M_2{}^j)^l \neq (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{2b} = \{M_1{}^i \times M_2{}^j \times (0) : (M_1{}^i)^l = (M_2{}^j)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{2c} = \{M_1{}^i \times M_2{}^j \times (0) : (M_1{}^i)^l \neq (0), (M_2{}^j)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{3a} = \{(0) \times M_2{}^j \times M_3{}^k : (M_2{}^j)^l = (0), (M_3{}^k)^l \neq (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{3b} = \{(0) \times M_2{}^j \times M_3{}^k : (M_2{}^j)^l = (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{3c} = \{(0) \times M_2{}^j \times M_3{}^k : (M_2{}^j)^l \neq (0), (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{4a} = \{M_1{}^i \times M_2{}^j \times M_3{}^k : (M_1{}^i)^l = (M_2{}^j)^l = (0), (M_3{}^k)^l \neq (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{4b} = \{M_1{}^i \times M_2{}^j \times M_3{}^k : (M_1{}^i)^l = (M_3{}^k)^l = (0), (M_2{}^j)^l \neq (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{4c} = \{M_1{}^i \times M_2{}^j \times M_3{}^k : (M_1{}^i)^l \neq (0), (M_2{}^j)^l = (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{4d} = \{M_1{}^i \times M_2{}^j \times M_3{}^k : (M_1{}^i)^l = (M_2{}^j)^l = (M_3{}^k)^l = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{4e} = \{M_1{}^i \times M_2{}^j \times M_3{}^k : (M_1{}^i)^l = (M_2{}^j)^{l+1} = (M_3{}^k)^{l+1} = (0), (M_2{}^j)^l \neq (0)$ and $(M_3{}^k)^l \neq (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{4f} = \{M_1{}^i \times M_2{}^j \times M_3{}^k : (M_1{}^i)^l \neq (0), (M_3{}^k)^l \neq (0), (M_2{}^j)^l = (M_1{}^i)^{l+1} = (M_3{}^k)^{l+1} = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_{4g} = \{M_1{}^i \times M_2{}^j \times M_3{}^k : (M_1{}^i)^l \neq (0), (M_2{}^j)^l \neq (0)$ and $(M_3{}^k)^l = (M_1{}^i)^{l+1} = (M_2{}^j)^{l+1} = (0)$ for some $l \in \mathbb{Z}^+\}$, $V_5 = \{(0) \times (0) \times M_3{}^k\}$, $V_6 = \{(0) \times M_2{}^j \times (0)\}$, $V_7 = \{M_1{}^i \times (0) \times (0)\}$, $V_8 = \{M_1{}^i \times R_2 \times (0)\}$, $V_9 = \{(0) \times R_2 \times M_3{}^k\}$, $V_{10} = \{M_1{}^i \times R_2 \times M_3{}^k\}$, $V_{11} = \{R_1 \times R_2 \times M_3{}^k\}$, $V_{12} = \{(0) \times M_2{}^j \times R_3\}$, $V_{13} = \{M_1{}^i \times M_2{}^j \times R_3\}$, $V_{14} = \{M_1{}^i \times (0) \times R_3\}$, $V_{15} = \{R_1 \times M_2{}^j \times R_3\}$, $V_{16} = \{R_1 \times (0) \times M_3{}^k\}$, $V_{17} = \{R_1 \times M_2{}^j \times M_3{}^k\}$, $V_{18} = \{R_1 \times M_2{}^j \times (0)\}$ and $V_{19} = \{M_1{}^i \times R_2 \times R_3\}$, for every $1 \leq i < n, 1 \leq j < m$ and $1 \leq k < r$. The induced subgraphs $\langle V_{1a}\rangle$, $\langle V_{1b}\rangle$, $\langle V_{1c}\rangle$, $\langle V_{2a}\rangle$, $\langle V_{2b}\rangle$, $\langle V_{2c}\rangle$, $\langle V_{3a}\rangle$, $\langle V_{3b}\rangle$, $\langle V_{3c}\rangle$, $\langle V_{4a}\rangle$, $\langle V_{4b}\rangle$, $\langle V_{4c}\rangle$, $\langle V_{4d}\rangle$, $\langle V_{4e}\rangle$, $\langle V_{4f}\rangle$, $\langle V_{4g}\rangle$, $\langle V_5\rangle$, $\langle V_6\rangle$ and $\langle V_7\rangle$ are complete graphs and $\langle V_8\rangle$, $\langle V_9\rangle$, $\langle V_{10}\rangle$, $\langle V_{11}\rangle$, $\langle V_{12}\rangle$, $\langle V_{13}\rangle$, $\langle V_{14}\rangle$, $\langle V_{15}\rangle$, $\langle V_{16}\rangle$, $\langle V_{17}\rangle$, $\langle V_{18}\rangle$ and $\langle V_{19}\rangle$ are totally disconnected.

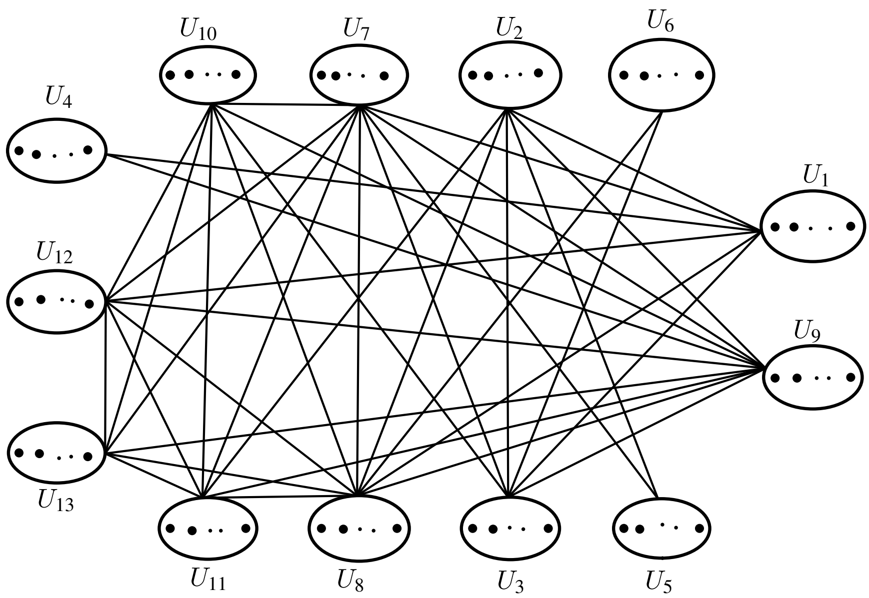

Figure 3.7

In view of Figure 3.7, the twin equivalence classes in $\mathbb{EAG}(R)$ are $U_1 = \{R_1 \times (0) \times (0)\} \cup V_{16} \cup V_{17} \cup V_{18}$, $U_2 = \{(0) \times (0) \times R_3\} \cup V_{12} \cup V_{13} \cup V_{14}$, $U_3 = \{(0) \times R_2 \times (0)\} \cup V_8 \cup V_9 \cup V_{10}$, $U_4 = \{(0) \times R_2 \times R_3\} \cup V_{19}$, $U_5 = \{R_1 \times R_2 \times (0)\} \cup V_{11}$, $U_6 = \{R_1 \times (0) \times R_3\} \cup V_{15}$, $U_7 = V_{1a} \cup V_{3a} \cup V_{4a} \cup V_5$, $U_8 = V_{2a} \cup V_{3c} \cup V_{4b} \cup V_6$, $U_9 = V_{1c} \cup V_{2c} \cup V_{4c} \cup V_7$, $U_{10} = V_{1b} \cup V_{4f}$, $U_{11} = V_{2b} \cup V_{4g}$, $U_{12} = V_{3b} \cup V_{4e}$ and $U_{13} = V_{4d}$ and the true twin equivalence classes are $U_7, U_8, U_9, U_{10}, U_{11}, U_{12}$ and U_{13} so that at least $|U_i| - 1$ vertices from U_i must contained in the local metric set for $\mathbb{EAG}(R)$ for all $i = 7$ to 13. Let $W = (\bigcup_{i=7}^{13} U_i \backslash \{J_i\}) \cup \{R_1 \times (0) \times (0), (0) \times R_2 \times (0), (0) \times (0) \times R_3\}$ where $J_i \in U_i$ and so $|W| = |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 5$. Then the adjacent vertices in $\mathbb{EAG}(R)$ have distinct local metric representations about W. Hence W is a local metric set of $\mathbb{EAG}(R)$ and $dim_l(\mathbb{EAG}(R)) \leq |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 5$. Suppose that $W = (\bigcup_{i=7}^{13} U_i \backslash \{J_i\}) \cup \{(0) \times R_2 \times (0), (0) \times (0) \times R_3\}$ where $J_i \in U_i$ so that the cardinality is $|\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 6$. Then $D(J_7|W) = D(J_{10}|W)$, $D(J_8|W) = D(J_{11}|W)$ and $D(J_{12}|W) = D(J_{13}|W)$. Hence W is not a local metric set of $\mathbb{EAG}(R)$. Similarly, $dim_l(\mathbb{EAG}(R)) \neq |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 6$

for all cases. Hence $dim_l(\mathbb{EAG}(R)) = |\mathbb{A}(R_1)|\,|\mathbb{A}(R_2)|\,|\mathbb{A}(R_3)| - 5$. From all cases, R_1, R_2 and R_3 are integral domains or R_1 is an integral domain, R_2 and R_3 are rings with unique nonzero proper ideal.
$(ii), (iii), (iv)$ and (v) follow from the proof of (i). □

The following is an instance of the previous theorem.

Example 3.6. (a) If $R \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, then $W = \{\mathbb{Z} \times (0) \times \mathbb{Z}, \mathbb{Z} \times (0) \times (0)\}$ is a local metric basis so that $dim_l(\mathbb{EAG}(R)) = 2$.
(b) Let $R \cong \mathbb{Z}[i] \times \mathbb{Z}[i] \times \mathbb{Z}_8$. Here (2) is the maximal ideal in $\mathbb{Z}_8$ and $W = \{(0) \times (0) \times (2), (0) \times \mathbb{Z}[i] \times (0), (0) \times (0) \times \mathbb{Z}_8\}$ is a local metric basis for $\mathbb{EAG}(R)$. Hence $dim_l(\mathbb{EAG}(R)) = 3$.
(c) Let $R \cong \mathbb{Z} \times \frac{\mathbb{Z}_2[X]}{(X^2)} \times \frac{\mathbb{Z}_2[X]}{(X^2)}$. In this example, (X) is a unique nonzero proper ideal in $\frac{\mathbb{Z}_2[X]}{(X^2)}$ and the local metric basis is $W = \{(0) \times \frac{\mathbb{Z}_2[X]}{(X^2)} \times (0), (0) \times (0) \times \frac{\mathbb{Z}_2[X]}{(X^2)}\}$. Then $dim_l(\mathbb{EAG}(R)) = 2$.
(d) Let $R \cong \mathbb{Z} \times \mathbb{Z}_8 \times \mathbb{Z}_8$. Then $W = \{(0) \times (0) \times (2), (0) \times (2) \times (0), (0) \times (0) \times \mathbb{Z}_8, (0) \times \mathbb{Z}_8 \times (0), (0) \times (2) \times (2), (0) \times (2) \times (4), (0) \times (4) \times (2)\}$ is a local metric basis for $\mathbb{EAG}(R)$ so that $dim_l(\mathbb{EAG}(R)) = 7$.
(e) Let $R \cong \frac{\mathbb{R}[X]}{(X^2)} \times \frac{\mathbb{R}[X]}{(X^2)} \times \frac{\mathbb{R}[X]}{(X^2)}$. Here (X) is a unique nonzero proper ideal in $\frac{\mathbb{R}[X]}{(X^2)}$ and the local metric basis for $\mathbb{EAG}(R)$ is $W = \{\frac{\mathbb{R}[X]}{(X^2)} \times (0) \times (0), (0) \times \frac{\mathbb{R}[X]}{(X^2)} \times (0), (0) \times (0) \times \frac{\mathbb{R}[X]}{(X^2)}\}$. Hence $dim_l(\mathbb{EAG}(R)) = 3$.
(f) Let $R \cong \frac{\mathbb{Z}[i]}{(1+i)^3} \times \frac{\mathbb{Z}[i]}{(1+i)^3} \times \frac{\mathbb{Z}[i]}{(1+i)^3}$. In this, $(1+i)$ is the maximal ideal in $\frac{\mathbb{Z}[i]}{(1+i)^3}$ such that $(1+i)^3 = (0)$ and $W = \{\frac{\mathbb{Z}[i]}{(1+i)^3} \times (0) \times (0), (0) \times \frac{\mathbb{Z}[i]}{(1+i)^3} \times (0), (0) \times (0) \times \frac{\mathbb{Z}[i]}{(1+i)^3}, (0) \times (0) \times (1+i), (0) \times (1+i) \times (0), (1+i) \times (0) \times (0), (1+i) \times (0) \times (1+i), (1+i) \times (0) \times (1+i)^2, (1+i)^2 \times (0) \times (1+i), (1+i) \times (1+i) \times (0), (1+i) \times (1+i)^2 \times (0), (1+i)^2 \times (1+i) \times (0), (0) \times (1+i) \times (1+i), (0) \times (1+i) \times (1+i)^2, (0) \times (1+i)^2 \times (1+i), (1+i) \times (1+i) \times (1+i), (1+i) \times (1+i) \times (1+i)^2, (1+i) \times (1+i)^2 \times (1+i), (1+i)^2 \times (1+i) \times (1+i), (1+i) \times (1+i)^2 \times (1+i)^2, (1+i)^2 \times (1+i) \times (1+i)^2, (1+i)^2 \times (1+i)^2 \times (1+i)\}$ is a local metric basis for $\mathbb{EAG}(R)$ so that $dim_l(\mathbb{EAG}(R)) = 22$.

Finally, we give an excel characterization of $\mathbb{Z}_n$ in the following corollary.

Corollary 3.7. *If $R \cong \mathbb{Z}_n$ and p, q, r be three distinct primes, then the following occurs.*

(i) *If $n = p^\alpha$, $\alpha \geq 1$, then*
- (a) *$\alpha \geq 2$ if and only if $dim_l(\mathbb{EAG}(R)) = \alpha - 2$.*
- (b) *$\alpha = 1$ if and only if $dim_l(\mathbb{EAG}(R))$ is undefined.*

(ii) *If $n = p^\alpha q^\beta, \alpha, \beta \geq 1$, then*
- (a) *Either $\alpha = \beta = 1$ or $\alpha = 1, \beta = 2$ if and only if $dim_l(\mathbb{EAG}(R)) = 1$.*
- (b) *$\alpha = 1, \beta \geq 3$ if and only if $dim_l(\mathbb{EAG}(R)) = \beta - 1$.*
- (c) *$\alpha = \beta = 2$ if and only if $dim_l(\mathbb{EAG}(R)) = 2$.*
- (d) *$\alpha \geq 2, \beta \geq 3$ if and only if $dim_l(\mathbb{EAG}(R)) = \alpha\beta - 2$.*

(iii) *If $n = p^\alpha q^\beta r^\gamma, \alpha, \beta, \gamma \geq 1$, then*
- (a) *Either $\alpha = \beta = \gamma = 1$ or $\alpha = 1, \beta = \gamma = 2$ if and only if $dim_l(\mathbb{EAG}(R)) = 2$.*
- (b) *$\alpha = \beta = 1, \gamma \geq 2$ if and only if $dim_l(\mathbb{EAG}(R)) = \gamma$.*
- (c) *$\alpha = 1, \beta \geq 2, \gamma \geq 3$ if and only if $dim_l(\mathbb{EAG}(R)) = \beta\gamma - 2$.*
- (d) *$\alpha = \beta = \gamma = 2$ if and only if $dim_l(\mathbb{EAG}(R)) = 3$.*
- (e) *$\alpha, \beta \geq 2, \gamma \geq 3$ if and only if $dim_l(\mathbb{EAG}(R)) = \alpha\beta\gamma - 5$.*

Proof. As R is an artinian PIR, then $R \cong \prod_{i=1}^{n} R_i$ where $R_i's$ are SPRs.
(i) (a) Assume that $n = p^\alpha$ and $\alpha \geq 2$. As R is a SPR, then by Theorem 3.2, the result holds. Conversely, assume that $dim_l(\mathbb{EAG}(R)) = \alpha - 2$. Let $\alpha = 1$. Since $\mathbb{EAG}(R)$ is an empty graph, $dim_l(\mathbb{EAG}(R))$ is undefined. Hence $\alpha \geq 2$.
(b) Follows from (a).

(ii) Here $\mathbb{A}(R)^* = \{(p^i)\} \cup \{(q^j)\} \cup (\{(p^i q^j)\} \backslash \{(p^\alpha q^\beta)\})$, for $1 \leq i \leq \alpha, 1 \leq j \leq \beta$. The result follows from Theorem 3.3.
(iii) In this case, $\mathbb{A}(R)^* = \{(p^i)\} \cup \{(q^j)\} \cup \{(r^k)\} \cup \{(p^i q^j)\} \cup \{(p^i r^k)\} \cup \{(q^j r^k)\} \cup (\{(p^i q^j r^k)\} \backslash \{(p^\alpha q^\beta r^\gamma)\})$, for $1 \leq i \leq \alpha, 1 \leq j \leq \beta, 1 \leq r \leq \gamma$. The proof follows from Theorem 3.5.

□

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