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Swot on Average degree square sum energy of graphs

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Abstract The average degree square sum matrix AD of a graph G is equal to order of G whose elements are defined as $\frac{d_i^2+d_j^2}{2}$ if $v_i \sim v_j$ and otherwise zero. In this paper, we introduce a new energy of graph under the name of average degree square sum energy. We also obtain characteristic polynomial of the average degree square sum of standard graphs and bounds. In addition, we apply graph operators to standard graphs ascertained the characteristic polynomials. Finally, we characterize the average degree square sum hyperenergetic, border-energetic and equi-energetic of some graphs.

1 Introduction

The concept of energy in a graph arose from Huckel theory in which the π-electron energy of a conjugated carbon molecule is computed which coincides with the energy of a graph. In discrete structures, there are several graph polynomials based on matrices such as adjacency matrix, Laplacian matrix, signless Laplacian matrix, distance matrix, degree sum matrix and degree exponent matrix are present. Motivated from these, we introduce and study the average degree square sum matrix of a graph G. Some of the new results to energies and directed graph [[11],[12]].

Let $A = (a_{ij})$ be an adjacency matrix of order n of a graph G. The characteristic polynomial of a graph G is denoted by $Ch(G, \lambda) = (\lambda I - G)$, where λ is an eigenvalue of a graph G. Hence, by [9], the energy of G is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid$.

Let $V(G)$ be the vertex set and $E(G)$ be an edge set of G. The degree of a vertex G is denoted by $d_u(G)$. The average degree square sum matrix of a graph G is denoted by $AD(G) = (s_{ij})$ and whose elements are defined as

$$s_{ij} = \begin{cases} \dfrac{d_i^2 + d_j^2}{2} & \text{if } v_i \sim v_j \\ 0 & \text{if otherwise} \end{cases} .$$

Here, the considered graphs are the simple, finite and undirected. Basic terminologies and notations can be found in [8].

2 Basic properties of largest average degree square sum eigenvalue

Here, we initiated to study few properties which are useful to further development.

Let us define the number p as

$$p = \sum_{i<j} \left(\frac{d_i^2 + d_j^2}{2} \right)^2$$

Proposition 2.1. *The first three coefficient of the polynomial $Ch(AD(G, \lambda))$ are as follows:*

(i) $a_0 = 1$,

(ii) $a_1 = 0$,

(iii) $a_2 = -p$.

Proof. (i) By the definition of characteristic poynomial, trivially, $a_0 = 1$.

(ii) We know that all principal diagonal entries of average degree square matrix are zero, so,

$$a_1 = tr(AD(G)) = 0.$$

(iii) We have ,

$$(-1)^2 a_2 = \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ji}a_{ij})$$
$$= -p.$$

□

Proposition 2.2. *If λ_1, λ_2 ,..., λ_n are the average degree square sum eigenvalues of AD(G), then*

$$\sum_{i=1}^{n} \lambda_i^2 = 2p.$$

Proof.

$$\sum_{i=1}^{n} \lambda_i^2 = tr([AD(G)]^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}d_{ji}$$
$$= 2\sum_{i<j}(d_{ij})^2 + \sum_{i=1}^{n}(d_{ij})^2$$
$$= \sum_{i<j}(d_i^2 + d_j^2)^2$$
$$= 2p.$$

□

Theorem 2.3 ([13]). *Let a_i and b_i are nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - (\sum_{i=1}^{n} a_i^2 b_i^2)^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2,$$

where $M_1 = max(a_i)$, $M_2 = max(b_i)$ and $m_1 = min(a_i)$, $m_2 = min(b_i)$ where $i = 1, 2, ..., n$.

Theorem 2.4 ([1]). *Let a_i and b_i are non-negative real numbers. Then*

$$\left| n\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \le \alpha(n)(A-a)(B-b),$$

where a, b, A and B are real constants such that $a \le a_i \le A$ and $b \le b_i \le B$ for each i, $1 \le i \le n$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Theorem 2.5 ([6]). *Let a_i and b_i are non-negative real numbers. Then*

$$\sum_{i=1}^{n} b_i^2 + C_1C_2 \sum_{i=1}^{n} a_i^2 \le (C_1 + C_2)\sum_{i=1}^{n} a_ib_i,$$

where C_1 and C_2 are real constants such that $C_1a_i \le b_i \le C_2a_i$ for each i, $1 \le i \le n$.

Theorem 2.6. *Let G be a r-regular graph of order n. Then G has only one positive average degree square sum eigenvalue $\lambda = r^2(n-1)$.*

Proof. Let G be a connected r-regular graph of order n and $\{v_1, v_2, ..., v_n\}$ be the vertex set of G. Let $d_i = r$ be the degree of v_i, $i = 1, 2, ...n$. Then the characteristic polynomial of $AD(G)$ is

$$Ch[AD(G), \lambda] = (\lambda - r^2(n-1))(\lambda + r^2)^{n-1}. \tag{2.1}$$

Therefore, the eigenvalues are $r^2(n-1)$ and $-r^2$ which repeats $(n-1)$ times.

□

Theorem 2.7. *Let G be any graph of order n and let λ_1 be the largest average degree square sum eigenvalue. Then*

$$\lambda_1 \leq \sqrt{\frac{2p(n-1)}{n}}.$$

Proof. By the Cauchy-Schwartz inequality [[**?**]], we have

$$\left(\sum_{i=1}^{n} a_i^2 b_i^2\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,$$

where a_i and b_i are non-negative real numbers. Now, substituting $a_i = 1$ and $b_i = \lambda_i$, we have

$$\left(\sum_{i=2}^{n} \lambda_i^2\right)^2 \leq (n-1) \sum_{i=2}^{n} b_i^2.$$

By using Propositions 2.1 and 2.2 in above inequality,

$$(-\lambda_1)^2 \leq (n-1)(2p - \lambda_1^2).$$

Hence,

$$\lambda_1 \leq \sqrt{\frac{2p(n-1)}{n}}.$$

Remark 2.8. If G be a regular graph, then

$$\lambda_1 = \sqrt{\frac{2p(n-1)}{n}}.$$

□

Remark 2.9. Let G be a r-regular graph of order n. Then $AD(G) = r^2J - r^2I$, where J is the the matrix of order n whose all entries are equal to one and I is an identity matrix of order n. The characteristic polynomial is given by

$$Ch[AD(G), \lambda] = (\lambda - r^2(n-1))(\lambda + r^2)^{n-1}.$$

Hence

$$\mathcal{E}[AD(G)] = 2r^2(n-1).$$

Remark 2.10. If G is a r-regular graph and its complement $\overline{G}$ is $(n-1-r)$ regular graph, then

$$Ch[AD(\overline{G}), \lambda] = (\lambda - (n-1)(n-1-r)^2)(\lambda + (n-1-r)^{2)^{n-1}}.$$

Thus,

$$\mathcal{E}[AD(\overline{G})] = 2(n-1-r)^2(n-1).$$

Theorem 2.11. *Let G be a graph of order n and size m. Then*

$$\mathcal{E}[AD(G)] \geq \sqrt{2np - \frac{n^2}{4}(\mid \lambda_1 \mid - \mid \lambda_2 \mid)^2}.$$

Proof. Let λ_1, λ_2, ..., λ_n are the average degree square sum eigenvalues of G. Substituting $a_i = 1$ and $b_i =| \lambda_i |$ in the theorem 2.3 we get

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} | \lambda_i |^2 - \left(\sum_{i=1}^{n} | \lambda_i |^2 \right)^2 \leq \frac{n^2}{4} (| \lambda_1 | - | \lambda_n |)^2$$

$$2pn - (\mathcal{E}[AD(G)])^2 \leq \frac{n^2}{4} (| \lambda_1 | - | \lambda_n |)^2$$

$$\mathcal{E}[AD(G)] \geq \sqrt{2np - \frac{n^2}{4} (| \lambda_1 | - | \lambda_n |)^2}$$

□

Theorem 2.12. *Let G be a graph of order n. Then*

$$\sqrt{2p} \leq \mathcal{E}[AD(G)] \leq \sqrt{2np}.$$

Proof. By the Cauchy-Schwartz inequality [[**?**]], we have

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

where a_i and b_i are non-negative real numbers. Now, substituting $a_i = 1$ and $b_i = \lambda_i$ we have

$$\left(\sum_{i=1}^{n} | \lambda_i | \right)^2 \leq \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} | \lambda_i |^2$$

$$(\mathcal{E}[AD(G)])^2 \leq 2pn.$$

Thus,

$$\mathcal{E}[AD(G)] \leq \sqrt{2pn}$$

and

$$\sum_{i=1}^{n} | \lambda_i |^2 \leq \left(\sum_{i=1}^{n} | \lambda_i | \right)^2$$

$$2p \leq (\mathcal{E}[AD(G)])^2$$

which implies

$$\mathcal{E}[AD(G)] \geq \sqrt{2p}.$$

□

Theorem 2.13. *Let G be a graph of order n and Δ be the absolute value of the determinant of $AD(G)$. Then*

$$\sqrt{2p + n(n-1)\Delta^{\frac{2}{n}}} \leq \mathcal{E}[AD(G)] \leq \sqrt{2np}.$$

Proof. We know that

$$(\mathcal{E}[AD(G)])^2 = \left(\sum_{i=1}^{n} | \lambda_i | \right)^2.$$

$$= \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{i<j} | \lambda_i || \lambda_j |$$

$$= 2p + 2 \sum_{i<j} | \lambda_i || \lambda_j |$$

$$(\mathcal{E}[AD(G)])^2 = 2p + \sum_{i \neq j} | \lambda_i || \lambda_j | \quad (2.2)$$

Since we know for non-negative numbers, the arithmetic mean is always greater than or equal to the geometric mean.

$$\frac{1}{n(n-1)}\sum_{i\neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$
$$= \left(\prod_{i=1} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$
$$= \prod_{i\neq j} |\lambda_i|^{\frac{2}{n}}$$
$$= \Delta^{\frac{2}{n}}$$

Therefore,

$$\sum_{i\neq j} |\lambda_i||\lambda_j| \geq n(n-1)\Delta^{\frac{2}{n}}.$$

from equation (2) we have,

$$\mathcal{E}[AD(G)] \geq \sqrt{2p + n(n-1)\Delta^{\frac{2}{n}}}.$$

Consider a non-negative quantity Y such that,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(|\lambda_i| - |\lambda_j|)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\lambda_i|^2 + |\lambda_j|^2 - 2|\lambda_i||\lambda_j|)$$

Let

$$Y = n\sum_{i=1}^{n} |\lambda_i|^2 + n\sum_{j=1}^{n} |\lambda_j|^2 - 2\sum_{i=1}^{n} |\lambda_i| \sum_{j=1}^{n} |\lambda_j|$$
$$Y = 4np - 2(\mathcal{E}[AD(G)])^2.$$

Since,

$$Y \geq 0$$
$$4np - 2(\mathcal{E}[AD(G)])^2 \geq 0$$
$$\mathcal{E}[AD(G)] \leq \sqrt{2np}$$

□

Corollary 2.14. *If G is a r-regular graph of order n, then*

$$\mathcal{E}[AD(G)] \leq 2nr^2\sqrt{n-1}.$$

Theorem 2.15. *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be a non-increasing arrangement of average degree square sum eigenvalues. Then*

$$\mathcal{E}[AD(G)] \geq \sqrt{2np - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$$

where $\alpha(n) = n\lfloor\frac{n}{2}\rfloor(1 - \frac{1}{n}\lfloor\frac{n}{2}\rfloor)$.

Proof. Let λ_1, λ_2, ..., λ_n are the average degree square sum eigenvalues of G. Substituting $a_i = |\lambda_i| = b_i$ and $a = |\lambda_n| = b$, $A = |\lambda_1| = B$ in the theorem 2.4

$$\left| n\sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

Since $\mathcal{E}[AD(G)] = \sum_{i=1}^{n} |\lambda_i|$ and $\sum_{i=1}^{n} |\lambda_i|^2 = 2p$ we get the required result. □

Theorem 2.16. *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be a non-increasing arrangement of average degree square sum eigenvalues. Then*

$$E[AD(G)] \geq \frac{|\lambda_1||\lambda_n| n + 2p}{|\lambda_1| + |\lambda_n|}$$

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the average degree square sum eigenvalues of G. Substituting $a_i = 1$ and $b_i = |\lambda_i|$, $C_1 = |\lambda_n|$, $C_2 = |\lambda_1|$ in Theorem 2.5,

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1^2 \leq (|\lambda_1| + |\lambda_n|) \left(\sum_{i=1}^{n} |\lambda_i| \right)$$

Since $\mathcal{E}[AD(G)] = \sum_{i=1}^{n} |\lambda_i|$ and $\sum_{i=1}^{n} |\lambda_i|^2 = 2p$ we get the required result. □

Definition 2.17. [8] The line graph $L(G)$ of a graph G is a graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G are adjacent.

The k^{th} iterated line graph [2, 3, 8] of G is defined as $L^k(G) = L(L^{k-1}(G))$, $k = 1, 2, 3..$ where $L^0(G) \cong G$ and $L^1(G) \cong L(G)$.

Remark 2.18 ([2, 3]). The line graph $L(G)$ of an r-regular graph of G of order n is an r_1 = $(2r - 2)$-regular graph of order $n_1 = \frac{nr}{2}$. Thus, $L^k(G)$ is an r_k-regular graph of order n_k given by

$$n_k = \frac{n}{2^k} \prod_{i=1}^{k-1} (2^i r - 2^{i+1} + 2) \quad and \quad r_k = 2^k r - 2^{k+1} + 2.$$

Theorem 2.19. *Let G be a r-regular graph of order n and let $L^k(G)$ be the r_k-regular graph of order n_k then average degree square sum energy of $L^k(G)$ is*

$$E[ADD(L^k(G))] = 2r_k^2(n-1) \quad where, \quad r_k = 2^k r - 2^{k+1} + 2.$$

Proof. The average degree square sum characteristic polynomial of $L^k(G)$ with vertex set n_k (Remarks 2.9 and 2.18) is given by

$$Ch[ADS(L^k(G)), \lambda] = [\lambda - (2^k r - 2^{k+1} + 2)^2 (n_k - 1)][\lambda + (2^k r - 2^{k+1} + 2)^2]^{n_k - 1}$$

Thus,

$$E[ADD(L^k(G))] = 2r_k^2(n-1) \quad where, \quad r_k = 2^k r - 2^{k+1} + 2.$$

□

Lemma 2.20 ([14]). *If a, b, c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

$$= (\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}[(\lambda - (n_1 - 1)a)(\lambda - (n_2 - 1)b) - n_1 n_2 cd]$$

3 Operated $AD(G)$ with illustrative graphs

This section instigate the operations of graphs.

Definition 3.1 ([8]). The subdivision graph $S(G)$ of a graph G is a graph with the vertex set $V(G) \cup E(G)$ and is obtained by inserting a new vertex of degree 2 into each edge of G.

Definition 3.2 ([15]). The semitotal line graph $T_1(G)$ of a graph G is a graph with the vertex set $V(G) \cup E(G)$ where two vertices of $T_1(G)$ are adjacent if and only if they corresponds to two adjacent edges of G or one is a vertex of G and another is an edge G incident with it in G.

Definition 3.3 ([15]). The semitotal point graph $T_2(G)$ of a graph G is a graph with the vertex set $V(G) \cup E(G)$ where two vertices of $T_2(G)$ are adjacent if and only if they corresponds to two adjacent vertices of G or one is a vertex of G and another is an edge G incident with it in G.

Definition 3.4 ([8]). The total graph $T(G)$ of a graph G is the graph whose the vertex set is $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are either adjacent or incident.

Definition 3.5 ([14]). The graph G^{+k} is a graph obtained from the graph G by attaching k pendant edges to each vertex of G. If G is a graph of order n and size m, then G^{+k} is graph of order $n + nk$ and size $m + nk$.

Definition 3.6 ([8]). The union of the graphs G_1 and G_2 is a graph $G_1 \cup G_2$ whose the vertex set is $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 3.7 ([8]). The join $G_1 + G_2$ of two graphs G_1 and G_2 is the graph obtained from G_1 and G_2 by joining every vertex of G_1 to all vertices of G_2.

Definition 3.8 ([8]). The product $G \times H$ of two graphs G and H is defined as follows:
Consider any two points $u = (u_1, u_2)$ and $u = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G \times H$ whenever $[u_1 = v_1 and u_2 adj v_2] or [u_2 = v_2 and u_1 adj v_1]$.

Definition 3.9 ([8]). The composition graph $G[H]$ of two graphs G and H is defined as follows:
Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$) in $V = V_1 \times V_2$. Then u and v are adjacent in $G[H]$ whenever $[u_1 adj v_1]$ or $[u_1 = v_1 and u_2 adj v_2]$.

Definition 3.10 ([8]). The corona $G \circ H$ of graphs G and H is a graph obtained from G and H by taking one copy of G and $\mid V(G) \mid$ copies of H and then joining by an edge each vertex of the i^{th} copy of H is named (H, i) with the i^{th} vertex of G.

Definition 3.11 ([4]). The jump graph $J(G)$ of a graph G is defined as a graph with the vertex set as $E(G)$ where the two vertices of $J(G)$ are adjacent if and only if they correspond to two nonadjacent edges of G.

Theorem 3.12. *Let G be a r-regular graph of order n and size m. Then*

$$Ch[AD(S(G))] = (\lambda + r^2)^{n-1}(\lambda + 4)^{\frac{nr}{2}-1}[\lambda^2 - (4(\frac{nr}{2} - 1) + r^2(n-1))\lambda + \frac{1}{4}(16r^2(n-1)(\frac{nr}{2} - 1) - \frac{n^2r}{2}(r^2+4)^2)].$$

Proof. The subdivision graph of a r-regular graph has two types of vertices. The n vertices with degree r and $\frac{nr}{2}$ vertices with degree 2. Hence,

$$AD[S(G)] = \begin{bmatrix} r^2(J_n - I_n) & \frac{(r^2+4)}{2} J_{n \times \frac{nr}{2}} \\ \frac{(r^2+4)}{2} J_{\frac{nr}{2} \times n} & 4(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(S(G))] &= \mid \lambda I - AD(S(G)) \mid \\ &= \begin{vmatrix} (\lambda + r^2)I_n - r^2 J_n & -\frac{(r^2+4)}{2} J_{n \times \frac{nr}{2}} \\ -\frac{(r^2+4)}{2} J_{\frac{nr}{2} \times n} & (\lambda+4)I_{\frac{nr}{2}} - 4J_{\frac{nr}{2}} \end{vmatrix}. \end{aligned}$$

Now by using Lemma 2.20, we get the desired result. □

Theorem 3.13. *Let G be a r- regular graph of order n and size m. Then*

$$Ch[AD(T_2(G))] = (\lambda + 4r^2)^{n-1}(\lambda + 4)^{m-1}[\lambda^2 - 4((m-1) + r^2(n-1))\lambda + 16r^2((n-1)(m-1)) - 4mn(r^2+1)^2)].$$

Proof. The semitotal point graph of a r-regular graph has wo types of vertices. The n vertices with degree $2r$ and m vertices with degree 2.

Hence,

$$AD(T_2) = \begin{bmatrix} 4r^2(J_n - I_n) & 2(r^2+1)J_{n\times m} \\ \\ 2(r^2+1)J_{m\times n} & 4(J_m - I_m) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(T_2)] &= \mid \lambda I - AD(T_2(G)) \mid \\ &= \begin{vmatrix} (\lambda+4r^2)I_n - 4r^2J_n & -2(r^2+1)J_{n\times m} \\ \\ -2(r^2+1)J_{m\times n} & (\lambda+4)I_m - 4J_m \end{vmatrix} \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.14. *Let G be a r- regular graph of order n and size m. Then,*

$$\begin{aligned} Ch[AD(T_1)] = {} & (\lambda + r^2)^{n-1}(\lambda+4r^2)^{m-1}[\lambda^2 - r^2(4(m-1)+(n-1))\lambda \\ & + \frac{1}{4}(16r^4(n-1)(m-1) - 25mnr^4]. \end{aligned}$$

Proof. The semitotal line graph of a r-regular graph has two types of vertices. The n vertices with degree r and m vertices with degree $2r$.

Hence,

$$AD(T_1) = \begin{bmatrix} r^2(J_n - I_n) & \frac{5r^2}{2}J_{n\times m} \\ \\ \frac{5r^2}{2}J_{m\times n} & 4r^2(J_m - I_m) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(S(G))] &= \mid \lambda I - AD(T_1(G)) \mid \\ &= \begin{vmatrix} (\lambda+r^2)I_n - r^2J_n & -\frac{5r^2}{2}J_{n\times m} \\ \\ -\frac{5r^2}{2}J_{m\times n} & (\lambda+4r^2)I_m - 4r^2J_m \end{vmatrix} \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.15. *Let G be a r-regular graph of order n and size m. Then*

$$Ch[AD(T(G))] = (\lambda - 4r^2(n+m-1))(\lambda+4r^2)^{n+m-1}.$$

Proof. The total graph of a r-regular graph is a regular graph of degree $2r$ with $n+m$ vertices. Hence the result follows from Equation (1).

□

Theorem 3.16. *Let G be a r- regular graph of order n and size m. Then*

$$\begin{aligned} Ch[AD(G^{+k})] = {} & (\lambda + (r+k)^2)^{n-1}(\lambda+1)^{nk-1}[\lambda^2 - ((nk-1)+(r+k)^2(n-1))\lambda \\ & + \frac{1}{4}(4(r+k)^2(n-1)(nk-1) - n^2k(1+(r+k)^2)^2)]. \end{aligned}$$

Proof. The graph G^{+k} of a r-regular graph has two types of vertices. The n vertices with degree $r+k$ and nk vertices with degree 1.

Hence,

$$AD(G^{+k}) = \begin{bmatrix} (r+k)^2(J_n - I_n) & \frac{((r+k)^2+1)}{2}J_{n\times m} \\ \\ \frac{((r+k)^2+1)}{2}J_{m\times n} & (J_m - I_m) \end{bmatrix}.$$

$$Ch[AD(G^{+k}))] = \mid \lambda I - AD(G^{+k}) \mid$$

$$= \begin{vmatrix} (\lambda + (r+k)^2)I_n - (r+k)^2 J_n & -\frac{((r+k)^2+1)}{2} J_{n\times m} \\ -\frac{((r+k)^2+1)}{2} J_{m\times n} & (\lambda+1)I_m - J_m \end{vmatrix}.$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.17. *If G is a r_1-regular graph of order n_1 and H is a r_2-regular graph of order n_2,then*

$$Ch[AD(G\cup H)] = Ch(AD(G))Ch(AD(H)) - (\lambda + r_1^2)^{n_1-1}(\lambda + r_2^2)^{n_2-1} n_1 n_2 \frac{(r_1^2+r_2^2)^2}{4}.$$

Proof. The graph $G \cup H$ of order $n_1 + n_2$ has two types of vertices, the n_1 vertices of degree r_1 and the remaining n_2 vertices are of degree r_2.

Hence,

$$AD(G\cup H) = \begin{vmatrix} AD(G) & \frac{(r_1^2+r_2^2)}{2} J_{n_1\times n_2} \\ \frac{(r_1^2+r_2^2)}{2} J_{n_2\times n_1} & AD(H) \end{vmatrix}$$

$$= \begin{vmatrix} r_1^2(J_{n_1} - I_{n_1}) & \frac{(r_1^2+r_2^2)}{2} J_{n_1\times n_2} \\ \frac{(r_1^2+r_2^2)}{2} J_{n_2\times n_1} & r_2^2(J_{n_2} - I_{n_2}) \end{vmatrix}.$$

$$Ch[AD(G\cup H)] = \mid \lambda I - AD(G\cup H) \mid$$

$$= \begin{vmatrix} (\lambda + r_1^2)I_{n_1} - r_1^2 J_{n_1} & -\frac{(r_1^2+r_2^2)}{2} J_{n_1\times n_2} \\ -\frac{(r_1^2+r_2)^2}{2} J_{n_2\times n_1} & (\lambda + r_2^2)I_m - r_2^2 J_{n_2} \end{vmatrix}.$$

Now, by using Lemma 2.20, we get

$$Ch[AD(G\cup H)] = (\lambda + r_1^2)^{n_1-1}(\lambda + r_2^2)^{n_2-1}[(\lambda - (n_1-1)r_1^2)(\lambda - (n_2-1)r_2^2) - \frac{n_1 n_2 (r_1^2+r_2^2)^2}{4}].$$

AS G and H are regular graphs of order n_1 and n_2 and degree r_1 and r_2 respectively, by equation (1) we have

$$Ch[AD(G)] = (\lambda - r_1^2(n_1-1))(\lambda + r_1^2)^{n_1-1}$$

and

$$Ch[AD(H)] = (\lambda - r_2^2(n_2-1))(\lambda + r_2^2)^{n_2-1}.$$

Hence the result follows. □

Theorem 3.18. *Let G be a r-regular graph of order n and size m. Then*

$$Ch[AD(G+H)] = (\lambda + R_1^2)^{n_1-1}(\lambda + R_2^2)^{n_2-1}[\lambda^2 - (R_2^2(n_2-1) + R_1^2(n_1-1))\lambda + \frac{1}{4}(R_1^2 R_2^2(n_1-1)(n_2-1) - n_1 n_2(R_1^2+R_2^2)^2).]$$

Proof. If G is a r_1-regular graph of order n_1 and H is a r_2-regular graph of order n_2, then $G+H$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and n_2 vertices with degree $R_2 = r_2 + n_1$. Hence

$$AD(G+H) = \begin{vmatrix} R_1^2(J_{n_1} - I_{n_1}) & \frac{(R_1^2+R_2^2)}{2} J_{n_1\times n_2} \\ \frac{(R_1^2+R_2^2)}{2} J_{n_2\times n_1} & R_2^2(J_{n_2} - I_{n_2}) \end{vmatrix}.$$

$$Ch[AD(G+H)] =\mid \lambda I - AD(G+H) \mid$$
$$= \begin{vmatrix} (\lambda + R_1^2)I_{n_1} - R_1^2 J_{n_1} & -\frac{(R_1^2+R_2^2)}{2} J_{n_1 \times n_2} \\ -\frac{(R_1^2+R_2^2)}{2} J_{n_2 \times n_1} & (\lambda + R_2^2)I_{n_2} - R_2^2 J_{n_2} \end{vmatrix}.$$

Now by using Lemma 2.20, we get the desired result. □

Theorem 3.19. *Let G be a r_1- regular graph of order n_1 and H be r_2 regular graph of order n_2. Then*

$$Ch[AD(G \times H)] = (\lambda - (r_1 + r_2)^2 (n_1 n_2 - 1))(\lambda + (r_1 + r_2)^2)^{n_1 n_2 - 1}.$$

Proof. If G be a r_1-regular graph of order n_1 and H be r_2 regular graph of order n_2, then $G \times H$ is a $(r_1 + r_2)$-regular graph with $n_1 n_2$ vertices. Hence the result follows from equation (1). □

Theorem 3.20. *Let G be a r_1- regular graph of order n_1 and H be a r_2 regular graph of order n_2. Then*

$$Ch[AD(G[H])] = (\lambda + (n_2 r_1 + r_2)^2)^{n_1 n_2 - 1} (\lambda - (n_2 r_1 + r_2)^2 (n_1 n_2 - 1)).$$

Proof. If G be a r_1- regular graph of order n_1 and H be a r_2 regular graph of order n_2, then $G[H]$ is a $(n_2 r_1 + r_2)$-regular graph with $n_1 n_2$ vertices. Hence the result follows from Equation (1). □

Theorem 3.21. *If G is a r_1-regular graph of order n_1 and H is a r_2-regular graph of order n_2, then*

$$Ch[AD(G \circ H)] = (\lambda + R_1^2)^{n_1 - 1} (\lambda + R_2^2)^{n_2 - 1} [\lambda^2 - (R_2^2 (n_1 n_2 - 1) + R_1^2 (n_1 - 1))\lambda$$
$$+ \frac{1}{4}(R_1^2 R_2^2 (n_1 - 1)(n_1 n_2 - 1) - n_1^2 n_2 (R_1^2 + R_2^2)^2)].$$

Proof. If G is a r_1-regular graph of order n_1 and H is a r_2-regular graph of order n_2, then $G \circ H$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and remaining $n_1 n_2$ vertices with degree $R_2 = r_2 + 1$.

Hence

$$AD(G \circ H) = \begin{bmatrix} R_1^2 (J_{n_1} - I_{n_1}) & \frac{(R_1^2+R_2^2)}{2} J_{n_1 \times n_1 n_2} \\ \frac{(R_1^2+R_2^2)}{2} J_{n_1 n_2 \times n_1} & R_2^2 (J_{n_1 n_2} - I_{n_1 n_2}) \end{bmatrix}.$$

$$Ch[AD(G \circ H)] =\mid \lambda I - AD(G \circ H) \mid$$
$$= \begin{vmatrix} (\lambda + R_1^2)I_{n_1} - R_1^2 J_{n_1} & -\frac{(R_1^2+R_2^2)}{2} J_{n_1 \times n_1 n_2} \\ -\frac{(R_1^2+R_2^2)}{2} J_{n_1 n_2 \times n_1} & (\lambda + R_2^2)I_{n_1 n_2} - R_2^2 J_{n_1 n_2} \end{vmatrix}.$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.22. *Let G be an r_1-regular graph of order n_1 and H be r_2 regular graph of order n_2. Then*

$$Ch[AD(G \times H)] = (\lambda - (r_1 + r_2)^2 (n_1 n_2 - 1))(\lambda + (r_1 + r_2)^2)^{n_1 n_2 - 1}$$

Proof. Let G be an r_1-regular graph of order n_1 and H be r_2 regular graph of order n_2. Then $G \times H$ is an $(r_1 + r_2)$- regular graph with $n_1 n_2$ vertices. Hence the result follows from Equation (1). □

Theorem 3.23. *If W_n is a wheel graph, then*

$$Ch[AD(W_n)] = (\lambda + 9)^{n-2} [\lambda^2 - 9(n-2)\lambda - \frac{(n-1)(3^2 + (n-1)^2)^2}{4}]$$

Proof. The graph W_n of order n has two types of verices, namely $n-1$ vertices are of degree 3 and central vertex has degree $n-1$.

Hence,

$$AD(W_n) = \begin{bmatrix} 9(J_{n-1} - I_{n-1}) & \frac{(3^2+(n-1)^2)}{2} J_{(n-1)\times 1} \\ \frac{(3^2+(n-1)^2)}{2} J_{1\times(n-1)} & (n-1)^2(J_1 - I_1) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(W_n)] &= \mid \lambda I - AD(W_n) \mid \\ &= \begin{vmatrix} (\lambda+9)I_{n-1} - 9J_{n-1} & -\frac{(3^2+(n-1)^2)}{2} J_{(n-1)\times 1} \\ -\frac{(3^2+(n-1)^2)}{2} J_{1\times(n-1)} & (\lambda+(n-1)^2)I_1 - (n-1)^2 J_1 \end{vmatrix}. \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.24. *If F_t^3 be a friendship graph, then*

$$Ch[AD(F_t^3)] = (\lambda+4)^{2t-1}[\lambda^2 - 4(2t-1)\lambda - \frac{2t(4+(2t)^2)^2}{4}].$$

Proof. The graph F_t^3 of order $2t+1$ has two types of verices namely, $2t$ vertices of degree 2 and 1 vertex of degree $2t$. Hence,

$$AD(F_t^3) = \begin{bmatrix} 4(J_{2t} - I_{2t}) & \frac{(4+(2t)^2)}{2} J_{2t\times 1} \\ \frac{(4+(2t)^2)}{2} J_{1\times 2t} & (2t)^2(J_1 - I_1) \end{bmatrix}.$$

$$\begin{aligned} Ch[F_t^3] &= \mid \lambda I - AD(F_t^3) \mid \\ &= \begin{vmatrix} (\lambda+4)I_{2t} - 4J_{2t} & -\frac{(4+(2t)^2)}{2} J_{2t\times 1} \\ -\frac{(4+(2t)^2)}{2} J_{1\times 2t} & (\lambda+(2t)^2)I_1 - (2t)^2 J_1 \end{vmatrix}. \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.25. *If $H_n - c$ is a helm without central vertex, then*

$$Ch[AD(H_n - c)] = (\lambda+9)^{n-2}(\lambda+1)^{n-2}[\lambda^2 - 10(n-2)\lambda + 9(n-2)^2 - 25(n-1)^2.$$

Proof. The graph $H_n - c$ of order $2(n-1)$ has two types of vertices namely, $n-1$ vertices are of degree 3 and remaining $(n-1)$ vertices has degree 1. Hence,

$$AD(H_n - c) = \begin{bmatrix} 9(J_{n-1} - I_{n-1}) & 5J_{(n-1)\times(n-1)} \\ 5J_{(n-1)\times(n-1)} & (J_{n-1} - I_{n-1}) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(H_n - c)] &= \mid \lambda I - AD(H_n - c) \mid \\ &= \begin{vmatrix} (\lambda+9)I_{n-1} - 9J_{n-1} & -5J_{(n-1)\times(n-1)} \\ -5J_{(n-1)\times(n-1)} & (\lambda+1)I_{n-1} - J_{n-1} \end{vmatrix}. \end{aligned}$$

Now by using Lemma 2.20, we get the desired result. □

Theorem 3.26. *If $H_n^{'} - c$ is a closed helm without central vertex, then*

$$Ch[AD(H_n^{'} - c)] = (\lambda - 9(2n-3))(\lambda+9)^{2n-3}.$$

Proof. The graph $H_n - c$ of order $2(n-1)$ has two types of vertices namely, $n-1$ vertices are of degree 3 and remaining $(n-1)$ vertices has degree 1. Hence,

$$AD(H_n - c) = \begin{bmatrix} 9(J_{n-1} - I_{n-1}) & 5J_{(n-1)\times(n-1)} \\ \\ 5J_{(n-1)\times(n-1)} & (J_{n-1} - I_{n-1}) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(H_n - c)] &= \mid \lambda I - AD(H_n - c) \mid \\ &= \begin{vmatrix} (\lambda+9)I_{n-1} - 9J_{n-1} & -5J_{(n-1)\times(n-1)} \\ \\ -5J_{(n-1)\times(n-1)} & (\lambda+1)I_{n-1} - J_{n-1} \end{vmatrix}. \end{aligned}$$

Now by using Lemma 2.20, we get the desired result. □

Theorem 3.27. *If $H_n^{'} - c$ is a closed helm without central vertex, then*

$$Ch[AD(H_n^{'} - c)] = (\lambda - 9(2n-3))(\lambda+9)^{2n-3}.$$

Proof. The closed helm without central vertex $H_n^{'} - c$ is 3-regular graph with $2(n-1)$ vertices. Hence the result follows fron equation (1). □

Theorem 3.28. *If $SF_n - c$ is a sun flower graph without central vertex, then*

$$\begin{aligned} Ch[AD(SF_n - c)] = (\lambda+9)^{n-2}(\lambda+4)^{n-2}[\lambda^2 - 13(n-2)\lambda + 36(n-2)^2 \\ - \frac{169(n-1)^2}{4}]. \end{aligned}$$

Proof. The sun flower graph $SF_n - c$ without central vertex is a graph of order $2(n-1)$, which has two types of vertices. The $n-1$ vertices have degree 3 and the remaining $n-1$ vertices have degree 2.

Hence,

$$AD(SF_n - c) = \begin{bmatrix} 9(J_{n-1} - I_{n-1}) & \frac{13}{2}J_{(n-1)\times(n-1)} \\ \\ \frac{13}{2}J_{(n-1)\times(n-1)} & 4(J_{n-1} - I_{n-1}) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(SF_n - c)] &= \mid \lambda I - AD(SF_n - c) \mid \\ &= \begin{vmatrix} (\lambda+9)I_{n-1} - 9J_{n-1} & -\frac{13}{2}J_{(n-1)\times(n-1)} \\ \\ -\frac{13}{2}J_{(n-1)\times(n-1)} & (\lambda+4)I_{n-1} - 4J_{n-1} \end{vmatrix}. \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.29. *If DC_n is a double cone then,*

$$\begin{aligned} Ch[AD(C_n)] = (\lambda+16)^{n-1}(\lambda+n^2)[\lambda^2 - (n^2 + 16(n-1))\lambda + 16n^2(n-1) \\ - \frac{n(16+n^2)^2}{2}]. \end{aligned}$$

Proof. The double cone is a graph of of order $n+2$ has two types of vertices. The n vertices have degree 4 and the remaining 2 vertices have degree n. Hence

$$AD(DC_n) = \begin{bmatrix} 16(J_n - I_n) & \frac{(16+n^2)}{2}J_{n\times 2} \\ \\ \frac{(16+n^2)}{2}J_{2\times n} & n^2(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(DC_n)] = \mid \lambda I - AD(DC_n) \mid$$
$$= \begin{vmatrix} (\lambda+16)I_n - 16J_n & -\frac{(16+n^2)}{2}J_{n\times 2} \\ -\frac{(16+n^2)}{2}J_{2\times n} & (\lambda+n^2)I_2 - n^2J_2 \end{vmatrix}.$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.30. *If B_b is a book graph, then*

$$Ch[AD(B_b)] = (\lambda+4)^{2b-1}(\lambda+(b+1)^2)[\lambda^2 - ((b+1)^2 + 4(2b-1))\lambda + 4(2b-1)(b+1)^2 - b(4+(b+1)^2)^2].$$

Proof. The Book graph B_b of order $2b+2$ has two types of vertices. The $2b$ vertices with degree 2 and 2 vertices are with degree $b+1$. Hence,

$$AD(B_b) = \begin{bmatrix} 4(J_{2b} - I_{2b}) & \frac{4+(b+1)^2}{2}J_{2b\times 2} \\ \frac{4+(b+1)^2}{2}J_{2\times 2b} & (b+1)^2(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(B_b)] = \mid \lambda I - AD(B_b) \mid$$
$$= \begin{vmatrix} (\lambda+4)I_{2b} - 4J_{2b} & -\frac{4+(b+1)^2}{2}J_{2b\times 2} \\ -\frac{4+(b+1)^2}{2}J_{2\times 2b} & (\lambda+(b+1)^2)I_2 - (b+1)^2J_2 \end{vmatrix}.$$

Now by using Lemma 2.20, we get the desired result. □

Theorem 3.31. *If B_t is a book with triangular pages, then*

$$Ch[AD(B_t)] = (\lambda+4)^{t-1}(\lambda+(t+1)^2)[\lambda^2 - ((t+1)^2 + 4(t-1))\lambda + 4(t-1)(t+1)^2 - \frac{t(4+(t+1)^2)^2}{2}].$$

Proof. The book B_t with triangular pages of order $t+2$ has two types of vertices. The t vertices have degree 2 and the remaining 2 vertices have degree $t+1$. Hence,

$$AD(B_t) = \begin{bmatrix} 4(J_t - I_t) & \frac{4+(t+1)^2}{2}J_{t\times 2} \\ \frac{4+(t+1)^2}{2}J_{2\times t} & (t+1)^2(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(B_t)] = \mid \lambda I - AD(B_t) \mid$$
$$= \begin{vmatrix} (\lambda+4)I_t - 4J_t & -\frac{4+(t+1)^2}{2}J_{t\times 2} \\ -\frac{4+(b+1)^2}{2}J_{2\times t} & (\lambda+(t+1)^2)I_2 - (t+1)^2J_2 \end{vmatrix}.$$

□

Theorem 3.32. *If L_n is a ladder graph, then*

$$Ch[AD(L_n)] = (\lambda+9)^{2n-5}(\lambda+4)^3[\lambda^2 + -(9(2n-5)+12)\lambda + 108(2n-5) - 169(2n-4).$$

Proof. The ladder graph L_n is a graph of order $2n$ and has two types of vertices. The 4 vertices have degree 2 and 2n-4 vertices have degree 3. Hence,

$$AD(L_n) = \begin{bmatrix} 9(J_{2n-4} - I_{2n-4}) & \frac{13}{2}J_{(2n-4)\times 4} \\ \\ \frac{13}{2}J_{4\times(2n-4)} & 4(J_4 - I_4) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(L_n)] &= \mid \lambda I - AD(L_n) \mid \\ &= \begin{vmatrix} (\lambda+9)I_{2n-4} - 9J_{2n-4} & -\frac{13}{2}J_{(2n-4)\times 4} \\ \\ -\frac{13}{2}J_{4\times(2n-4)} & (\lambda+4)I_4 - 4J_4 \end{vmatrix}. \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.33. *If Pr_n is a prism graph, then*

$$Ch[AD(Pr_n)] = (\lambda+9)^{2n-1}(\lambda - 9(2n-1)).$$

Proof. The prism Pr_n is 3-regular graph with $2n$ vertices. Hence, the result follows from equation (1).

□

Theorem 3.34. *If T_n is a triangular snake, then*

$$\begin{aligned} Ch[AD(T_n)] = (\lambda+4)^n(\lambda+16)^{n-3}[\lambda^2 &- (16(n-3)+4n)\lambda + 64n(n-3) \\ &- 25(n+1)(n-2)]. \end{aligned}$$

Proof. The triangular snake T_n has two types of vertices. The $n+1$ vertices have degree 2 and the remaining $n-2$ vertices have degree 4. Hence,

$$AD(T_n) = \begin{bmatrix} 4(J_{n+1} - I_{n+1}) & 5J_{(n+1)\times(n-2)} \\ \\ 5J_{(n-2)\times(n+1)} & 16(J_{n-2} - I_{n-2}) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(T_n)] &= \mid \lambda I - AD(T_n) \mid \\ &= \begin{vmatrix} (\lambda+4)I_{n+1} - 4J_{n+1} & -5J_{(n+1)\times(n-2)} \\ \\ -5J_{(n-2)\times(n+1)} & (\lambda+16)I_{n-2} - 16J_{n-2} \end{vmatrix}. \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.35. *If Q_n is a quadrilateral snake, then*

$$\begin{aligned} Ch[AD(Q_n)] = (\lambda+4)^{2n-1}(\lambda+16)^{n-3}[\lambda^2 &- (16(n-3)+4(2n-1))\lambda \\ &+ 64(2n-1)(n-3) - 50n(n-2)]. \end{aligned}$$

Proof. The quadrilateral snake Q_n of degree $3n-2$ has two types of vertices. The 2n vertices have degree 2 and the remaining $n-2$ vertices have degree 4. Hence,

$$AD(Q_n) = \begin{bmatrix} 4(J_{2n} - I_{2n}) & 5J_{(2n)\times(n-2)} \\ \\ 5J_{(n-2)\times(2n)} & 16(J_{n-2} - I_{n-2}) \end{bmatrix}.$$

$$\begin{aligned} Ch[AD(Q_n)] &= \mid \lambda I - AD(Q_n) \mid \\ &= \begin{vmatrix} (\lambda+4)I_{2n} - 4J_{2n} & -5J_{(2n)\times(n-2)} \\ \\ -5J_{(n-2)\times(2n)} & (\lambda+16)I_{n-2} - 16J_{n-2} \end{vmatrix}. \end{aligned}$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.36. *If G is an r-regular graph of order n, then*

$$Ch[AD(J(G))] = (\lambda + r_1^2(\frac{nr}{2} - 1))(\lambda - r_1^2)^{(\frac{nr}{2}-1)} \quad where, \quad r_1 = \frac{(n-4)r}{2} + 1.$$

Proof. The jump graph $J(G)$ is r-regular graph is $r_1 = (\frac{(n-4)r}{2} + 1)$-regular graph with $\frac{nr}{2}$ vertices. Hence, the result follows from Equation (1). □

Theorem 3.37. *If S_n is a star graph, then*

$$Ch[AD(S_n)] = (\lambda + 1)^{n-2}[\lambda^2 - (n-2)\lambda - \frac{(n-1)(1+(n-1)^2)^2}{4}].$$

Proof. The graph S_n of order n has two types of verices namely, $n-1$ vertices are of degree 1 and central vertex has degree $n-1$. Hence,

$$AD(S_n) = \begin{bmatrix} (J_{n-1} - I_{n-1}) & \frac{1+(n-1)^2}{2} J_{(n-1)\times 1} \\ \\ \frac{1+(n-1)^2}{2} J_{1\times(n-1)} & (n-1)^2(J_1 - I_1) \end{bmatrix}.$$

$$Ch[AD(S_n)] = \mid \lambda I - AD(S_n) \mid$$
$$= \begin{vmatrix} (\lambda+1)I_{n-1} - J_{n-1} & -\frac{1+(n-1)^2}{2} J_{(n-1)\times 1} \\ \\ -\frac{1+(n-1)^2}{2} J_{1\times(n-1)} & (\lambda + (n-1)^2)I_1 - (n-1)^2 J_1 \end{vmatrix}.$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.38. *If $S_{n,n}$ is a double star graph, then*

$$Ch[AD(S_{n,n})] = (\lambda+1)^{2n-3}(\lambda+n^2)[\lambda^2 - ((2n-3)+n^2)\lambda + (2n-3)n^2 - (n-1)(n^2+1)^2].$$

Proof. The graph $S_{n,n}$ of order $2n$ has two types of verices namely, $2n-1$ vertices are of degree 1 and remaining two of degree n. Hence,

$$AD(S_{n,n}) = \begin{bmatrix} (J_{2n-2} - I_{2n-2}) & \frac{n^2+1}{2} J_{(2n-2)\times 2} \\ \\ \frac{n^2+1}{2} J_{2\times(2n-2)} & n^2(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(S_{n,n})] = \mid \lambda I - AD(S_{n,n}) \mid$$
$$= \begin{vmatrix} (\lambda+1)I_{2n-2} - J_{2n-2} & -\frac{(n^2+1)}{2} J_{(2n-2)\times 2} \\ \\ -\frac{(n^2+1)}{2} J_{2\times(2n-2)} & (\lambda+n^2)I_2 - n^2 J_2 \end{vmatrix}.$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.39. *If $K_{m,n}$ is a complete biparite graph, then*

$$Ch[AD(K_{m,n})] = (\lambda+n^2)^{m-1}(\lambda+m^2)^{n-1}[\lambda^2 - (m^2(n-1) + n^2(m-1))\lambda$$
$$+ (m-1)(n-1)m^2n^2 - \frac{mn(m^2+n^2)^2}{4}].$$

Proof. The graph $K_{m,n}$ of order $m+n$ has two types of verices namely, m vertices are of degree n and n of degree m. Hence,

$$AD(K_{m,n}) = \begin{bmatrix} n^2(J_m - I_m) & \frac{m^2+n^2}{2} J_{m\times n} \\ \\ \frac{m^2+n^2}{2} J_{n\times m} & m^2(J_n - I_n) \end{bmatrix}.$$

$$Ch[AD(K_{m,n})] = \mid \lambda I - AD(K_{m,n}) \mid = \begin{vmatrix} (\lambda + n^2)I_m - n^2 J_m & -\frac{m^2+n^2}{2} J_{m\times n} \\ -\frac{m^2+n^2}{2} J_{m\times n} & (\lambda + m^2)I_n - m^2 J_n \end{vmatrix}.$$

Now, by using Lemma 2.20, we get the desired result. □

Theorem 3.40. *If P_n is a path graph, then*

$$Ch[AD(P_n)] = (\lambda + 4)^{n-3}(\lambda + 1)[\lambda^2 - (4(n-3) + 1)\lambda + 4(n-3) - \frac{25(n-2)}{2}].$$

Proof. The graph P_n of order n has two types of verices namely, $n-2$ vertices are of degree 2 and remaining two end vertices of degree 1. Hence,

$$AD(P_n) = \begin{bmatrix} 4(J_{n-2} - I_{n-2}) & \frac{5}{2} J_{(n-2)\times 2} \\ \frac{5}{2} J_{2\times(n-2)} & (J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(P_n)] = \mid \lambda I - AD(P_n) \mid = \begin{vmatrix} (\lambda + 4)I_{n-2} - 4J_{n-2} & -\frac{5}{2} J_{(n-2)\times 2} \\ -\frac{5}{2} J_{2\times(n-2)} & (\lambda + 1)I_2 - J_2 \end{vmatrix}.$$

Now, by using Lemma 2.20, we get the desired result. □

A dumbbell is the graph obtained from two disjoint cycles by joining them by a path.

Theorem 3.41. *If $D_{n,n}$ is a dumbbell graph, then*

$$Ch[AD(D_{n,n})] = (\lambda + 4)^{2n-3}(\lambda + 9)[\lambda^2 - (4(2n-3) + 9)\lambda + 36(2n-3) - 169(n-1)].$$

Proof. The graph $D_{n,n}$ of order $2n$ has two types of verices namely, $2n-2$ vertices are of degree 2 and remaining two of degree 3. Hence,

$$AD(D_{n,n}) = \begin{bmatrix} 4(J_{2n-2} - I_{2n-2}) & \frac{13}{2} J_{(2n-2)\times 2} \\ \frac{13}{2} J_{2\times(2n-2)} & 9(J_2 - I_2) \end{bmatrix}.$$

$$Ch[AD(D_{n,n})] = \mid \lambda I - AD(D_{n,n}) \mid = \begin{vmatrix} (\lambda + 4)I_{2n-2} - 4J_{2n-2} & -\frac{13}{2} J_{(2n-2)\times 2} \\ -\frac{13}{2} J_{2\times(2n-2)} & (\lambda + 9)I_2 - 9J_2 \end{vmatrix}.$$

Now by using Lemma 2.20, we get the desired result. □

4 Hyperenergetic graphs

A graph G with n vertices is said to be hyperenergetic [10] if $\mathcal{E}(G) \geq 2n-2$, and to be nonhyperenergetic if $\mathcal{E}(G) \leq 2n-2$. A non-complete graph whose energy is equal to $2n-2$ is called borderenergetic [7].

Definition 4.1. A graph G of order n is said to be average degree square sum hyperenergetic if $AD(G) \geq 2(n-1)^3$.

Definition 4.2. A graph G of order n is said to be average degree square sum non-hyperenergetic if $AD(G) \leq 2(n-1)^3$.

Definition 4.3. A non-complete graph of order n whose energy is equal to $2(n-1)^3$ is called average degree square sum borderenergetic.

Definition 4.4. Two graphs G_1 and G_2 are said to be average degree square sum equienergetic if they have same average degree square sum energy. That is, $\mathcal{E}[AD(G_1)] = \mathcal{E}[AD(G_2)]$.

Theorem 4.5. *If G is an r-regular graph of order n, then $\overline{G}$ is*

(i) average degree square sum borderenergetic for $r = 0$,

(ii) average degree square sum non-hyperenergetic for $r \geq 1$.

Proof. The graph $\overline{G}$ is $(n-1-r)$-regular graph.

$$Ch[AD(\overline{G}), \lambda] = (\lambda - (n-1)(n-1-r)^2)(\lambda + (n-1-r)^{2)^{n-1}}$$

Thus ,

$$\mathcal{E}[AD(\overline{G})] = 2(n-1-r)^2(n-1)$$

From Definition 4.1, the graph G is average degree square sum hyperenergetic if $\mathcal{E}(\overline{G}) > 2(n-1)^3$.
That is, if $2(n-1-r)^2(n-1) \geq 2(n-1)^3$ This inequality does not hold for any value of r, whereas the two quantities are equal when $r = 0$. Hence, $\overline{G}$ is average degree square sum borderenergetic for $r = 0$ and average degree square sum nonhyperenergetic for $r \geq 1$. □

Theorem 4.6. *The graph $L(K_n)$ is average degree square sum borderenergetic for $n = 2, 3$ and average degree square sum non-hyperenergetic for $n \geq 4$.*

Proof. The complete graph K_n is $(n-1)$-regular graph of order n. Thus,

$$Ch[AD(K_n), \lambda] = (\lambda - (n-1)^3)(\lambda + (n-1)^2)^{n-1}$$

The line graph of K_n is $L(K_n)$ is $(2n-4)$-regular graph of order $n_1 = \frac{nr}{2}$ and,

$$Ch[AD(L(K_n), \lambda] = (\lambda - 2(n-2)^2(n(n-1)-2))(\lambda + 4(n-2)^2)^{\frac{n(n-1)-2}{2}}$$

Hence,

$$\mathcal{E}[AD(L(K_n))] = 4(n-2)^2(n(n-1)-2)$$

Clearly, $\mathcal{E}[AD(L(K_n))] \leq 2(\frac{n(n-1)}{2} - 1)^3$ for $n \geq 4$ and equality holds for $n = 2, 3$.
Hence, $L(K_2)$, $L(K_3)$ are average degree square sum borderenergetic and $L(K_n)$ is average degree square sum nonhyperenergetic for $n \geq 4$. □

Theorem 4.7. *If G is an r-regular graph of order n, then $J(G)$ is*

(i) average degree square sum borderenergetic for $r = 1$,

(ii) average degree square sum non-hyperenergetic for $r \geq 2$.

Proof. The jump graph $J(G)$ is r-regular graph is $r_1 = (\frac{(n-4)r}{2} + 1)$-regular graph with $\frac{nr}{2}$ vertices.

$$Ch[AD(J(G))] = (\lambda + r_1^2(\frac{nr}{2} - 1))(\lambda - r_1^2)^{(\frac{nr}{2}-1)} \quad where, \quad r_1 = \frac{(n-4)r}{2} + 1$$

Hence,

$$\mathcal{E}[AD(J(G)] = 2r_1^2(\frac{nr}{2} - 1)$$
$$= \frac{((n-4)r+2)^2(nr-2)}{4}$$

$\mathcal{E}[AD(J(G))] \leq 2(\frac{nr}{2} - 1)^3$ for $r \geq 2$ and equality holds for $r = 1$. □

Theorem 4.8. *If G is an r-regular graph of order n, then $T(G)$ is average degree square sum non-hyperenergetic.*

Proof. The total graph $T(G)$ of an r-regular graph G is a regular graph of degree $2r$ with $n + \frac{nr}{2}$ vertices. Then,

$$Ch[AD(T(G))] = (\lambda - 4r^2(n+m-1))(\lambda + 4r^2)^{n+m-1}$$

Hence,

$$\mathcal{E}(AD(T(G))) = 4r^2(n(r+2)-2)$$

$\mathcal{E}[AD(T(G))] \leq 2(n + \frac{nr}{2} - 1)^3$ for all r. Thus $T(G)$ is average degree square sum nonhyperenergetic. □

5 Conclusion

We conclude with the following observations. In this paper, we have obtain the characteristic polynomial of the average degree square sum matrix of graphs obtained by some graphs operations. Further, bounds for both largest average degree square sum eigenvalue and average degree square sum energy of graphs are established. Also, obtained sharp bounds. Characterized average degree square sum hyperenergetic, borderenergetic and equienergetic of few graphs.

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