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CONFORMALLY BERWALD FINSLER SPACE WITH SPECIAL (α, β) -METRIC

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Abstract During this paper, we discover the required necessary and sufficient conditions for a Finsler space with the metric $L = \alpha - \beta^2/\alpha$ to be a Berwald space and also to be a Berwald space, where α could be a Riemannian metric and β may be a differential one-form. In the present paper, we study the conformal change of Berwald space with the above mentioned special (α, β)-metric. In Finsler space we see special (α, β)-metrics such as Randers metric, Kropina metric and Matsumoto metric.,etc.

1 Introduction

The conformal change theory of Finsler spaces has been introduced by M. S. Knebelman in 1929 and anon M. Hashiguchi developed such theory which was supported Matsumoto's approach to Finsler geometry. The conformal change theory of two-dimensional Finsler space has been studied by M. Matsumoto and supported on the above work, B. N. Prasad and D. K. Diwedi discussed the speculation of conformal change of three-dimensional Finsler space.

The word Douglas space has been introduced by M. Matsumoto and S. Bacso as the generalized Berwald space from the view point of geodesic equations. It's remarkable that a Finsler space could be a Douglas space or is of Douglas type if and provided that the Douglas tensor vanishes identically.

The primary part of the current paper is dedicated to study the condition for the Finsler space with special (α, β)-metric to be a Berwald type. The second part is dedicated to find the condition for the Finsler space with special (α, β)-metric to be Berwald space and eventually, we apply the conformal change of Finsler space with the special (α, β)-metric of Douglas type. We study conformal change of Berwald space with special (α, β)-metric $L = \alpha - \beta^2/\alpha$ and also we obtain the conditions for Finsler space with an (α, β)-metric $L^2 = 2\alpha\beta$ to be conformally Berwald.

The concept of (α, β)-metric was introduced in 1972 by M. Matsumoto and studied by M. Hachiguchi(1975), Y. Ichijyo(1975), S. Kikuchi(1979), C.Shibata(1984). The examples of the (α, β)-metric are Randers metric, Kropina metric and Matsumoto metric. Z. Shen extended the notion of dually flatness [16] to Finsler metrics.

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) \quad = \quad 0 \tag{1.1}$$

where $(x^i(t))$ are the coordinates of $c(t)$ and $G^i = G^i(x, y)$ are defined by

$$G^i \quad = \quad \frac{g^{il}}{4}\{[F^2]_{x^k x^l} y^k - [F^2]_{x^l}\} \tag{1.2}$$

where $g_{ij} = \frac{1}{2}[L^2]_{y^i y^j}(x, y)$ and $(g^{ij}) := (g_{ij})^{-1}$. The local functions $G^i = G^i(x, y)$ define a global vector field on TM. G is called the $spray$ of F and G^i are called the $spray$ coefficients.

2 Preliminaries

To find the condition for the Finsler space with special (α, β)-metric to be Berwald space and at last, we apply the conformal change of Finsler space with the special (α, β)-metric of Douglas type. We study conformal change of Berwald space with special (α, β)-metric $L = \alpha - \beta^2/\alpha$ and also we obtain the conditions for Finsler space with an (α, β)-metric $L^2 = 2\alpha\beta$ to be conformally Berwald.

A Finsler metric on a manifold M is a C^∞ function $F : TM \to [0, \infty)$ satisfies the following properties:

- Regularity: L is C^∞ on $TM \backslash \{0\}$;
- Positively homogeneity: $L(x, \lambda y) = \lambda L(x, y), for \lambda > 0$;
- Strong convexity: The fundamental tensor $g_{ij}(x, y)$ is positive for all $(x, y) \in TM \backslash \{0\}$; where, $g_{ij} = \frac{1}{2}[L^2]_{y^i y^j}(x, y)$.

The concept of (α, β) was introduced in 1972 by M. Matsumoto and studied by many authors like ([7],[8],[12],[9],[13],[3]).The Finsler space $F^n = (M, L)$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$.

Dually flat Finsler metrics on an open subset in R^n can be characterized by a simple PDE. ([11])A Finsler metric $L = L(x, y)$ on an open subset $u \subset R^n$ is dually flat and it satisfies the following conditions:

$$[L^2]_{x^k x^l} y^k - 2[L^2]_{x^l} = 0 \tag{2.1}$$

In this case, $H = H(x, y)$ in (1.1) is given by $H = -\frac{1}{6} L^2{}_{x^m} y^m$

A Finsler metric $L = L(x, y)$ is locally projectively flat if at every point there is a coordinate system (x^i) in which all geodesics are straight lines, or equivalently. The spray coefficients are in the following form

$$G^i = P y^i \tag{2.2}$$

where $P = P(x, y)$ is a local scalar function.

Projectively flat metrics on an open subset in R^n can be characterized by a simple PDE. ([5])A Finsler metric $L = L(x, y)$ on an open subset $u \subset R^n$ is projectively flat and it satisfies the following equations:

$$L_{x^k x^l} y^k - 2L_{x^l} = 0. \tag{2.3}$$

In this case, Local function $P = P(x, y)$ in (2.2) is given by $P = \frac{L_{x^m} y^m}{2L}$

A Finsler metric is said to be dually flat and projectively flat on an open subset $u \subset R^n$ if the spray coefficients G^i. Therefore there are Finsler metrics on an open subset in R^n which are dually flat and projectively flat.

3 THE CONDITION TO BE A DOUGLAS SPACE

The geodesics of a Finsler space $F^n = (M^n, L)$ are given by the differential equations:

$$(d^2 x^i)/dt^2 + 2G^i(x, dx/dt) = 0 \tag{3.1}$$

Where $2G^i(x, y) = \gamma_j{}^i{}_k(x, y) y^j y^k$ and $\gamma_j{}^i{}_k(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with relevancy to x^i.

A Finsler space F^n is claimed to be of Douglas type if

$$D^{ij} \equiv G^i(x, y)y^j - G^j(x, y)y^i \tag{3.2}$$

are homogeneous polynomials in y^i of degree three. Let $hp(r)$ denote the homogeneous polynomial in y^i of degree r.

We use the subsequent definition in future.

Definition 3.1. The Finsler space F^n is of Douglas type if and as long as the Douglas tensor $D_{i\ jk}^{h} = C_{i\ jk}^{h} - 1/(n+1)(G_{ijk}y^h + G_{ij}\delta_k^h + G_{jk}\delta_i^h + G_k i\delta_j^h)$

Vanishes identically, where $G_{i\ jk}^{h} = \dot{\partial}_k G_{i\ j}^{h}$ is $h\nu$-curvature tensor of the Berwald connection.

The covariant differentiation with relation to the Levi-Cavita connection $\left\{ j \, {}^{i} \, k \right\}(x)$ of R^n is denoted by $(|)$. We use the symbols as follows:

$$r_{ij} = 1/2(b_{(i|j)} + b_{(j|i)}) \qquad s_{ij} = 1/2(b_{(i|j)} + b_{(j|i)}) \tag{3.3}$$

$$s_j = b_r s^{rj} \qquad s_j^i = a^{ir} s_{rj} \tag{3.4}$$

The functions $G^i(x,y)$ of F^n with (α,β)-metric are written within the form

$$2G^i = \left\{ 0 \, {}^{i} \, 0 \right\} + 2B^i \tag{3.5}$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} S_0^i + C^*[\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i)] \tag{3.6}$$

Where $L_\alpha = \partial L/\alpha, L_\beta = \partial L/\partial, L_{\alpha\alpha} = \partial L/\partial\alpha\partial\alpha$, the subscript 0 means contraction by y^i and that we put

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})} \tag{3.7}$$

We use the subsequent lemma. If $\alpha^2 \equiv 0(mod\beta)$, that is,$a_i j(x) y^i y^j$ contains $b_i y^i$ an element, then the dimension capable n is equal to 2 and b^2 vanishes. During this case, we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.

4 Conclusion

M. Matsumoto and S. Bacso introduced the notion of Douglas metric as a generalized Berwald space as geodesic equations. Douglas metrics are viewed as generalized Berwald metrics. The study of Douglas metrics will increase our understanding on the geometrical meaning of non-Riemannian quantities. In the present paper, we found the conditions for a Finsler space with special (α,β)-metric $L = \alpha - \beta^2/\alpha$ to be a Douglas space and also to be a Berwald space. Further, we found the conditions for a conformally transformed Douglas space with the above mentioned special (α,β)-metric to be a Douglas space.

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