Palestine Journal of Mathematics

Vol. 10(Special Issue I)(2021) , 140–143

ON ANTI-INVERSE SEMIRINGS WITH IDENTITY a+a.b = b

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Communicated by V. Lokesha

MSC 2010 Classifications: Primary 16Y60

Keywords and phrases: Anti-inversesemirings, Singular semirings, Semilattice.

Abstract: Here we concentrated on properties of semiring satisfying an identity a+a.b = b. It has been verified, if S be a mono semiring then (S, +) is left as well as right singular if and only if S is Anti - inverse semiring. $\forall$ a and b $\in$ S, a relation $\leq$ is defined on S in which a $\leq$ b $\Leftrightarrow$ a + a.b = b. and if 'e' be the additive and multiplicative identity then (S, +, .) be a t.o.s and so 'e' is the maximal element of mono semiring S. Investigated results on structural properties of Anti - inverse semirings using the above identity.

1 Introduction

In modern algebra, a mathematical structure is said to be a semiring similar to a ring, but it is not necessary that each element must have an additive inverse. In the literature, various concepts of semirings have been considered like properties of rings such as cancelling or added. Vandiver [7] introduced the concept of semiring in 1934 and Anti-inverse elements in a semigroups was introduced by J.C.Sharp [2] in 1982. Later Blagojevic [1] studied over its structural properties. Semirings have so many applications in data encryption like automata theory, optimization theory. Rajeswari A. [3] proved some results on the identity a+a.b = b. We extended the results of Sheela N. and Rajeswari A. [6] work.
Overview the recent development of Anti-inverse semirings and its various structures by applying idempotent semirings.The absorbing identity a+a.b = b and structural properties of semirings.

2 Preliminaries

Definition 2.1: A nonempty set S is said to be a semiring considering binary operations on S as addition '+' and multiplication '.' and is defined such that for all a,b in S.
i. (S, +) and (S, .) are the semigroups.
ii. Distributive law holds on multiplication over addition on either side. i.e.,
a.[b+c] = [a.b] + [a.c] and [a+b].c = [a.c] + [b.c].
Examples:
i. Any ring (S, +, .) is a semiring.
ii. The non-negative real numbers and rational numbers form semirings.

Definition 2.2: A semiring S is said to be Anti-inverse semiring if for every element a $\in$ S $\exists$ an element b $\in$ S such that a $*$ b $*$ a = b and b $*$ a $*$ b=a, then a and b are Anti-inverse elements.

$*$	a	b
a	a	a
b	b	b

Since a $*$ a $*$ a = a, b $*$ b $*$ b = b this implies a and b are having their own Anti-inverses.
a $*$ b $*$ a = b and b $*$ a $*$ b =a, a and b are Anti-inverses. Where $*$ is any binary operation, usually addition and substraction.
Definition 2.3: (S, +, .) , A semiring is a mono semiring if a+b = ab $\forall$ a, b$\in$ S.
Definition 2.4: If (S, .) is a semigroup then it is said to be left (right) singular if a.b = a (a.b = b)

∀ a, b ∈ S respectively.
Definition 2.5: (S, +, .) is idempotent if a.a = a.
Definition 2.6: (S, +, .) is band if every element in S is idempotent.

Remark 2.2.1: We write a $*b*$ a as aba if $*$ is a binary operation for multiplication throughout the paper.
Theorem 2.3.1 : If (S, +, .) be Anti-inverse and mono semiring satisfies a + a.b = b then (S, +) is left as well as right singular.

Proof: Consider a + a.b = a + a+ b because of Anti-inverse a+a.b = (a + a + a) + b + a =a+b+a = b this ⇒ S satisfies the identity.
Since S is Anti-inverse, consider aba=b and S is mono semiring thus a b + a = b this ⇒ a (b+a) a = b, then by mono semiring a+ b+ a +a = b which is b+a=b, hence (S, +) is a left singular.
If aba = b because of mono semiring, we write (a+b). a = b, this ⇒ a.a+b.a =b , thus a+a+b+a = b.
So a+b = b this ⇒ (S, +) is a right singular.
Coversely suppose iff (S, +) is left as well as right singular and mono semiring.
Now aba = (a+b) a = ba[since S is right singular] = b+a [since S is mono semiring] =b [since S is left singular].
Similarly we can prove that bab=a. Thus (S, .) is an Anti-inverse semigroup.
Also consider a+b+a = b+a = b.
Thus (S, +, .) is Anti-inverse semiring.

Theorem 2.3.2:
A semiring (S, +, .) in which (S, .) be Anti-inverse semigroup,if S has an identity a+a.b = b then S is Simple ⇔ ab+acb=ab ∀ a, b, c ∈ S.

Proof: Suppose a.1.b + acb = ab this ⇒ a (1+c) b = ab. If a=b=1 then 1+c = 1 ∀ as in S. Thus,S is Simple.
Conversely,consider ab+acb =ab+ a (bcb) b = ab (1+ (bcb) bb) = ab (1+c) = ab(1) = ab.

Theorem 2.3.3: If (S, +, .) be a semiring in which (S, .) be idempotent and Anti-inverse semigroup. If S satisfies the identity a+a.b = b then the conditions stated below on S is inter related.
(i) ∀ a,b,c ∈ S, ab+acb= ab.
(ii) (S, +) is idempotent semigroup.
(iii) (S, +) is right regular.

Proof: To prove (i) implies (ii)
Consider ab +acb = ab multiplied both sides by 'a' we have aba+acba = aba⇒ b + (bab) cba =b ⇒ b+baca = b
⇒ b + (cbc) c = b ⇒ b+cbc = b ⇒ by anti-inverse property we get b+b = b ∀ b ∈ S.
Thus (S, +) is idempotent.
Conversely, if S is idempotent semigroup.
b+b = b multiplied both the sides by 'a' then ab+ab=ab ⇒ ab+a (cbc)=ab (since S is Anti-inverse) ⇒ ab+accbc = ab ⇒ ab +acb=ab. This proves (i).
To prove (iii) Assume (ii)
Consider aba = a (a + a.b) a by distributivity aba = (a.a+aa.b) a = (a + a.b) a = ba ⇒ (S,.) is right regular semigroup.
In a similar way it can be easily prove that (S, +) is right regular semigroup.
Conversely, if S is right regular
aba = ba ⇒ abab = bab ⇒ a.a = a ⇒ (S,.) is idempotent semigroup
Similarly (S, +) is idempotent semigroup ⇒ S is idempotent.
Assume (iii), if S is right regular.
a + a.b = b ⇒ bab + a (cbc) = aba ⇒ abaab +ac.cbc = aba ⇒ abab + acb = abbab
⇒ aab + acb =abab ⇒ ab +acb = ab ⇒ ab +acb = ab. Proving (i).
To prove (iii) Assume (ii)

Consider aba = a(a + a.b)a
by distributivity aba= (a.a+aa.b) a = (a + a.b) a = ba $\Rightarrow$ (S, .) is right regular semigroup.
Similarly we can prove that (S, +) is right regular semigroup.
Conversely, if S is right regular
aba = ba$\Rightarrow$ abab = $\Rightarrow$a.a = a $\Rightarrow$ (S, .) is idempotent semigroup
Similarly (S, +) is idempotent semigroup $\Rightarrow$ S is idempotent.
Assume (iii), if S is right regular.
a + a.b = b $\Rightarrow$ bab + a (cbc) = aba $\Rightarrow$ abaab +ac.cbc = aba $\Rightarrow$ abab + acb = abbab $\Rightarrow$ aab + acb =abab $\Rightarrow$ ab +acb =abab.b=aab= $\Rightarrow$ ab +acb = ab. Proving (i).

Theorem 2.3.4: If S is simple and Anti-inverse semigroup with a + a.b = b implies a = b.

Proof: Consider a + a.b = b Multiplied both sides by 'b', bab + bab.b = bbb bab (1 + b) = bbb then bab.1=b by Anti-inverse, we get a=b.

Theorem 2.3.5: If (S, +, .) be a semiring with a + a.b = b then S is simple $\Leftrightarrow$ S is additive idempotent if S is Anti-inverse and multiplicatively idempotent.

Proof: Since S is simple
Thus 1+b=1 for all b in S, thereexist a in S in which a + a.b = b $\Rightarrow$ a + bab.b =aba $\Rightarrow$ a + bab = babba=baba=a.a because S is Anti-inverse a + a = a
Hence S is additive idempotent.
Conversely suppose S is additive idempotent.
b + b = b, $\forall$ b $\in$ S
b + a + a.b = b put b = 1, 1+a+a=1 $\Rightarrow$ 1+a+a=1 => 1+a=1.Thus S is simple semiring.

Theorem 2.3.6: Let a simple semiring (S, +, .) in which (S, .) is Anti-inverse semiring with a + a.b = b $\Rightarrow$ (S, +, .) is a semilattice.

Proof For each b in S, consider b+b =a + a.b+a.b.a =a + a.b (1+a) = a + ab.1 = a + a.b =b implies(S, +) is idempotent semigroup and hence band.
a+b = a+a + a.b then a+b = (b + a + b) + a +a.b= b+a+(a + a.b)+a + a.b = b + a + a + (a.b + a + a.b)
= b+(a+a+a) = b+a and hence 'b' on either sides we get ab+abb = bb $\Rightarrow$ ab+ (bab) bb =bb $\Rightarrow$ ab + ba (bbb) = bb $\Rightarrow$ ab+bab =bb $\Rightarrow$ ab+a = bb $\Rightarrow$ b =bb $\Rightarrow$ (S, .) is idempotent.
Idempotent Anti-inverse semiring is commutative. Thus (S, +, .) is a commutative band and hence S is semilattice.

Conclusion: In this paper we have proved some structural properties of Anti-inverse semirings with identity a + a.b = b.

Acknowledgment : I would like to thank Visvesvaraya Technological University, Belagavi, Karnataka, India for helping in my research work.

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Received: Dec 21, 2020.
Accepted: Feb 28, 2021.