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MATRIX BASED UNITARY RAMANUJAN SUMS TRANSFORMS

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Abstract In this paper, we study the Unitary Ramanujan sums (URS) transform by means of matrix multiplication. The URS are orthogonal in nature and therefore offer excellent energy conservation capability. The 1D and 2D forward URS transforms are easy to calculate, but their inverse transforms are not defined in the literature for non-even function (mod M). We solved this problem by using matrix multiplication in this paper.

Keywords: Ramanujan Sums (RS); Fourier transform (FT); Gaussian white noise

1 Introduction

The Unitary Ramanujan Sums (URS) are proposed by S.Ramanujan in 1918 [1] and applied to time frequency analysis, signal processing, moment invariants and shape recognition recently. The URS are orthogonal in nature and therefore offer excellent energy conservation and is similar to the Fourier Transform (FT). The URS are operated on integers and hence can obtain a reduced quantization error implementation. Even though the URS transform has so many important properties, it does not have the inverse URS transform for non-even function (mod M) signals. In this paper, we analyze the URS transforms by means of matrix multiplication, which can invert the URS transform easily. We derive both the forward and inverse URS transforms for 1D signals and 2D images. A few examples are also tested and our method can recover the 1D signals and 2D images perfectly without any errors. The organization of this paper is as follows. Section 2 presents a short review of the URS transform and proposes the matrix based URS transforms for 1D signals and 2D images. The inverse URS transforms recover the signals and images perfectly without any errors. Finally section 3 concludes the paper and proposes future research directions about the URS transform.

2 MATRIX - BASED UNITARY RAMANUJAN SUMS TRANSFORM

The URS transform has been used as means of representing arithmetical functions by an infinite series expansion .The basis of this transform is the building block of number theoretic- functions. The URS are sums of the n^{th} powers of q^{th} primitive roots of unity ,defined as

$$C^*_{N(L)} = \sum_{x=1 gcd(x,N)^*=1}^{N} e^{2i\Pi xL/N} \tag{2.1}$$

Where

$$gcd(x,N)^* = 1 \tag{2.2}$$

means that the Unitary greatest common divisor (GCD) is unity, i,e, x and N are co-prime . An alternate computation of URS be given as

$$C^*_{N(L)} = \mu^*\left(\frac{N}{gcd(N,L)}\right)\frac{\varphi^*(N)}{\varphi^*\left(\frac{N}{gcd(N,L)}\right)} \tag{2.3}$$

Let $N = \prod_i N_i^{\alpha_i}\,(N_i prime)$ Then, we have $\varphi^i(N) = N\prod_i\left(1, \frac{1}{N}_i\right)$. The unitary Mobius function $\mu^*(n)$ is equal to 0. If N contains a square free number If $N = 1$ and (-1^k), if n is a product of k distinct prime numbers. We tabulate $C^*_N(L)$ with $N \in [1, 15]$ in Table. The URS have the following multiplicative property $C^*_{NN^1}(L) = C^*_N(L)C^*_N(L), if gcd(N, N^1) = 1$

$\sum_{L=1}^{NN^1} C_N^*(L)C_N^*(L) = 0$, if $N = N^1$
$\sum_{L=1}^{N} C_N^{*2}(L)C_N^*(L) = N_\varphi(N)$ otherwise
We can also derive $C_N^*(L)$ by using Euler formula $e^{ix} = cosx + isinx$ and basic trigonometric identities

$$C_1^*(L) = 1 \tag{2.4}$$

$$C_2^*(L) = cosL\pi \tag{2.5}$$

$$C_3^*(L) = 2cos(2/3)L\pi \tag{2.6}$$

$$C_5^*(L) = 2cos(2/5)L\pi + 2cos(4/5)L\pi \tag{2.7}$$

$$C_6^*(L) = 2cos(1/3)L\pi \tag{2.8}$$

$$C_7^*(L) = 2cos(2/7)L\pi + 2cos(4/7)L\pi + 2cos(6/7)L\pi \tag{2.9}$$

$$C_{10}^*(L) = 2cos(1/5)L\pi + 2cos(3/5)L\pi \tag{2.10}$$

$$C_{11}^*(L) = 2cos(2/11)L\pi + 2cos(4/11)L\pi + 2cos(6/11)L\pi + 2cos(8/11)L\pi + 2cos(10/11)L\pi \tag{2.11}$$

$$C_{13}^*(L) = 2cos(2/13)L\pi + 2cos(4/13)L\pi + 2cos(6/13)L\pi + 2cos(8/13)L\pi + 2cos(10/13)L\pi + 2cos(12/1 \tag{2.12}$$

$$C_{14}^*(L) = 2cos(2/14)L\pi + 2cos(6/14)L\pi + 2cos(10/14)L\pi \tag{2.13}$$

$$r(N) = \frac{1}{\varphi^*(N)} \frac{1}{M} \sum_{m=0}^{M} x(m) C_{N(m)}^* \tag{2.14}$$

Where E_M is the set of all even functions (mod M). A signal is called even signal (mod M) if $x_M = x_M\,(gcd(n, M))$ for any N. It is easily shown that every even function (mod M) is a periodic function, but the converse does not hold. This means that for an ordinary input signal, its forward 1D URS exists, but the inverse transform cannot be calculated by using the above formula. In this paper, we represent the 1D and 2D forward and inverse URS transforms by means of matrix multiplication. Let us define the matrix

$$A(N, j) = \frac{1}{\varphi^*(N)M} C_N^*(mod(j-1, N) + 1) \tag{2.15}$$

Where $q, j \in [1, M]$ and mod (.) means the modular operation. The input signal can be represented as $(x(1), x(2), ...x(M))^\tau$,

where T means the transpose of the vector.

The forward 1D URS transform of a signal X can be realized as $Y = AX$
Where $(Y(1), Y(2), ...Y(M))^\tau$, .

The inverse 1D URS transform can be obtained as $X = A^{-1}Y$

Where A^{-1} means the inverse of matrix A. It has been proved that the determinant of the MXM matrix C^* ,whose N, j entry is the Unitary Ramanujan sum $C_{N(L)}^*$ is $det\,[C_N^*(j)] = M!$
.

Therefore $det[A(N, j)] = \frac{M!}{M^M \sum_{N=1}^{M} \varphi^*(N)} \neq 0$

This means that the matrix $A(N, j)$ in always non- singular. We can use QR decomposition to decompose matrix A, i.e, $A = QR$. Therefore

$$X = R^{-1}Q^{\tau}Y \tag{2.16}$$

We calculate the 1D forward and inverse URS transforms of three 1D signals that are used extensively in signal de noising literature. Figure 1 shows the one input signal in the first row, i.e, forward URS. Their inverse URS transform signal in the second row. The difference between the original signals and their reconstructed signals by using the method proposed in this chapter in the last row. It can be seen that the error introduced in the transforms is nearly zero.

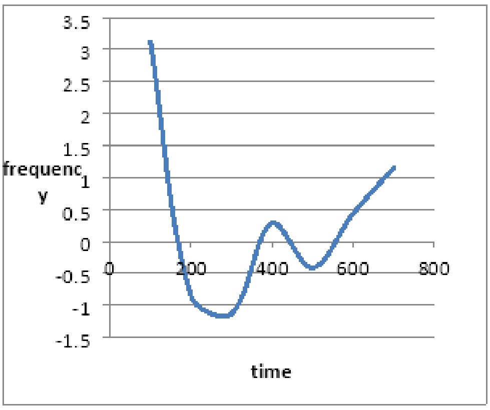

Figure 1. Forward URS

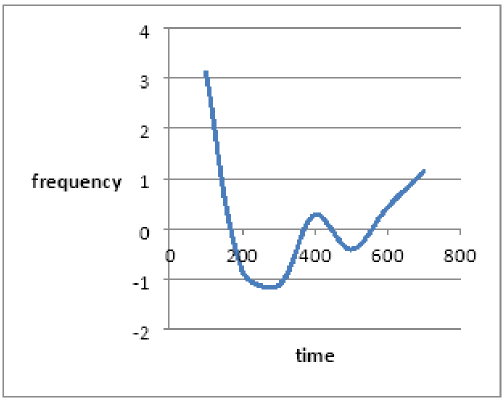

Figure 2. Inverse URS

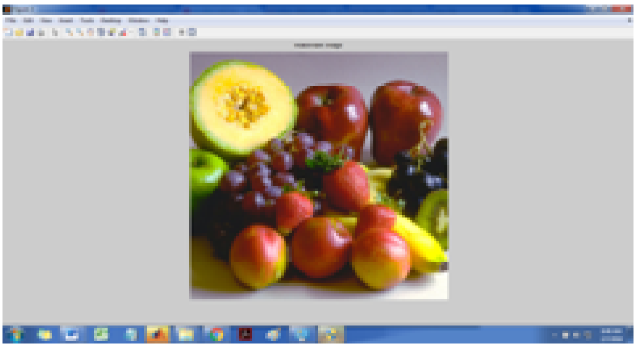

Figure 2. The forward 2D URS transform for fruits image

For 2D images ,we can perform the forward URS transform as

$$Y(x, N) = \frac{1}{\varphi^*(x)\varphi^*(L)} \frac{1}{ML} \sum_{m=1}^{M} \sum_{L=1}^{N} x(m, L) C_N^*(m) C_N^*(L) \tag{2.17}$$

This can be written in the Matrix form

$$Y = AXA^{\tau} \tag{2.18}$$

Where A is defined as in equation ...and $X = (x(m, L))$ for $m \in [1, M]$ and $L \in [1, N]$. The inverse 2D URS transform can be gives as

$$X = A^{-1}Y(A^{-1})^{\tau} \tag{2.19}$$

As before , let $A = QR$ then

$$X = R^{-1}Q^{\tau}Y(R^{-1})^{\tau} \tag{2.20}$$

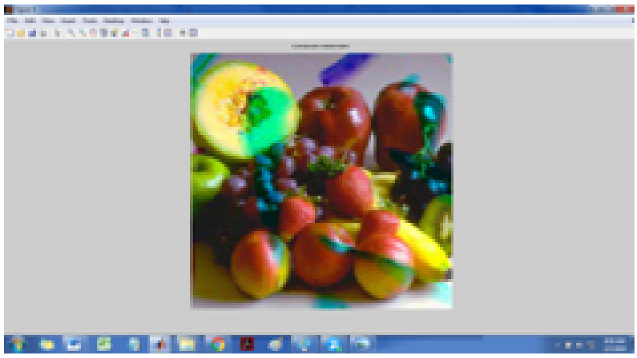

Figure 3. The inverse 2D URS transform for fruits image

We tested fruits image for our 2D URS transform. Figure 1 shows original fruit image, its URS coefficients and the reconstructed image. Our inverse 2D URS transform can recover the input image perfectly without errors. The computational complexity of our matrix based URS transform is as follows. For 1D signal, both the forward and backward 1D URS transforms need $O(M^2)$ flops of operations, where M is signal length. For 2D images, the forward 2D URS transforms needs $O(M^2N + MN^2)$ flops of operations. The inverse 2D URS transform also requires $O(M^2N + MN^2)$ flops of operations.

Table 1. Comparison of existing localization algorithms

N															
1	1														
2	-1	1													
3	-1	-1	2												
5	-1	-1	-1	-1	4										
6	1	-1	-2	-1	1	2									
7	-1	-1	-1	-1	-1	-1	6								
10	1	-1	1	-1	-4	-1	1	-1	1	4					
11	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	10				
13	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	12		
14	1	-1	1	-1	1	-1	-6	-1	1	-1	1	-1	1	6	
15	1	1	-2	1	-4	-2	1	1	-2	-4	1	-2	1	1	8

3 CONCLUSION

In this paper, we have studied the 1D and 2D forward and inverse URS transform by means of matrix multiplication. Our method can find the 1D and 2D inverse URS transform for any kind of signals and images. Currently, there is no existing inverse URS transform of the literature for non even function of (mod M). Future research direction about the URS transform is given below. We would like to apply the 1D and 2D URS transforms to signal, image or video compression. This is because the URS transform has very good property to compress the energy of the input signals, images or videos into a few number of URS coefficients. We would also extend the URS transforms to 3D data cube.

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