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Locally Projectively Flat Special (α, β)-Metric

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Abstract In this article, the author studies the projectively flat special (α, β)-metric $F = c_1\alpha + c_2\beta + \beta^2/\alpha$. Then they conclude that α is locally projectively flat and further shows β is parallel with respect to α.

1 Introduction

One of important problems in Finsler geometry is to study the geometric properties of locally projectively flat Finsler manifolds. Locally projectively flat metrics are of scalar flag curvature, namely, the flag curvature is a scalar function of tangent vectors, independent of the tangent planes containing the tangent vector.

It is Hilbert's Fourth Problem which characterizes the (not-necessarily-reversible) distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are geodesics. Regular distance functions with straight geodesics are projectively flat Finsler metrics. A Finsler metric function F in a differentiable manifold M is called an (α, β)-metric, if F is a positively homogeneous function of degree one of a Riemannian metric $\alpha^2 = a_{ij}y^iy^j$ and a non-vanishing 1- form $\beta = b_iy^i$ on M. matsumoto metric was introduced physically by using the gradient of slope, speed and gravity. But this metric is expressed as an infinite series form for $|\beta| < |\alpha|$.

Locally projectively flat Finsler metrics compose an important group of metrics in Finsler geometry. The characterization of these metrics is the regular case of the Hilbert's Fourth Problem. Distance functions induced by a Finsler metric are regarded as smooth ones. The Hilbert Fourth Problem in the smooth case is characterize projectively flat Finsler metrics on an open subset in $\mathbb{R}^n$. A Finsler metric on an open subset $U \subset \mathbb{R}^n$ is said to be projectively flat if all geodesics are straight in U. A Finsler metric on a manifold M is said to be locally projectively flat if at any point, there is a local coordinate system (x^i) in which F is projectively flat. By the Beltrami's theorem, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. However the situation is much more complicated for Finsler metrics. J. Douglas' a famous theorem said that a Finsler metric on a manifold $M(dimM \geq 3)$ is locally projectively flat if and only if F is a Douglas metric of scalar curvature.

A change $F \to \bar{F}$ of a Finsler metric on a same underlying manifold M is called *projective*, if any geodesic in (M, F) remains to be a geodesic in (M, F) and vice versa. A Finsler space is called *projectively flat* if it is projective to a locally Minkowski space.

The well-known Funk metric $F = F(x)$ on a strongly convex domain in $\mathbb{R}^n$ is projectively flat with the constant flag curvature $K = -1/4$. When the domain is the unit ball $B^n \subset \mathbb{R}^n$, the Funk metric is given by,

$$F = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}.$$

which is in such form as

$$F = \alpha + \beta,$$

where α is a Riemannian metric and $\beta = b_iy^i$ is a 1-form. Finsler metrics in such a form are called the Randers metrics. It is known that a Randers metric $F = \alpha + \beta$ is projectively flat if and only if α is projectively flat and β is closed.

It is well-known that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed. For a general (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, if β is parallel with respect to α, then F is locally projectively flat if and only if α is locally projectively flat.

A Finsler metric $F = F(x, y)$ is said to be locally Minkowskian if at every point, there is a local coordinate domain in which the metric $F = F(y)$ is independent of its position x. In this case, all geodesics are linear lines $x^i(t) = ta^i + b^i$. A Finsler metric $F = F(x, y)$ is said to be *locally projectively flat* if at every point, there is a local coordinate domain in which the geodesics are straight lines, namely, $x^i(t) = f(t)a^i + b^i$. The main purpose of this paper is to study locally projectively flat special (α, β)-metrics.

2 Preliminaries

A Finsler metric is a scalar field $L(x, y)$ which satisfies the following three conditions:

(i) It is defined and differential for any point of $TM\backslash\{0\}$,

(ii) It is positively homogeneous of first degree in y^i, that is,
$L(x, \lambda y) = \lambda L(x, y)$, for any positive number λ,

(iii) It is regular, that is,
$g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$,
constitute the regular matrix g_{ij}, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

The manifold M^n equipped with a fundamental function $L(x, y)$ is called Finsler space $F^n = (M^n, L)$

There is a class of Finsler metrics defined by a Riemannian metric and 1-form on a manifold, which is relatively simple with interesting curvature properties called (α, β)-metrics and these metrics are computable.

The Finsler space $F^n = (M^n, L)$ is said to be have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_i(x)y^i$, where α is a Riemannian metric and β is differentiable 1-form.

The space $\mathbb{R}^n = (M^n, \alpha)$ is called the *associated Riemannian space* and the covariant vector field b_i is the *associated vector field.*
An (α, β)-metric is expressed in the following form,

$$L = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by,

$$\|\beta_x\|_\alpha = \sup_{y\epsilon T_xM} \beta(x, y), \alpha(x, y) = a_{ij}(x)b_i(x)b_j(x).$$

In order to define L, β must satisfy the condition $\|\beta_x\|_\alpha < b_0$ for all $x\epsilon M$.
Let G^i and G^i_α denote the spray coefficient of F and α respectively, given as follows

$$G^i = \frac{g^{il}}{4}\left\{[F^2]_{x^ky^l}y^k - [F^2]_{x^k}\right\},$$

$$G^i_\alpha = \frac{a^{il}}{4}\left\{[\alpha^2]_{x^ky^l}y^k - [\alpha^2]_{x^l}\right\},$$

where $(g_{ij}) = (1/2([F^2]_{y^iy^j}))$ and $(a^{ij}) = (a_{ij})$. We have the following

Lemma 2.1. *The spray coefficients G^i are related to G^i_α by*

$$G^i = G^i\alpha + \alpha Q s^i_0 + J\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + H\{-2Q\alpha s_0 + r_{00}\}\left\{b^i - s\frac{y^i}{\alpha}\right\}, \qquad (2.1)$$

where,

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ J &= \frac{\phi^{'}(\phi - s\phi)}{2\phi[(\phi - s\phi) + (b^2 - s^2)\phi'']}, \\ H &= \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi''))}. \end{aligned} \tag{2.2}$$

where, $s = \beta/\alpha, b = \|\beta_x\|_\alpha, s_{ij} = 1/2(b_{i|j} - b_{j|i}), s_{l0} = s_{li}y^i, s_0 = s_{l0}b^l, r_{ij} = 1/2(b_{i|j} - b_{j|i})$ *and* $r_{00} = r_{ij}y^iy^j$.

Lemma 2.2. *An* (α, β)*-metric* $F + \alpha\phi(s)$*, where* $s = \beta/\alpha$*, is projectively flat on an open subset* $U \subset \mathbb{R}^n$ *if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2Q\alpha s_0 + r_{00})(b_l\alpha - sy_l) = 0, \tag{2.3}$$

where, $y_l = a_{lj}y^j$.

3 Locally Projectively Flat (α, β)-metric

In this section, we consider an (α, β)-metric in the following form:

$$F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}, \quad c_2 \neq 0. \quad \phi(s) = c_1 + c_2 s + s^2, \quad s = \beta/\alpha. \tag{3.1}$$

where $s < 1$ so that ϕ must be a positive function. Let b_0 be the largest number such that

$$\phi(s) - s\phi^{'} + (b^2 - s^2)\phi^{''}(s) > 0, \qquad |s| \leq b < b_0,$$

theta is,

$$c_1 + 2b^2 - 3s^2 > 0, \quad |s| \leq b < b_0.$$

Lemma 3.1. *Let* $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ *is a Finsler metric if and only if* $\|\beta_x\|_\alpha < \sqrt{c_1}$.

Proof. If $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ is a Finsler metric, then

$$c_1 + 2b^2 - 3s^2 > 0.$$

Let $s = b$, we obtain $b < \sqrt{c_1}, \forall b < b_0$. Let $b \to b_0$, then $b_0 < \sqrt{c_1}$. So $\|\beta_x\|_\alpha < \sqrt{c_1}$. Now , if

$$|s| \leq b < \sqrt{c_1},$$

then

$$c_1 + c^2 - 3s^2 > 0.$$

Thus, $F = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ is a Finsler metric. From Lemma 2.1, the spray coefficients G^i of F are given by 2.1 with

$$\begin{aligned} Q &= \frac{c_2 + 2s}{c_1 - s^2} = \frac{\alpha(\alpha c_2 + 2\beta)}{c_1\alpha^2 - \beta^2}, \\ J &= \frac{(c_2 + 2s)(c_1 - s^2)}{2(c_1 + c_2 s + s^2)(c_1 + 2b^2 - 3s^2)}, \\ &= \frac{\alpha(c_2\alpha + 2\beta)(c_1\alpha^2 - \beta^2)}{2(c_1\alpha^2 + c_2\alpha + \beta^2)(c_1\alpha^2 + 2b^2\alpha^2 - 3\beta^2)}, \\ H &= \frac{1}{c_1 + 2b^2 - 3s^2} = \frac{\alpha^2}{c_1\alpha^2 + 2b^2\alpha^2 - 3\beta^2}. \end{aligned}$$

Eq. 2.3 is reduced as follows,

$$\begin{aligned} 0 &= (a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 \frac{\alpha(\alpha c_2 + 2\beta)}{c_1\alpha^2 - \beta^2} s_{l0} \\ &+ \frac{\alpha^3}{c_1\alpha^2 + 2b^2\alpha^2 - 3\beta^2} \left\{ -2\alpha s_0 \left[\frac{\alpha(\alpha c_2 + 2\beta)}{c_1\alpha^2 - \beta^2} \right] + r_{00} \right\} \left[b_l\alpha - \frac{\beta}{\alpha} y_l \right] \end{aligned} \tag{3.2}$$

Lemma 3.2. *If $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, then α is projectively flat.*

Proof. If $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, then

$$a_{ml}\alpha^2 G^m_\alpha = y_m y_l G^m_\alpha. \tag{3.3}$$

Contracting 3.3 with a^{il}, we have

$$\alpha^2 G^i_\alpha = y_m y^i G^m_\alpha.$$

Let $\lambda(x, y) = y_m G^m_\alpha / \alpha^2$, then

$$G^i_\alpha = \lambda y^i.$$

Therefore, α is projectively flat.
From 3.2, we can prove the following

Theorem 3.3. *The Finsler special metric $F = c_1\alpha + c_2\beta + \beta^2/\alpha$, $\quad c_2 \neq 0$ is locally projectively flat if and only if*

(i) β is parallel with respect to α,

(ii) α is locally projectively flat, that is, of constant curvature.

Proof. Assume that F is locally projectively flat. Firstly, rewrite (3.2) as a polynomial in y^i. Which gives

$$\begin{aligned} 0 &= (c_1\alpha^2 - \beta^2)[(c_1 + 2b^2)\alpha^2 - 3\beta^2](a_{ml}\alpha^2 - y_m y_l)G^m_\alpha \\ &+ \alpha^4(\alpha c_2 + 2\beta)[(c_1 + 2b^2)\alpha^2 - 3\beta^2]s_{l0} \\ &+ \alpha^2 \left\{ -2\alpha^2 s_0(\alpha c_2 + 2\beta) - r_{00}(c_1\alpha^2 - \beta^2) \right\} (b_l\alpha^2 - \beta y^l) \end{aligned} \tag{3.4}$$

The coefficient of α must be zero (note: α^{even} is a polynomial in y^i). We have

$$\begin{aligned} 2\beta^2(2c_1 + b^2)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha &= \alpha^4(\alpha c_2 + 2\beta)(c_1 + 2b^2)s_{l0} \\ &- \left[2\alpha^4 s_0(\alpha c_2 + 2\beta) + r_{00}\beta^2 \right] (b_l\alpha^2 - \beta y_l). \end{aligned} \tag{3.5}$$

$$\begin{aligned} \left[3\beta^4 + \alpha^4(1 + 2b^2)c_1 \right] (a_{ml}\alpha^2 - y_m y_l)G^m_\alpha &= \alpha^4(\alpha c_2 + 2\beta)3\beta^2 s_{l0} \\ &- r_{00}c_1\alpha^4(b_l\alpha^2 - \beta y_l). \end{aligned} \tag{3.6}$$

Contract (3.5) and (3.6) with b^l yields

$$\begin{aligned} 2\beta^2(2c_1 + b^2)(b_m\alpha^2 - y_m\beta)G^m_\alpha &= \alpha^4(\alpha c_2 + 2\beta)(c_1 + 2b^2)s_0 \\ &- \left[2\alpha^4 s_0(\alpha c_2 + 2\beta) + r_{00}\beta^2 \right] (b^2\alpha^2 - \beta^2). \end{aligned} \tag{3.7}$$

$$\begin{aligned} \left[3\beta^4 + \alpha^4(1 + 2b^2)c_1 \right] (b_m\alpha^2 - y_m\beta)G^m_\alpha &= \alpha^4(\alpha c_2 + 2\beta)3\beta^2 s_{l0} \\ &- r_{00}c_1\alpha^4(b^2\alpha^2 - \beta^2). \end{aligned} \tag{3.8}$$

(3.7) multiplied by α^4– (3.8) multiplied by $2\beta^2$ which gives,

$$2\beta^2(b_m\alpha^2 - y_m\beta)G^m_\alpha = \alpha^4 s_0. \tag{3.9}$$

Here α^4 is not divisible by β^2, β^2 is not divisible by α^4. thus s_0 is divisible by β^2 and $(b_m\alpha^2 - y_m\beta)G^m_\alpha$ is divisible by α^4. Therefore, there are scalar functions $\tau = \tau(x), \chi = \chi(x)$ such that

$$s_0 = \tau\beta^2, \tag{3.10}$$

$$(b_m\alpha^2 - y_m\beta)G^m_\alpha = \chi\alpha^2. \tag{3.11}$$

Then (3.9) becomes

$$2\beta^2\chi\alpha^4 = \alpha^4\tau\beta^2. \tag{3.12}$$

Thus, $\tau = \tau(x)$. Then (3.8) becomes

$$\left[\alpha^4(1+2b^2)c_1 + 3\beta^4(1-(\alpha c_2+2\beta))\right]\chi = -r_{00}c_1(b^2\alpha^2 - \beta^2).$$

Remember that $(b^2\alpha^2 - \beta^2)$ is not divisible by $\left[\alpha^4(1+2b^2)c_1 + 3\beta^4(1-(\alpha c_2+2\beta))\right]$. Thus $\chi = 0$.

$$r_{00} = 0. \tag{3.13}$$

and

$$s_0 = 0. \tag{3.14}$$

Then plugging (3.13) and (3.14) into (3.5) and (3.6), we have

$$2\beta^2(2c_1+b^2)(a_{ml}\alpha^2 - y_my_l)G^m_\alpha - \alpha^4(\alpha c_2+2\beta)(c_1+2b^2)s_{l0} = 0. \tag{3.15}$$

and

$$\left[3\beta^4 + \alpha^4(1+2b^2)c_1\right](a_{ml}\alpha^2 - y_my_l)G^m_\alpha - \alpha^4(\alpha c_2+2\beta)3\beta^2 s_{l0} = 0. \tag{3.16}$$

So,

$$\begin{vmatrix} 2\beta^2(2c_1+b^2) & -\alpha^4(\alpha c_2+2\beta)(c_1+2b^2) \\ \left[3\beta^4+\alpha^4(1+2b^2)c_1\right] & -\alpha^4(\alpha c_2+2\beta)3\beta^2 \end{vmatrix} = 0$$

Thus,

$$(a_{ml}\alpha^2 - y_my_l)G^m_\alpha \tag{3.17}$$

$$s_{l0} = 0 \tag{3.18}$$

Then by Lemma 2.1 and Lemma 3.2, α is projectively flat. And by (3.14) and (3.18), $b_{i|j} = 0$. Thus β is parallel with respect to α.

On the contrary, if β is parallel with respect to α and α is locally projectively flat, then by Lemma 2.1, we have seen that F is locally projectively flat.

4 Conclusion

A Finsler metric on an open domain in $\mathbb{R}^n$ is said to be projectively flat if its geodesics are straight lines. Hilbert's Fourth Problem in the regular case is to study and characterize projectively flat metrics on a convex open domain in $\mathbb{R}^n$. Every locally projectively flat Finsler metric is of scalar flag curvature, namely, the flag curvature is a scalar function on the tangent bundle, which might not necessarily be constant as in the Riemannian case. Thus locally projectively form a rich class of Finsler metrics.

The main ambition of this paper is, we studied and characterize locally projectively flat Finsler (α, β)-metrics.

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