Palestine Journal of Mathematics

Vol. 12(Special Issue II)(2023) , 13–19

LEAST COMMON MULTIPLE OF PRODUCT GRAPHS

Reji T, Ruby R and Sneha B

Communicated by S. Monikandan

MSC 2010 Classifications: 05C38, 05C51, 05C70.

Keywords and phrases: Graph decomposition, Least common multiple, Cartesian product.

Abstract A graph G without isolated vertices is a least common multiple of two graphs H_1 and H_2 if G is a smallest graph, in terms of number of edges, such that there exists a decomposition of G into edge disjoint copies of H_1 and H_2. The collection of all least common multiples of H_1 and H_2 is denoted by $LCM(H_1, H_2)$ and the size of a least common multiple of H_1 and H_2 is denoted by $lcm(H_1, H_2)$. In this paper $lcm(P_4, C_m \square P_n)$, $lcm(P_4, W_m \square P_n)$ and $lcm(P_4, W_m \square C_n)$ are determined where the product is the cartesian product.

1 Introduction

All graphs considered in this paper are assumed to be simple and to have no isolated vertices. The size of a graph G is the number of edges of G denoted by $|E(G)|$. A graph H is said to divide a graph G if there exists a set of subgraphs of G, each isomorphic to H, whose edge sets partition the edge set of G. Such a set of subgraphs is called an H-*decomposition* of G. G is said to be H-*decomposable* if G has an H- *decomposition* and write $H|G$.

A graph G is called a common multiple of two graphs H_1 and H_2 if both $H_1|G$ and $H_2|G$. A graph G is a least common multiple of H_1 and H_2 if G is a common multiple of H_1 and H_2 and no other common multiple has fewer edges. Several authors have investigated the problem of finding least common multiples of pairs of graphs H_1 and H_2; that is graphs of minimum size which are both H_1 and H_2 decomposable. The problem was introduced by Chartrand et.al in [4] and they showed that every two nonempty graphs have a least common multiple. The problem of finding the size of least common multiples of graphs has been studied for several pairs of graphs: cycles and stars [4, 13, 14], paths and complete graphs [9], pairs of cycles [8], pairs of complete graphs [3], complete graphs and a 4-cycle [2], pairs of cubes [1], complete graph and star [11] and paths and stars [7]. Pairs of graphs having a unique least common multiple were investigated by several authors [6, 12, 10]. Least common multiple of digraphs were considered in [5].

An obvious necessary condition for the existence of a graph G which is a common multiple of H_1 and H_2 is that both $|E(H_1)|$ and $|E(H_2)|$ divide $|E(G)|$. This condition is not always sufficient. Therefore, we may ask: Given two graphs H_1 and H_2, for which value of q does there exist a graph G having q edges which is a common multiple of the graphs H_1 and H_2? Adams, Bryant and Maenhaut [2] gave a complete solution to this problem in the case where H_1 is the 4-cycle and H_2 is a complete graph; Bryant and Maenhaut [3] gave a complete solution to this problem in the case where H_1 is the complete graph K_3 and H_2 is a complete graph. Thus the problem to find least common multiple of H_1 and H_2 is to find the least positive integer q such that there exists a graph G having q edges which is both H_1 and H_2 decomposable. We denote the set of all least common multiples of H_1 and H_2 by $LCM(H_1, H_2)$. The size of a least common multiple of H_1 and H_2 is denoted by $lcm(H_1, H_2)$. Since every two nonempty graphs have a least common multiple, $LCM(H_1, H_2)$ is nonempty. The number of elements in the set $LCM(H_1, H_2)$ is greater than one for many pairs of graphs. For example both P_7 and C_6 are least common multiples of P_4 and P_3.

In fact, Chartrand et.al [6] proved that for every positive integer n there exist two graphs having exactly n least common multiples. In [9] it was shown that every least common multiple of two connected graphs is connected and that every least common multiple of two 2-connected graphs is 2-connected. But this is not the case for disconnected graphs. For example if we take $H_1 = 2K_2$, $H_2 = C_5$, then $G_1 = 2C_5$ and G_2 which is the graph obtained by identifying two vertices in two copies of C_5, are in $LCM(H_1, H_2)$ of which G_1 is disconnected while G_2 is connected.

2 Main Result

The cartesian product of two graphs G and H denoted by $G \,\Box\, H$ is a graph with vertex set $V(G) \times V(H)$ for which $\{(x, u), (y, v)\}$ is an edge if $x = y$ and $\{u, v\} \in E(H)$ or $\{x, y\} \in E(G)$ and $u = v$. The graph $G \,\Box\, H$ has $|V(G)||V(H)|$ vertices and $|V(G)||E(H)| + |V(H)||E(G)|$ edges. In this section graphs that belong to $LCM(P_4, C_m \,\Box\, P_n)$, $LCM(P_4, W_m \,\Box\, P_n)$ and $LCM(P_4, W_m \,\Box\, C_n)$ are constructed and hence computed the lcm of the respective pairs of graphs. Let G^t for $t = 1, 2, 3$ denote the t-th copy of the graph G. Also let v^t and e^t denote a vertex and an edge in G^t.

2.1 lcm of P_4 and $C_m \,\Box\, P_n$

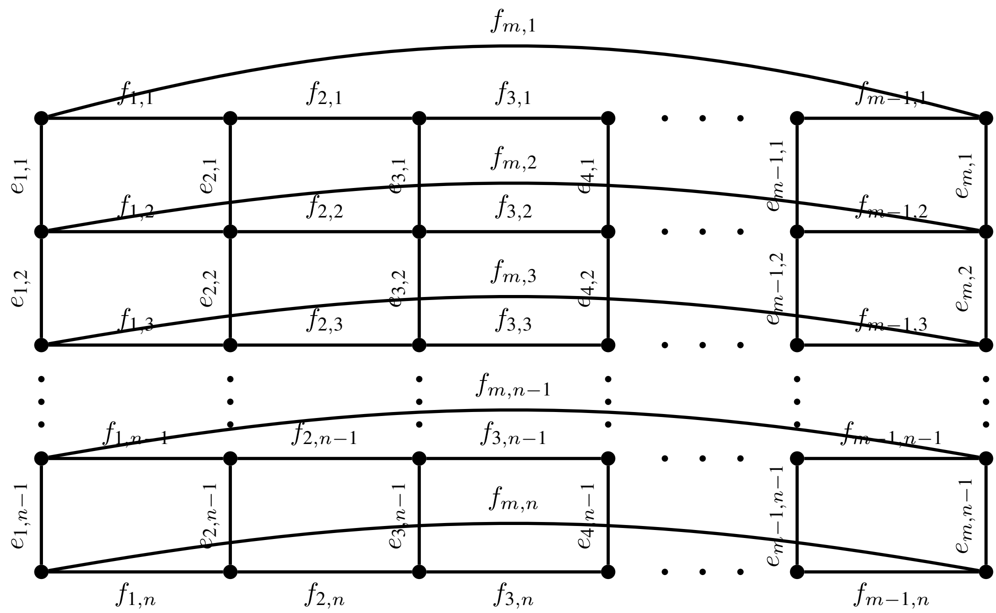

Figure 1. $\mathbf{C_m \,\Box\, P_n}$

Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be the vertices of C_m and P_n respectively. $C_m \times \{b_j\}$, $1 \leq j \leq n$ are the C_m-fibers and $\{a_i\} \times P_n$, $1 \leq i \leq m$ are the P_n-fibers in $C_m \,\Box\, P_n$. Label the vertices and edges of the j-th C_m-fiber, $C_m \times \{b_j\}$ as $\{v_{1,j}, v_{2,j}, \ldots, v_{m,j}\}$, $\{f_{1,j}, f_{2,j}, \ldots, f_{m,j}\}$ and that of the i-th P_n-fiber, $\{a_i\} \times P_n$ as $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$, $\{e_{i,1}, e_{i,2}, \ldots, e_{i,n-1}\}$.

Theorem 2.1. $lcm(P_4, C_m \,\Box\, P_n) = \begin{cases} 2mn - m & \textit{if } m \equiv 0 \pmod 3 \textit{ or } n \equiv 2 \pmod 3 \\ 6mn - 3m & \textit{otherwise} \end{cases}$

Proof. Least common multiple of P_4 and $C_m \,\Box\, P_n$ is the number of edges in the graph of least size that is both P_4-decomposable and $C_m \,\Box\, P_n$-decomposable. We consider various cases for m and n in modulo 3 and will construct in each case a graph of least size that is both P_4-decomposable and $C_m \,\Box\, P_n$-decomposable.

Case 1: $n = 2,\ m \in \mathbb{N},\ m \geq 3$

The graph $G = C_m \,\Box\, P_2$ has $3m$ edges. A P_4-decomposition of G is given by the following copies of P_4: $(f_{i,1}, e_{i,1}, f_{i,2})$, $1 \leq i \leq m$. Thus G is P_4-decomposable and hence

$$lcm(P_4, C_m \,\Box\, P_2) = 3m.$$

Case 2: $m = 3, n \in \mathbb{N},\ n \geq 3$

In this case $G = C_3 \,\square\, P_n$, which has $6n - 3$ edges. A P_4-decomposition of G is obtained as follows:

$$\{(f_{1,j}, e_{2,j}, f_{2,j+1}), 1 \leq j \leq n-1\}, \{(e_{1,j}, f_{3,j}, e_{3,j-1}), 2 \leq j \leq n-1\},$$

$$(e_{1,1}, f_{3,1}, e_{3,1}), \qquad (f_{1,n}, f_{3,n}, e_{3,n-1})$$

Thus G is P_4-decomposable and hence $lcm(P_4, C_3 \,\square\, P_n) = 6n - 3$.

Case 3: $m = 3k,\ k \geq 2$

Subcase 3.1: $n = 3l,\ l \geq 1$

The graph $G = C_{3k} \,\square\, P_{3l}$ has $3k(3l-1) + (3l)(3k)$ edges and hence $|E(G)| \equiv 0 \pmod 3$. The $3l - 1$ edges of the i-th P_n-fiber of G, where $1 \leq i \leq m$, together with the edge $f_{i,n}$ of the n-th C_m-fiber makes a P_{3l+1}, which is P_4-decomposable. For $1 \leq j \leq n-1$, the j-th C_m-fiber contains $3k$ edges and hence it is P_4-decomposable. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k} \,\square\, P_{3l}) = 3k(3l-1) + (3l)(3k)$.

Subcase 3.2: $n = 3l + 1,\ l \geq 1$

In this case $G = C_{3k} \,\square\, P_{3l+1}$ and $|E(G)| = 3k(3l) + (3l+1)(3k) \equiv 0 \pmod 3$. Here each C_m-fiber has $3k$ edges and each P_n-fiber has $3l$ edges and hence every C_m-fiber and P_n-fiber are P_4-decomposable. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k} \,\square\, P_{3l+1}) = 3k(3l) + (3l+1)(3k)$.

Subcase 3.3: $n = 3l + 2,\ l \geq 1$

Here $G = C_{3k} \,\square\, P_{3l+2}$ and it has $3k(3l+1)+(3l+2)(3k)$ edges which is a multiple of three. The j-th C_m-fiber, where $1 \leq j \leq n-2$, has $3k$ edges and hence it is P_4-decomposable. The first $3l$ edges of the i-th P_n-fiber, where $1 \leq i \leq m$ makes a P_{3l+1}, which is P_4-decomposable. Consider the edges of the $(n-1)$-th and n-th C_m-fibers and the edges $\{e_{i,n-1}, 1 \leq i \leq m\}$. Then $\{(f_{i,n-1}, e_{i,n-1}, f_{i,n}),\ 1 \leq i \leq m\}$ gives a copy of P_4 for each i. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k} \,\square\, P_{3l+2}) = 3k(3l+1) + (3l+2)(3k)$.

Case 4: $m = 3k + 1,\ k \geq 1$

Subcase 4.1: $n = 3l,\ l \geq 1$

The graph $G = C_{3k+1} \,\square\, P_{3l}$ has $(3k+1)(3l-1)+(3l)(3k+1)$ edges and hence $|E(G)| \equiv 2 \pmod 3$. The first $3k$ edges of the j-th C_m-fiber, where $1 \leq j \leq n-1$, makes a P_{3k+1}, which is P_4-decomposable. The $3l-1$ edges of the i-th P_n-fiber, where $2 \leq i \leq m-1$, together with the edge $f_{i-1,n}$ of the n-th C_m-fiber makes a P_{3l+1}, which is P_4-decomposable. Now $\{(e_{1,j}, f_{m,j}, e_{m,j}),\ 1 \leq j \leq n-1\}$ gives a copy of P_4 for each j. The edges $\{f_{m-1,n}, f_{m,n}\}$ are left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{1,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m-1,n}$ with the vertex $v^3_{m-1,n}$. The left out edges $\{f^t_{m-1,n}, f^t_{m,n}; t = 1,2,3\}$ in the three copies of G will make a P_7 in H, which is P_4-decomposable. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+1} \,\square\, P_{3l}) = 3((3k+1)(3l-1) + (3l)(3k+1))$.

Subcase 4.2: $n = 3l + 1,\ l \geq 1$

In this case $G = C_{3k+1} \,\square\, P_{3l+1}$ which has $(3k+1)(3l) + (3l+1)(3k+1)$ edges and hence $|E(G)| \equiv 1 \pmod 3$. The first $3k$ edges of the j-th C_m-fiber, where $1 \leq j \leq n$, makes a P_{3k+1}, which is P_4-decomposable. For $2 \leq i \leq m-1$, the i-th P_n-fiber, has $3l$ edges and hence it is P_4-decomposable. Now $\{(e_{1,j}, f_{m,j}, e_{m,j}),\ 1 \leq j \leq n-1\}$ gives a copy of P_4 for each j. The edge $f_{m,n}$ is left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{m,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m,n}$ with the vertex $v^3_{1,n}$. The left out edges $\{f^t_{m,n}; t = 1,2,3\}$ in the three copies of G will make a P_4 in H. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+1} \,\square\, P_{3l+1}) = 3((3k+1)(3l) + (3l+1)(3k+1))$.

Subcase 4.3: $n = 3l + 2,\ l \geq 1$

Here $G = C_{3k+1} \,\square\, P_{3l+2}$ and $|E(G)| = (3k+1)(3l+1)+(3l+2)(3k+1)$, which is a multiple of three. The first $3k$ edges of the j-th C_m-fiber, where $1 \leq j \leq n-2$, makes a P_{3k+1}, which is P_4-decomposable. The first $3l$ edges of the i-th P_n-fiber, where $2 \leq i \leq m-1$ makes a P_{3l+1}, which is P_4-decomposable. $\{(e_{1,j}, f_{m,j}, e_{m,j}),\ 1 \leq j \leq n-1\}$ gives a copy of P_4 for each j. Consider the edges of the $(n-1)$-th and n-th C_m-fibers and the edges $\{e_{i,n-1}, 1 \leq i \leq m\}$. Then

$\{(f_{i,n-1}, e_{i,n-1}, f_{i,n}),\ 1 \leq i \leq m\}$ gives a copy of P_4 for each i. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k+1} \square P_{3l+2}) = (3k+1)(3l+1) + (3l+2)(3k+1)$.

Case 5: $m = 3k+2,\ k \geq 1$

Subcase 5.1: $n = 3l,\ l \geq 1$

For the graph $G = C_{3k+2} \square P_{3l}$, $|E(G)| = (3k+2)(3l-1) + (3l)(3k+2) \equiv 1 \pmod 3$. The $3k+2$ edges of the j-th C_m-fiber, where $1 \leq j \leq n-1$, together with the edge $e_{m,j}$ of the m-th P_n-fiber, makes $3k+3$ edges, which is P_4-decomposable. The $3l-1$ edges of the i-th P_n-fiber, where $1 \leq i \leq m-1$, together with the edge $f_{i,n}$ of the n-th C_m-fiber makes a P_{3l+1}, which is P_4-decomposable. The edge $f_{m,n}$ is left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{m,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m,n}$ with the vertex $v^3_{1,n}$. The left out edges $\{f^t_{m,n}; t = 1, 2, 3\}$ in the three copies of G will make a P_4 in H. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+2} \square P_{3l}) = 3((3k+2)(3l-1) + (3l)(3k+2))$.

Subcase 5.2: $n = 3l+1,\ l \geq 1$

In this case $G = C_{3k+2} \square P_{3l+1}$ which has $(3k+2)(3l) + (3l+1)(3k+2)$ edges and hence $|E(G)| \equiv 2 \pmod 3$. The $3k+2$ edges of the j-th C_m-fiber, where $1 \leq j \leq n-1$, together with the edge $e_{m,j}$ of the m-th P_n-fiber, makes $3k+3$ edges, which is P_4-decomposable. For $1 \leq i \leq m-1$, the i-th P_n-fiber, has $3l$ edges and hence it is P_4-decomposable. The first $3k$ edges of the n-th C_m-fiber makes a P_{3k+1}, which is P_4-decomposable. The edges $\{f_{m-1,n}, f_{m,n}\}$ are left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{1,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m-1,n}$ with the vertex $v^3_{m-1,n}$. The left out edges $\{f^t_{m-1,n}, f^t_{m,n}; t = 1, 2, 3\}$ in the three copies of G will make a P_7 in H, which is P_4-decomposable. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+2} \square P_{3l+1}) = 3((3k+2)(3l) + (3l+1)(3k+2))$.

Subcase 5.3: $n = 3l+2,\ l \geq 1$

The graph $G = C_{3k+1} \square P_{3l+2}$ has $(3k+2)(3l+1) + (3l+2)(3k+2)$ edges, which is a multiple of three. The $3k+2$ edges of the j-th C_m-fiber, where $1 \leq j \leq n-2$, together with the edge $e_{m,j}$ of the m-th P_n-fiber, makes $3k+3$ edges, which is P_4-decomposable. The first $3l$ edges of the i-th P_n-fiber, where $1 \leq i \leq m-1$ makes a P_{3l+1}, which is P_4-decomposable. Consider the edges of the $(n-1)$-th and n-th C_m-fibers and the edges $\{e_{i,n-1}, 1 \leq i \leq m\}$. Then $\{(f_{i,n-1}, e_{i,n-1}, f_{i,n}),\ 1 \leq i \leq m\}$ gives a copy of P_4 for each i. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k+2} \square P_{3l+2}) = (3k+2)(3l+1) + (3l+2)(3k+2)$. □

Theorem 2.2. *$C_m \square P_n$ is P_4-decomposable if and only if $m \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$.*

2.2 lcm of P_4 and $W_m \square P_n$

Let W_m denote the wheel graph of order m, which contains a cycle C_{m-1} and a vertex called hub, which is adjacent to every vertex of C_{m-1}. $|E(W_m)| = 2m-2$. Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be the vertices of W_m and P_n respectively, where a_m is the hub vertex of W_m. $W_m \times \{b_j\}$, $1 \leq j \leq n$ are the W_m-fibers and $\{a_i\} \times P_n$, $1 \leq i \leq m$ are the P_n-fibers in $W_m \square P_n$. Label the vertices and edges of the j-th W_m-fiber, $W_m \times \{b_j\}$ as $\{v_{1,j}, v_{2,j}, \ldots, v_{m,j}\}$, $\{f_{1,j}, f_{2,j}, \ldots, f_{m-1,j}, g_{1,j}, g_{2,j}, \ldots, g_{m-1,j}\}$ where $\{f_{1,j}, f_{2,j}, \ldots, f_{m-1,j}\}$ are the edges of the cycle in the j-th W_m-fiber and $\{g_{1,j}, g_{2,j}, \ldots, g_{m-1,j}\}$ are the edges connecting the hub and the vertices of the cycle in the j-th W_m-fiber. The vertices and edges of the i-th P_n-fiber, $\{a_i\} \times P_n$ are labelled as $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$ and $\{e_{i,1}, e_{i,2}, \ldots, e_{i,n-1}\}$ respectively.

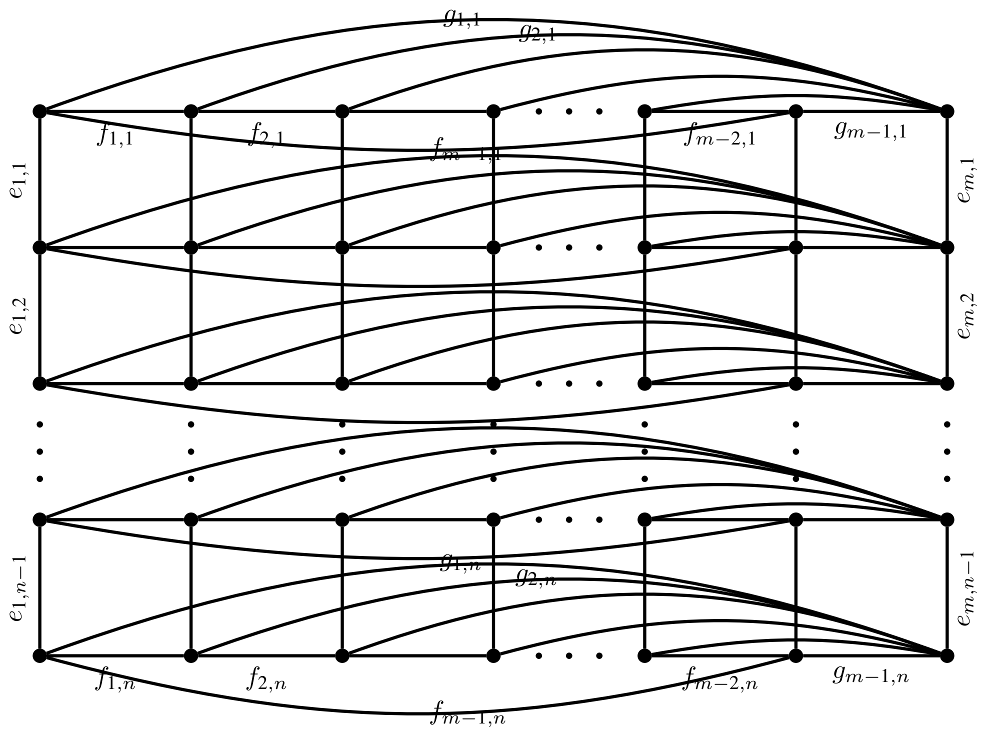

Figure 2. $\mathbf{W_m \,\Box\, P_n}$

Theorem 2.3. $lcm(P_4, W_m \,\Box\, P_n) = \begin{cases} 3mn - 2n - m & \text{if } 2m + n \equiv 0 \pmod 3 \\ 3(3mn - 2n - m) & \text{otherwise} \end{cases}$

Proof. Let P' be the path $v_{1,1}f_{1,1}v_{2,1}f_{2,1}\dots f_{m-2,1}v_{m-1,1}g_{m-1,1}v_{m,1}$, which is contained in the first W_m-fiber, P'' : $v_{m,1}e_{m,1}v_{m,2}e_{m,2}\dots v_{m,n-1}e_{m,n-1}v_{m,n}$, the m-th P_n-fiber and P''' : $v_{m,n}g_{m-1,n}v_{m-1,n}f_{m-2,n}\dots v_{2,n}f_{1,n}v_{1,n}$, the path contained in the last W_m-fiber.

Let $G = W_m \,\Box\, P_n$. Then $|E(G)| = m(n-1) + n(2m-2) = 3mn - 2n - m$. Consider the edges of $G^* = (W_m \,\Box\, P_n)\backslash\{P', P'', P'''\}$. Copies of P_4 are obtained as follows :
For a fixed j, $1 \le j \le n-2$, $\{(g_{i,j}, e_{i,j}, f_{i,j+1}),\ 1 \le i \le m-2\}$, $\{(f_{m-1,j}, e_{m-1,j}, g_{m-1,j+1})\}$,

$$\{(g_{i,n-1}, e_{i,n-1}, g_{i,n}),\ 1 \le i \le m-2\}, (f_{m-1,n-1}, e_{m-1,n-1}, f_{m-1,n}).$$

Thus G^* is P_4-decomposable. The paths P', P'' and P''' makes the path P^* of length $2m+n-3$ in $W_m \,\Box\, P_n$. Thus $W_m \,\Box\, P_n$ is P_4-decomposable if P^* is P_4-decomposable and this happens if $2m + n \equiv 0 \pmod 3$.

If $2m + n \equiv 1$ *or* $2 \pmod 3$, take three copies of G namely G^1, G^2, G^3 and in each copy of G, the subgraph G^* has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{1,1}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{1,1}$ with the vertex $v^3_{1,1}$. Then the path P^* in the three copies of G will make a path of length $3(2m+n-3)$ in H, which is P_4-decomposable and so is H. Thus $lcm(P_4, W_m \,\Box\, P_n) = |E(W_m \,\Box\, P_n)|$ if $2m+n \equiv 0 \pmod 3$ and $3|E(W_m \,\Box\, P_n)|$ otherwise. □

Theorem 2.4. *$W_m \,\Box\, P_n$ is P_4-decomposable if and only if* $2m + n \equiv 0 \pmod 3$.

2.3 lcm of P_4 and $W_m \,\Box\, C_n$

Let $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_n$ be the vertices of W_m and C_n respectively, where a_m is the hub vertex of W_m. $W_m \times \{b_j\}$, $1 \le j \le n$ are the W_m-fibers and $\{a_i\} \times C_n,\ 1 \le i \le m$ are the C_n-fibers in $W_m \,\Box\, C_n$. Label the vertices and edges of the j-th W_m-fiber, $W_m \times \{b_j\}$ as in the above case of $W_m \square P_n$. The vertices and edges of the i-th C_n-fiber, $\{a_i\} \times C_n$ are labelled as $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$, $\{e_{i,1}, e_{i,2}, \ldots, e_{i,n}\}$.

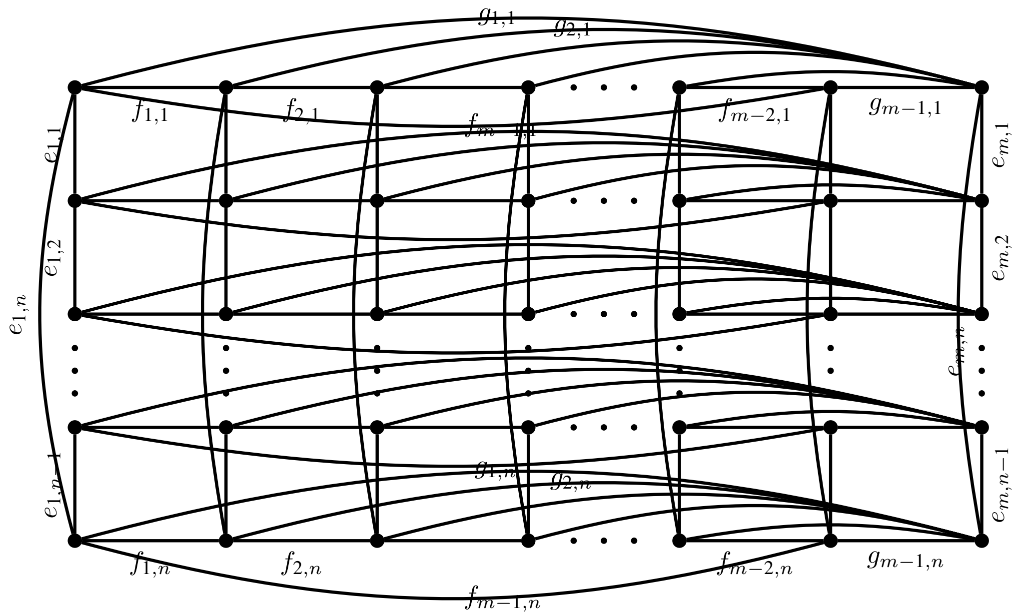

Figure 3. $\mathbf{W_m \square C_n}$

Theorem 2.5. $lcm(P_4, W_m \square C_n) = \begin{cases} 3mn - 2n & \text{if } n \equiv 0 \pmod 3 \\ 3(3mn - 2n) & \text{otherwise} \end{cases}$

Proof. Let $G = W_m \square C_n$. Then $|E(G)| = mn + n(2m-2) = 3mn - 2n$. Copies of P_4 are obtained as follows :
For a fixed j, $2 \le j \le n-2$, $\{(g_{i,j}, e_{i,j}, f_{i,j+1}),\ 1 \le i \le m-2\}$, $\{(f_{m-1,j}, e_{m-1,j}, g_{m-1,j+1})\}$,

$$\{(g_{i,1}, e_{i,n}, f_{i,n}), (f_{i,1}, e_{i,1}, f_{i,2}), (g_{i,n-1}, e_{i,n-1}, g_{i,n}); 1 \le i \le m-2\},$$

$$(f_{m-1,1}, e_{m-1,1}, g_{m-1,2}), (f_{m-1,n-1}, e_{m-1,n-1}, f_{m-1,n}), (e_{m-1,n}, g_{m-1,1}, e_{m,n})$$

The path P^* of length n consisting of the edges $\{e_{m,1}, e_{m,2}, \ldots, e_{m,n-1}, g_{m-1,n}\}$ is left out. Thus $W_m \square C_n$ is P_4-decomposable if P^* is P_4-decomposable and this happens if $n \equiv 0 \pmod 3$.

If $n \equiv 1$ *or* $2 \pmod 3$, take three copies of G namely G^1, G^2, G^3 having the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{m,1}$ with the vertex $v^2_{m,1}$ and the vertex $v^2_{m-1,n}$ with the vertex $v^3_{m-1,n}$. Then the path P^* in the three copies of G will make a path of length $3n$ in H, which is P_4-decomposable and so is H. Thus $lcm(P_4, W_m \square C_n) = |E(W_m \square C_n)|$ if $n \equiv 0 \pmod 3$ and $3|E(W_m \square C_n)|$ otherwise. □

Theorem 2.6. *$W_m \square C_n$ is P_4-decomposable if and only if $n \equiv 0 \pmod 3$.*

Acknowledgement

The authors are thankful to Prof. M. I. Jinnah, formerly University of Kerala, India for his support and for suggesting this topic.

References

[1] P.Adams, D. Bryant, S.I. El-Zanati, C.Vanden Eynden and B. Maenhaut, Least common multiples of cubes, *Bull. Inst. Combin. Appl.*, **38** (2003) 45-49.

[2] P. Adams, D. Bryant and B. Maenhaut, Common multiples of complete graphs and a 4-cycle, *Discrete Math.* **275** (2004) 289-297; doi.org/10.1016/j.disc.2002.11.001

[3] D. Bryant and B. Maenhaut, Common multiples of complete graphs, *Proc. London Math. Soc.* **86**(2) (2003) 302-326; doi.org/10.1112/S0024611502013771

[4] G. Chartrand, L. Holley, G. Kubicki and M. Schultz, Greatest common divisors and least common multiples of graphs , *Period. Math. Hungar* **27**(2) (1993) 95-104; doi.org/10.1007/bf01876635

[5] G. Chartrand and F. Saba, On least common multiple of digraphs, *Util. Math.* **49** (1996) 45-63.

[6] G. Chartrand, G. Kubicki, C.M. Mynhardt and F. Saba, On graphs with a unique least common multiple, *Ars Combin.* **46** (1997) 177-190.

[7] Z-C Chen and T-W Shyu, Common multiples of paths and stars, *Ars Combin.* **146** (2019) 115-122; hdl.handle.net/11536/152715

[8] O.Favaron and C.M. Mynhardt, On the sizes of least common multiples of several pairs of graphs, *Ars Combin.* **43** (1996) 181-190.

[9] C.M. Mynhardt and F. Saba, On the sizes of least common multiples of paths versus complete graphs, *Util. Math.* **46** (1994) 117-128.

[10] T. Reji, On graphs that have a unique least common multiple with matchings, *Far East J. Appl. Math.* **18**(3) (2005) 281-288.

[11] T. Reji and J. Varughese. Least common multiple of graphs. *Discrete Math. Algorithms Appl.* **8**(2) (2016) 1650032 1-8; doi.org/10.1142/S1793830916500324.

[12] T Reji, J. Varughese and R Ruby. On graphs that have a unique least common multiple. *Cubo* **24** (1) (2022) 53-62; doi.org/10.4067/S0719-06462022000100053.

[13] C. Sunil Kumar, Least common multiple of a cycle and a star, *Electron. Notes Discrete Math.* **15** (2003) 204-206; doi.org/10.1016/S1571-0653(04)00581-5

[14] P. Wang, On the sizes of least common multiples of stars versus cycles, *Util. Math.* **53** (1998) 231-242.

Author information

Reji T, Ruby R and Sneha B, Department of Mathematics, Government College, Chittur, Palakkad, Kerala-678104, India.
E-mail: `rejiaran@gmail.com`, `rubymathpkd@gmail.com` (Corresponding author), `sneharbkrishnan@gmail.com`