Palestine Journal of Mathematics

Vol. 12(Special Issue II)(2023) , 1–6

$K_n(\lambda)$ IS FULLY $\{P_4, C_6\}$-DECOMPOSABLE

R. Chinnavedi and R. Sangeetha

Communicated by A. Muthusamy

MSC 2010 Classifications: 05C38, 05C70.

Keywords and phrases: Decomposition, Complete multigraph, Path, Cycle.

Abstract Let P_{k+1} denote a path of length k, C_ℓ denote a cycle of length ℓ, and $K_n(\lambda)$ denote the complete multigraph on n vertices in which every pair of distinct vertices is joined by λ edges. In this paper, we proved that the necessary conditions are also sufficient for a $\{P_4, C_6\}$-decomposition of $K_n(\lambda)$.

1 Introduction

All graphs considered here are finite and undirected with no loops. For the standard graph-theoretic terminology the reader is referred to [2]. A simple graph in which every pair of distinct vertices is joined by an edge is called a *complete graph*, denoted by K_n. If more than one edge joining two vertices are allowed, the resulting object is called a *multigraph*. Let $K_n(\lambda)$ denote the *complete multigraph* on n vertices in which every pair of distinct vertices is joined by λ edges. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = a$ and $|Y| = b$, such a graph is denoted by $K_{a,b}$. In $K_{a,b}(\lambda)$, we label the vertices in the partite set X as $\{x_1, x_2, \dots, x_a\}$ and Y as $\{x_{a+1}, x_{a+2}, \dots, x_{a+b}\}$. A *cycle* is a closed trail with no repeated vertex other than the first and last vertex. A cycle with ℓ edges is denoted by C_ℓ. A *path* is an open trail with no repeated vertex. A path with k edges is denoted by P_{k+1}. The complete bipartite graph $K_{1,m}$ is called a *star* and is denoted by S_m. For $m \geq 3$, the vertex of degree m in S_m is called the *center* and any vertex of degree 1 in S_m is called an *end vertex*.

Let G be a graph and let G_1 be a subgraph of G. Then $G \backslash G_1$ is obtained from G by deleting the edges of G_1. Let G_1 and G_2 be subgraphs of G. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We say that G_1 and G_2 are *edge-disjoint* if they have no edge in common. If G_1 and G_2 are edge-disjoint, we denote their union by $G_1 + G_2$. A *decomposition* of a graph G is a collection of edge-disjoint subgraphs $G_1, G_2, \dots, G_n$ of G such that every edge of G is in exactly one G_i. Here it is said that G is *decomposed* or *decomposable* into $G_1, G_2, \dots, G_n$. If G has a decomposition into p_1 copies of $G_1, \dots, p_n$ copies of G_n, then we say that G has a $\{p_1 G_1, \dots, p_n G_n\}$-decomposition. If such a decomposition exists for all values of $p_1, \dots, p_n$ satisfying necessary conditions, then we say that G has a $\{G_1, \dots, G_n\}_{\{p_1, \dots, p_n\}}$-decomposition or G is fully $\{G_1, \dots, G_n\}$-decomposable.

In [8], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of $\{pG_1, qG_2\}$-decomposition of $K_n(\lambda)$, when $(G_1, G_2) \in \{(P_n, S_{n-1}), (C_n, S_{n-1}), (P_n, C_n)\}$. In [10], Shyu gave the necessary conditions for a $\{pP_{k+1}, qC_\ell\}$-decomposition of K_n and proved that K_n is fully $\{P_{k+1}, C_k\}$-decomposable, when k is even, n is odd, $n \geq 5k + 1$ and settled the case $k = 4$ completely. In [11], Shyu proved that K_n is fully $\{P_4, C_3\}$-decomposable. In [7], Jeevadoss and Muthusamy proved that K_n is fully $\{P_{k+1}, C_k\}$-decomposable, when k is even and n is odd with $n > 4k$. In [6], Ilayaraja and Muthusamy proved that K_n is fully $\{P_4, C_4\}$-decomposable. In [4], the authors proved that $K_n(\lambda)$ is fully $\{P_4, C_4\}$-decomposable. In [9], Sarvate and Zhang obtained necessary and sufficient conditions for the existence of a $\{pP_3, qC_3\}$-decomposition of $K_n(\lambda)$, when $p = q$. In [12], Shyu gave the necessary conditions for a $\{pC_k, qP_{k+1}, rS_k\}$-decomposition of K_n and proved that K_n is fully $\{C_4, P_5, S_4\}$-decomposable, when n is odd. In [3], the authors gave the necessary conditions for a $\{pP_{k+1}, qC_\ell\}$-decomposition of $K_n(\lambda)$ and proved that $K_n(\lambda)$ is fully $\{P_5, C_6\}$-decomposable. In this paper we prove that $K_n(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

2 Preliminaries

For convenience we denote $V(K_n(\lambda)) = \{x_1, x_2, \ldots, x_n\}$. The notation $(x_1 x_2 \cdots x_\ell)$ denotes a cycle with vertices $x_1, x_2, \ldots, x_\ell$ and edges $x_1x_2, x_2x_3, \ldots, x_{\ell-1}x_\ell, x_\ell x_1$, and $[x_1 x_2 \cdots x_{k+1}]$ is a path with vertices $x_1, x_2, \ldots, x_{k+1}$ and edges $x_1x_2, x_2x_3, \ldots, x_kx_{k+1}$.

We recall here some results on P_{k+1} and C_ℓ-decompositions that are useful for our proofs.

Theorem 2.1. (**Bryant***et al.*[1]) *Let λ, n and ℓ be integers with $n, \ell \geq 3$ and $\lambda \geq 1$. There exists a decomposition of $K_n(\lambda)$ into C_ℓ if and only if $\ell \leq n$, $\lambda(n-1)$ is even and ℓ divides $\lambda\binom{n}{2}$. There exists a decomposition of $K_n(\lambda)$ into C_ℓ and a perfect matching if and only if $\ell \leq n$, $\lambda(n-1)$ is odd and ℓ divides $\lambda\binom{n}{2} - \frac{n}{2}$.*

Theorem 2.2. (**Tarsi** [13]) *A necessary and sufficient conditions for the existence of a P_{k+1}-decomposition of $K_n(\lambda)$ is $\lambda\binom{n}{2} \equiv 0 \pmod{k}$ and $n \geq k+1$.*

Theorem 2.3. (**Hung-Chih Lee**[5]) *For positive integers λ, a, b and ℓ with $\lambda a \equiv \lambda b \equiv \ell \equiv 0 \pmod{2}$ and $min\{a, b\} \geq \frac{\ell}{2} \geq 2$, the multigraph $K_{a,b}(\lambda)$ is C_ℓ-decomposable if one of the following conditions holds: (i) λ is odd and ℓ divides ab, (ii) λ is even and ℓ divides $2ab$ (iii) λ is even and λa or λb is divisible by ℓ.*

Theorem 2.4. (**Truszczynski**[14]) *Let k be a positive integer and let a and b be positive even integers such that $a \geq b$. $K_{a,b}(\lambda)$ has a P_{k+1}-decomposition if and only if $a \geq \lceil \frac{k+1}{2} \rceil, b \geq \lceil \frac{k}{2} \rceil$ and $\lambda ab \equiv 0 \pmod{k}$.*

In [3], the authors discussed the necessary conditions for a $\{pP_{k+1}, qC_\ell\}$-decomposition of $K_n(\lambda)$ when $\lambda \geq 1$, which is as follows:

Theorem 2.5. (**Chinnavedi***et al.* [3]) *Let λ, n, k and ℓ be positive integers such that $n \geq \max\{\ell, k+1\}$. If $K_n(\lambda)$ can be decomposed into p copies of P_{k+1} and q copies of C_ℓ for nonnegative integers p and q, then (i) $pk + q\ell = \lambda\binom{n}{2}$ (ii) $p \neq 1$ if n is odd or n and λ are both even and (iii) $p \geq \frac{n}{2}$ if λ is odd and n is even.*

We prove that the above necessary conditions are sufficient for $k = 3$ and $\ell = 6$ in Theorem 3.5.

3 Main Result

In this section, we discuss a $\{P_4, C_6\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$, when $\lambda \geq 1$. By Theorem 2.5, it is enough to consider the $\{pP_4, qC_6\}$-decomposition of $K_n(\lambda)$, for $n \geq 6$.

Remark 3.1. Necessary conditions for the existence of a $\{P_4, C_6\}_{\{p,q\}}$-decomposition in $K_n(\lambda)$ are satisfied when $n \equiv 0, 1, 3, 4 \pmod{6}$ if $\lambda \geq 1$ and $n \equiv 2, 5 \pmod{6}$ if $\lambda \equiv 0, 3 \pmod{6}$, (or $\lambda \equiv 0 \pmod{3}$). i.e., there does not exist nonnegative integers p and q satisfying $3p + 6q = \lambda\binom{n}{2}$ when $n \equiv 2, 5 \pmod{6}$ if $\lambda \equiv 1, 2, 4, 5 \pmod{6}$.

Remark 3.2. If G is C_ℓ-decomposable and $\ell \equiv 0 \pmod{k}$, then G is fully $\{P_{k+1}, C_\ell\}$- decomposable as each C_ℓ can be decomposed into $\frac{\ell}{k}$ copies of P_{k+1}.

The proof of the following two lemmas are immediate from Theorem 2.3 and Remark 3.2.

Lemma 3.3. *If p and q are nonnegative integers such that $3p + 6q = 24$, then $K_{6,4}$ is fully $\{P_4, C_6\}$-decomposable.*

Lemma 3.4. *If p and q are nonnegative integers such that $3p + 6q = 36$, then $K_{6,6}$ is fully $\{P_4, C_6\}$-decomposable.*

We now prove our main result.

Theorem 3.5. *For any nonnegative integers p and q and any integer $n \geq 6$, there exists a $\{P_4, C_6\}_{\{p,q\}}$-decomposition of $K_n(\lambda)$ if and only if (i) $3p + 6q = \lambda\binom{n}{2}$, (ii) $p \geq \frac{n}{2}$, if λ is odd and n is even (iii) $p \neq 1$ if otherwise.*

Proof. The necessary part follows from Theorem 2.5. By Theorem 2.1 and Remark 3.2, $K_n(\lambda)$ is fully $\{P_4, C_6\}$-decomposable if $\lambda\binom{n}{2} \equiv 0 \pmod 6$, n is odd or n and λ are both even. First we prove the result for $6 \le n \le 17$ with λ is odd; then we use induction to settle the remaining cases. As we discuss $\{pP_4, qC_6\}$-decompositions of $K_n(\lambda)$ for all possible choices of p and q, we have the following cases:

Case 1: $n = 6$.

If $\lambda = 1$, then $(p, q) \in \{(3, 1), (5, 0)\}$. The graph K_6 can be decomposed into $3P_4$: $[x_2x_5x_1x_3]$, $[x_1x_4x_2x_6]$, $[x_4x_6x_3x_5]$ and a C_6 : $(x_1x_2x_3x_4x_5x_6)$. By Theorem 2.2, K_6 is $\{5P_4, 0C_6\}$-decomposable.

If $\lambda \ge 3$, then we write $K_6(\lambda) = K_6(\lambda-1) + K_6 = \frac{\lambda-1}{2}K_6(2) + K_6$. By Theorem 2.1 and Remark 3.2, the graph $K_6(2)$ is $\{P_4, C_6\}_{\{p,q\}}$-decomposable. Therefore $K_6(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Case 2: $n = 7$.

If $\lambda = 1$, then $(p, q) \in \{(3, 2), (5, 1), (7, 0)\}$. The graph K_7 can be decomposed into $3P_4$: $[x_3x_7x_4x_6], [x_3x_1x_7x_2]$, $[x_2x_5x_7x_6]$ and $2C_6$: $(x_1x_2x_3x_4x_5x_6)$, $(x_1x_4x_2x_6x_3x_5)$. For $(p, q) \in \{(5, 1), (7, 0)\}$, a $\{pP_4, qC_6\}$-decomposition of K_7 follows easily from a $\{3P_4, 2C_6\}$-decomposition of K_7 as C_6 can be decomposed into 2 copies of P_4. By Theorem 2.1 and Remark 3.2, the graph $K_7(2)$ is fully $\{P_4, C_6\}$-decomposable.

If $\lambda \ge 3$, then we write $K_7(\lambda) = K_7(\lambda-1) + K_7 = \frac{\lambda-1}{2}K_7(2) + K_7$. Therefore $K_7(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Case 3: $n = 8$.

If $\lambda = 3$, then $(p, q) \in \{(4, 12), (6, 11), (8, 10), \ldots, (28, 0)\}$ (we see that the values of p increases by 2 and the values of q decreases by 1). The graph $K_8(3)$ can be decomposed into $4P_4$: $[x_2x_3x_5x_6], [x_7x_3x_8x_4], [x_8x_4x_2x_3], [x_5x_6x_7x_1]$ and $12C_6$: $(x_7x_1x_3x_2x_8x_5), (x_7x_1x_5x_6x_8x_4)$, 2 copies of $(x_8x_2x_4x_7x_3x_5), (x_8x_3x_1x_5x_7x_6)$, 3 copies of $(x_1x_8x_7x_2x_6x_4), (x_1x_2x_5x_4x_3x_6)$. For $(p, q) \in \{(6, 11), (8, 10), (10, 9), \ldots, (28, 0)\}$, a $\{pP_4, qC_6\}$-decomposition of $K_8(3)$ follows easily from a $\{4P_4, 12C_6\}$-decomposition of $K_8(3)$ as C_6 can be decomposed into 2 copies of P_4. By Theorem 2.1 and Remark 3.2, the graph $K_8(6)$ is fully $\{P_4, C_6\}$-decomposable.

If $\lambda \ge 9$, then we write $K_8(\lambda) = K_8(\lambda-3) + K_8(3) = \frac{\lambda-3}{6}K_8(6) + K_8(3)$. Therefore $K_8(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Case 4: $n = 9$.

By Theorem 2.1 and Remark 3.2, the graph $K_9(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Case 5: $n = 10$.

If $\lambda = 1$, then $(p, q) \in \{(5, 5), (7, 4), (9, 3), \ldots, (15, 0)\}$. We write $K_{10} = (K_{10}\backslash K_6) + K_6$. The graph $K_{10}\backslash K_6$ can be decomposed into $2P_4$: $[x_{10}x_7x_8x_9]$, $[x_8x_6x_9x_7]$ and $4C_6$: $(x_{10}x_1x_9x_2x_8x_3)$, $(x_{10}x_2x_7x_1x_8x_4)$, $(x_{10}x_5x_9x_3x_7x_6)$, $(x_{10}x_9x_4x_7x_5x_8)$. By combining these copies of P_4 and C_6 along with the copies of P_4 and C_6 in K_6, we get the decompositions $(p, q) \in \{(5, 5), (7, 4)\}$. For $(p, q) \in \{(9, 3), (11, 2), (13, 1), (15, 0)\}$, a $\{pP_4, qC_6\}$-decomposition of K_{10} follows easily from a $\{7P_4, 4C_6\}$-decomposition of K_{10} as C_6 can be decomposed into 2 copies of P_4. By Theorem 2.1 and Remark 3.2, the graph $K_{10}(2)$ is fully $\{P_4, C_6\}$-decomposable.

If $\lambda \ge 3$, then we write $K_{10}(\lambda) = K_{10}(\lambda-1) + K_{10} = \frac{\lambda-1}{2}K_{10}(2) + K_{10}$. Therefore $K_{10}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Case 6: $n = 11$.

If $\lambda = 3$, then $(p, q) \in \{(3, 26), (5, 25), (7, 24), \ldots, (55, 0)\}$. We write $K_{11}(3) = (K_{11}(3)\backslash K_7(3)) + K_7(3)$. The graph $K_{11}(3)\backslash K_7(3)$ can be decomposed into $17C_6$: $(x_8x_9x_5x_{10}x_{11}x_2), (x_8x_{10}x_4x_{11}x_9x_7), (x_{11}x_9x_4x_8x_{10}x_1), (x_3x_9x_1x_8x_{10}x_{11}), (x_8x_9x_{11}x_6x_{10}x_3)$, $(x_8x_5x_{11}x_{10}x_2x_9)$, 3 copies $(x_{11}x_8x_6x_9x_{10}x_7)$, 2 copies of $(x_{11}x_1x_{10}x_2x_9x_3), (x_8x_4x_{10}x_5x_9x_7)$, $(x_{11}x_2x_8x_1x_9x_4), (x_{10}x_3x_8x_5x_{11}x_6)$. By combining these copies of P_4 and C_6 along with the copies of P_4 and C_6 in $K_7(3)$, we get the decompositions $(p, q) \in \{(3, 26), (5, 25), (7, 24)$, $(9, 23), (11, 22), (13, 21), (15, 20), (17, 19), (19, 18), (21, 17)\}$. For $(p, q) \in \{(23, 16), (25, 15)$, $(27, 14), \ldots, (55, 0)\}$, a $\{pP_4, qC_6\}$-decomposition of $K_{11}(3)$ follows easily from a $\{21P_4, 17C_6\}$ -decomposition of $K_{11}(3)$ as C_6 can be decomposed into 2 copies of P_4. By Theorem 2.1 and Remark 3.2, the graph $K_{11}(6)$ is fully $\{P_4, C_6\}$-decomposable.

If $\lambda \ge 9$, then we write $K_{11}(\lambda) = K_{11}(\lambda-3) + K_{11}(3) = \frac{\lambda-3}{6}K_{11}(6) + K_{11}(3)$. Therefore $K_{11}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Case 7: $n = 12$.

By taking $K_{12}(\lambda) = 2K_6(\lambda) + \lambda K_{6,6}$, we get all the possible decompositions.

Case 8: $n = 13$.

By Theorem 2.1 and Remark 3.2, the graph $K_{13}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Case 9: $n = 14$.

By taking $K_{14}(\lambda) = K_8(\lambda) + K_6(\lambda) + 2\lambda K_{4,6}$, we get all the possible decompositions.

Case 10: $n = 15$.

By taking $K_{15}(\lambda) = K_9(\lambda) + K_7(\lambda) + 2\lambda K_{4,6}$, we get all the possible decompositions.

Case 11: $n = 16$.

By taking $K_{16}(\lambda) = K_{10}(\lambda)+K_6(\lambda)+\lambda K_{4,6}+\lambda K_{6,6}$, we get all the possible decompositions.

Case 12: $n = 17$.

By Theorem 2.1 and Remark 3.2, the graph $K_{17}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

Now we prove the result for $n > 17$. We apply by mathematical induction on n and split the proof into six cases as follows.

$n \equiv 0 \pmod 6$. Let $n = 6r$, with $r \geq 3$. Assume that $K_{6t}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable if $1 \leq t < r$. Write $K_{6r}(\lambda) = K_{6(r-2)}(\lambda) + K_{12}(\lambda) + K_{6(r-2),12}(\lambda) = K_{6(r-2)}(\lambda) + K_{12}(\lambda) + (r-2)K_{6,12}(\lambda) = K_{6(r-2)}(\lambda) + K_{12}(\lambda) + (3r-6)\lambda K_{6,4}$. Suppose the nonnegative integers p and q satisfy the obvious necessary conditions for a $\{pP_4, qC_6\}$-decomposition in $K_{6r}(\lambda)$. Then we have $3p + 6q = \frac{\lambda(6r)\times(6r-1)}{2} = \frac{\lambda}{2}(36r^2 - 6r) = \lambda(18r^2 - 3r) = 18\lambda r^2 - 3\lambda r = 18\lambda r^2 - 3\lambda r + 144\lambda - 144\lambda = 18\lambda r^2 - 75\lambda r + 78\lambda + 66\lambda + 72\lambda r - 144\lambda = \lambda(18r^2 - 75r + 78) + 66\lambda + 72\lambda r - 144\lambda = \frac{\lambda}{2}(36r^2 - 150r + 156) + 66\lambda + 72\lambda r - 144\lambda = \frac{\lambda}{2}(36r^2 - 78r - 72r + 156) + 66\lambda + 72\lambda r - 144\lambda = \frac{\lambda}{2}(6r - 12) \times (6r - 13) + 66\lambda + 72\lambda r - 144\lambda = \frac{\lambda}{2}(6r-12)\times(6r-12-1)+66\lambda+72\lambda r-144\lambda = \frac{\lambda}{2}(6(r-2)\times 6(r-2)-1)+66\lambda+24\lambda(3r-6) = \frac{\lambda}{2}(6(r-2) \times 6(r-2) - 1) + \frac{132\lambda}{2} + 4 \times 6\lambda(3r-6) = \frac{\lambda}{2}(6(r-2) \times 6(r-2) - 1) + \frac{\lambda}{2}(132) + (3r-6)\lambda 4 \times 6 = \frac{\lambda}{2}(6(r-2) \times 6(r-2) - 1) + \frac{\lambda}{2}(12 \times 11) + (3r-6)24\lambda = (3p_1 + 6q_1) + (3p_2 + 6q_2) + (3p_3 + 6q_3)$. By the induction hypothesis and Case 7, the graphs $K_{6(r-2)}(\lambda)$ and $K_{12}(\lambda)$ are fully $\{P_4, C_6\}$-decomposable. By Lemma 3.3 and Remark 3.2, the graph $K_{6,4}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable and hence $K_{6r}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

$n \equiv 1 \pmod 6$. Let $n = 6r + 1$, with $r \geq 3$. Assume that $K_{6t+1}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable if $1 \leq t < r$. Write $K_{6r+1}(\lambda) = K_{6(r-2)+1}(\lambda) + K_{13}(\lambda) + K_{6(r-2),12}(\lambda) = K_{6(r-2)+1}(\lambda) + K_{13}(\lambda) + (r-2)K_{6,12}(\lambda) = K_{6(r-2)+1}(\lambda) + K_{13}(\lambda) + (3r-6)\lambda K_{6,4}$. Suppose the nonnegative integers p and q satisfy the obvious necessary conditions for a $\{pP_4, qC_6\}$-decomposition in $K_{6r+1}(\lambda)$. Then we have $3p + 6q = \frac{\lambda(6r+1)\times(6r)}{2} = \frac{\lambda}{2}(36r^2 + 6r) = \lambda(18r^2 + 3r) = 18\lambda r^2+3\lambda r = 18\lambda r^2+3\lambda r+144\lambda-144\lambda = 18\lambda r^2-69\lambda r+66\lambda+78\lambda+72\lambda r-144\lambda = \lambda(18r^2 - 69r + 66) + 78\lambda + 72\lambda r - 144\lambda = \frac{\lambda}{2}(36r^2 - 138r + 132) + 78\lambda + 72\lambda r - 144\lambda = \frac{\lambda}{2}(36r^2-72r-66r+132)+78\lambda+72\lambda r-144\lambda = \frac{\lambda}{2}(6r-11)\times(6r-12)+78\lambda+72\lambda r-144\lambda = \frac{\lambda}{2}(6r-12+1)\times(6r-12)+78\lambda+24\lambda r-144\lambda = \frac{\lambda}{2}(6(r-2)+1\times 6(r-2))+78\lambda+24\lambda(3r-6) = \frac{\lambda}{2}(6(r-2) + 1 \times 6(r-2) + 1 - 1) + \frac{156\lambda}{2} + 4 \times 6\lambda(3r-6) = \frac{\lambda}{2}(6(r-2) + 1 \times 6(r-2) + 1 - 1) + \frac{\lambda}{2}(156) + (3r-6)\lambda 4 \times 6 = \frac{\lambda}{2}(6(r-2) + 1 \times 6(r-2) + 1 - 1) + \frac{\lambda}{2}(13 \times 12) + (3r-6)24\lambda = (3p_1 + 6q_1) + (3p_2 + 6q_2) + (3p_3 + 6q_3)$. By the induction hypothesis and Case 8, the graphs $K_{6(r-2)+1}(\lambda)$ and $K_{13}(\lambda)$ are fully $\{P_4, C_6\}$-decomposable. By Lemma 3.3 and Remark 3.2, the graph $K_{6,4}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable and hence $K_{6r+1}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

$n \equiv 2 \pmod 6$. Let $n = 6r + 2$, with $r \geq 3$. Assume that $K_{6t+2}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable if $1 \leq t < r$. Write $K_{6r+2}(\lambda) = K_{6(r-1)}(\lambda) + K_8(\lambda) + K_{6(r-1),8}(\lambda) = K_{6(r-1)}(\lambda)+K_8(\lambda)+(r-1)K_{6,8}(\lambda) = K_{6(r-1)}(\lambda)+K_8(\lambda)+2(r-1)\lambda K_{6,4}$. Suppose the nonnegative integers p and q satisfy the obvious necessary conditions for a $\{pP_4, qC_6\}$-decomposition in $K_{6r+2}(\lambda)$. Then we have $3p+6q = \frac{\lambda(6r+2)\times(6r+1)}{2} = \frac{\lambda}{2}(36r^2+18r+2) = \lambda(18r^2+9r+1) = 18\lambda r^2 + 9\lambda r + \lambda = 18\lambda r^2 + 9\lambda r + 49\lambda - 48\lambda = 18\lambda r^2 - 39\lambda r + 21\lambda + 28\lambda + 48\lambda r - 48\lambda = \lambda(18r^2 - 39r + 21) + 28\lambda + 48\lambda r - 48\lambda = \frac{\lambda}{2}(36r^2 - 78r + 42) + 28\lambda + 48\lambda r - 48\lambda = \frac{\lambda}{2}(36r^2 - 42r - 36r + 42) + 28\lambda + 48\lambda r - 48\lambda = \frac{\lambda}{2}(6r-6) \times (6r-7) + 28\lambda + 48\lambda r - 48\lambda = \frac{\lambda}{2}(6r-6)\times((6r-6)-1)+28\lambda+48\lambda r-48\lambda = \frac{\lambda}{2}(6(r-1)\times 6(r-1)-1)+28\lambda+24\lambda(2(r-1)) = \frac{\lambda}{2}(6(r-1) \times 6(r-1) - 1) + \frac{56\lambda}{2} + 4 \times 6\lambda(2(r-1)) = \frac{\lambda}{2}(6(r-1) \times 6(r-1) - 1) + \frac{\lambda}{2}(56) + 2(r-1)\lambda 4 \times 6 = \frac{\lambda}{2}(6(r-1) \times 6(r-1) - 1) + \frac{\lambda}{2}(8 \times 7) + 2(r-1)24\lambda = (3p_1 + 6q_1) + (3p_2 + 6q_2) + (3p_3 + 6q_3)$. By the induction hypothesis and Case 3, the graphs $K_{6(r-1)}(\lambda)$ and $K_8(\lambda)$ are fully $\{P_4, C_6\}$-decomposable. By Lemma 3.3 and Remark 3.2, the graph $K_{6,4}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable and hence $K_{6r+2}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

$n \equiv 3 \pmod 6$. Let $n = 6r+3$, with $r \geq 3$. Assume that $K_{6t+3}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable if $1 \leq t < r$. Write $K_{6r+3}(\lambda) = K_{6(r-1)+1}(\lambda) + K_9(\lambda) + K_{6(r-1),8}(\lambda) = K_{6(r-1)+1}(\lambda) + K_9(\lambda) + (r-1)K_{6,8}(\lambda) = K_{6(r-1)+1}(\lambda) + K_9(\lambda) + 2(r-1)\lambda K_{6,4}$. Suppose the nonnegative integers p and q satisfy the obvious necessary conditions for a $\{pP_4, qC_6\}$-decomposition in $K_{6r+3}(\lambda)$. Then we have $3p+6q = \frac{\lambda(6r+3)\times(6r+2)}{2} = \frac{\lambda}{2}(36r^2+30r+6) = \lambda(18r^2+15r+3) = 18\lambda r^2+15\lambda r+3\lambda = 18\lambda r^2+15\lambda r+51\lambda-48\lambda = 18\lambda r^2-33\lambda r+15\lambda+36\lambda+48\lambda r-48\lambda = \lambda(18r^2-33r+15)+36\lambda+48\lambda r-48\lambda = \frac{\lambda}{2}(36r^2-66r+30)+36\lambda+48\lambda r-48\lambda = \frac{\lambda}{2}(36r^2-36r-30r+30)+36\lambda+48\lambda r-48\lambda = \frac{\lambda}{2}(6r-5)\times(6r-6)+36\lambda+48\lambda r-48\lambda = \frac{\lambda}{2}(6r-6+1)\times(6r-6)+36\lambda+48\lambda r-48\lambda = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1)+1-1)+36\lambda+24\lambda(2r-2) = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1)+1-1)+\frac{72\lambda}{2}+4\times 6\lambda(2r-2) = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1)+1-1)+\frac{\lambda}{2}(72)+(2r-2)\lambda 4\times 6 = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1)+1-1)+\frac{\lambda}{2}(9\times 8)+(2r-2)24\lambda = (3p_1+6q_1)+(3p_2+6q_2)+(3p_3+6q_3)$. By the induction hypothesis and Case 4, the graphs $K_{6(r-1)+1}(\lambda)$ and $K_9(\lambda)$ are fully $\{P_4, C_6\}$-decomposable. By Lemma 3.3 and Remark 3.2, the graph $K_{6,4}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable and hence $K_{6r+3}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

$n \equiv 4 \pmod 6$. Let $n = 6r+4$, with $r \geq 3$. Assume that $K_{6t+4}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable if $1 \leq t < r$. Write $K_{6r+4}(\lambda) = K_{2(3r-1)}(\lambda) + K_6(\lambda) + K_{4,6}(\lambda) + K_{6(r-1),6}(\lambda) = K_{2(3r-1)}(\lambda) + K_6(\lambda) + \lambda K_{4,6} + (r-1)\lambda K_{6,6}$. Suppose the nonnegative integers p and q satisfy the obvious necessary conditions for a $\{pP_4, qC_6\}$-decomposition in $K_{6r+4}(\lambda)$. Then we have $3p+6q = \frac{\lambda(6r+4)\times(6r+3)}{2} = \frac{\lambda}{2}(36r^2+42r+12) = \lambda(18r^2+21r+6) = 18\lambda r^2+21\lambda r+6\lambda = 18\lambda r^2+21\lambda r+42\lambda-36\lambda = 18\lambda r^2-15\lambda r+3\lambda+15\lambda+24\lambda+36\lambda r-36\lambda = \lambda(18r^2-15r+3)+15\lambda+24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(36r^2-30r+6)+15\lambda+24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(36r^2-18r-12r+6)+15\lambda+24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(6r-2)\times(6r-3)+15\lambda+24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(6r-2)\times(6r-2)-1+15\lambda+24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(2(3r-1)\times 2(3r-1)-1)+15\lambda+24\lambda+36\lambda(r-1) = \frac{\lambda}{2}(2(3r-1)\times 2(3r-1)-1)+\frac{30\lambda}{2}+24\lambda+6\times 6\lambda(r-1) = \frac{\lambda}{2}(2(3r-1)\times 2(3r-1)-1)+\frac{\lambda}{2}(30)+24\lambda+(r-1)\lambda 6\times 6 = \frac{\lambda}{2}(2(3r-1)\times 2(3r-1)-1)+\frac{\lambda}{2}(6\times 5)+24\lambda+(r-1)36\lambda = (3p_1+6q_1)+(3p_2+6q_2)+(3p_3+6q_3)+(3p_4+6q_4)$. By the induction hypothesis and Case 1, the graphs $K_{2(3r-1)}(\lambda)$ and $K_6(\lambda)$ are fully $\{P_4, C_6\}$-decomposable. By Lemmas 3.3, 3.4 and Remark 3.2, the graphs $K_{6,4}(\lambda)$ and $K_{6,6}(\lambda)$ are fully $\{P_4, C_6\}$-decomposable and hence $K_{6r+4}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable.

$n \equiv 5 \pmod 6$. Let $n = 6r+5$, with $r \geq 3$. Assume that $K_{6t+5}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable if $1 \leq t < r$. Write $K_{6r+5}(\lambda) = K_{6(r-1)+1}(\lambda) + K_{11}(\lambda) + K_{6(r-1),10}(\lambda) = K_{6(r-1)+1}(\lambda) + K_{11}(\lambda) + (r-1)K_{6,10}(\lambda) = K_{6(r-1)+1}(\lambda) + K_{11}(\lambda) + (r-1)\lambda K_{6,4} + (r-1)\lambda K_{6,6}$. Suppose the nonnegative integers p and q satisfy the obvious necessary conditions for a $\{pP_4, qC_6\}$-decomposition in $K_{6r+5}(\lambda)$. Then we have $3p+6q = \frac{\lambda(6r+5)\times(6r+4)}{2} = \frac{\lambda}{2}(36r^2+54r+20) = \lambda(18r^2+27r+10) = 18\lambda r^2+27\lambda r+10\lambda = 18\lambda r^2+60\lambda r-33\lambda r+70\lambda-60\lambda = 18\lambda r^2-33\lambda r+15\lambda+55\lambda+24\lambda r-24\lambda+36\lambda r-36\lambda = \lambda(18r^2-33r+15)+55\lambda+24\lambda r-24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(36r^2-66r+30)+55\lambda+24\lambda r-24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(36r^2-36r-30r+30)+55\lambda+24\lambda r-24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(6r-5)\times(6r-6)+55\lambda+24\lambda r-24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(6r-6+1)\times(6r-6)+55\lambda+24\lambda r-24\lambda+36\lambda r-36\lambda = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1))+55\lambda+24\lambda(r-1)+36\lambda(r-1) = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1)+1-1)+\frac{110\lambda}{2}+4\times 6\lambda(r-1)+6\times 6\lambda(r-1) = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1)+1-1)+\frac{\lambda}{2}(110)+(r-1)\lambda 4\times 6+(r-1)\lambda 6\times 6 = \frac{\lambda}{2}(6(r-1)+1\times 6(r-1)+1-1)+\frac{\lambda}{2}(11\times 10)+(r-1)24\lambda+(r-1)36\lambda = (3p_1+6q_1)+(3p_2+6q_2)+(3p_3+6q_3)+(3p_4+6q_4)$. By the induction hypothesis and Case 6, the graphs $K_{6(r-1)+1}(\lambda)$ and $K_{11}(\lambda)$ are fully $\{P_4, C_6\}$-decomposable. By Lemmas 3.3, 3.4 and Remark 3.2, the graphs $K_{6,4}(\lambda)$ and $K_{6,6}(\lambda)$ are fully $\{P_4, C_6\}$-decomposable and hence $K_{6r+5}(\lambda)$ is fully $\{P_4, C_6\}$-decomposable. □

Acknowledgments

The second author thanks the University Grants Commission, New Delhi for its financial support (NO. F. MRP-6292/15(SERO/UGC)). Further authors thank the anonymous referee for the helpful remarks and suggestions.

References

[1] D. Bryant, D. Horsley, B. Maenhaut and B. R. Smith, Cycle decompositions of complete multigraphs, *J. Combin. Des.* **19**, 42–69 (2011).

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan Press, London, (1976).

[3] R. Chinnavedi and R. Sangeetha, $K_n(\lambda)$ is fully $\{P_5, C_6\}$-Decomposable, *Contrib. Discrete Math.* **17** (1), 1–12 (2022).

[4] R. Chinnavedi and R. Sangeetha, $K_n(\lambda)$ is fully $\{P_4, C_4\}$-Decomposable, *Indian J. Discrete Math.* **6** (2), 129–139 (2020).

[5] Hung-Chih Lee, Decomposition of the complete bipartite multigraph into cycles and stars, *Discrete Math.* **338**, 1362–1369 (2015).

[6] M. Ilayaraja and A. Muthusamy, $\{pP_4, qC_4\}$-Decomposition of complete graphs, *Indian J. Discrete Math.* **4** (2), 13–33 (2018).

[7] S. Jeevadoss and A. Muthusamy, Decomposition of complete bipartite graphs into paths and cycles, *Discrete Math.* **331**, 98–108 (2014).

[8] H. M. Priyadharsini and A. Muthusamy, (G_m, H_m)-multifactorization of λK_m, *J. Combin. Math. Combin. Comput.* **69**, 145–150 (2009).

[9] D. G. Sarvate and L. Zhang, Decomposition of a λK_v into equal number of K_3^s and P_3^s, *Bull. Inst. Combin. Appl.* **67**, 43–48 (2013).

[10] T. W. Shyu, Decompositions of complete graphs into paths and cycles, *Ars Combin.* **97**, 257–270 (2010).

[11] T. W. Shyu, Decomposition of complete graphs into paths of length three and triangles, *Ars Combin.* **107**, 209–224 (2012).

[12] T. W. Shyu, Decompositions of complete bipartite graphs and complete graphs into paths, stars and cycles with four edges each, *Discuss. Math. Graph Theory* **41**, 451–468 (2021).

[13] M. Tarsi, Decomposition of complete multigraph into simple paths: Nonbalanced handcuffed designs, *J. Combin. Theory Ser A* **34**, 60–70 (1983).

[14] M. Truszczynski, Note on the decomposition of $\lambda K_{m,n}(\lambda K^*_{m,n})$ into paths, *Discrete Math.* **55**, 89–96 (1985).

Author information

R. Chinnavedi and R. Sangeetha, Department of Mathematics, A.V.V.M. Sri Pushpam College (Affiliated to Bharathidasan University), Poondi, Thanjavur, Tamil Nadu, India-613 503.
E-mail: chinnavedi571991@gmail.com, jaisangmaths@yahoo.com