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RUPTURE DEGREE OF SOME CLASSES OF GRAPHS

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Abstract Computer or communication networks are so designed that they do not get disrupted under external attack and moreover these are easily reconstructed when they do get disrupted. Many graph theoretical parameters are used to describe the stability and reliability of communication networks. Among the parameters rupture degree is comparatively better parameter to measure the vulnerability of networks. In this paper we have determined the exact values of the rupture degree of the Wheel graph and also that of cartesian product of graphs such as $P_m \square P_n$ and $C_m \square C_n$, where P_n is a path of order n and C_n is a cycle of order n.

1 Introduction

A communication network is composed of processors and communication links. Network designers attach importance the reliability and stability of a network. If the network begins losing communication then there is a loss in its effectivenesss. This event is called as the vulnerability of communication networks.In a communication network, vulnerability measures the resistance of the network after a breakdown of some of its processors or communication links[8].

The vulnerability of communication networks measures the resistance of a network to a disruption in operation after the failure of certain processors and communication links. Cable cuts, processors interruptions, software errors, hardware failures or transmission failures at various points can interrupt service for a long period of time. But network designs require greater degree of stability and reliability or less vulnerability in communication networks. Thus communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconfiguration of the network. A communication network can be modeled as a graph G whose vertices represent the processors and whose edges represent the lines of communication. Many graph theoretical parameters have been used in the past to describe the stability and reliability of communication networks. Among them, two basic parameters, connectivity and edge connectivity have been extensively used. The higher the connectivity (edge connectivity) of G, the more stable it is considered to be. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently, several other parameters such as toughness, scattering number, integrity, tenacity, rupture degree, neighbour-integrity, neighbour-scattering number and their edge-analogues have been introduced to cope with this problem.

In an analysis of the vulnerability of networks to disruption, three important quantities, there may be others, that are considered seriously are

(i) the number of elements that are not functioning,

(ii) the number of remaining connected sub networks,

(iii) the size of a largest remaining group within which mutual communication can still occur [10].

The rupture degree takes into account both the number of components left after external attack and the size of the largest remaining component. A network with minimum rupture degree performs better under external attack. Thus the less the rupture degree of a network the more the stable it is considered to be. In [10], Li et al. have obtained some basic results on the rupture degree and they have proved that the rupture degree is a better parameter to measure the vulnerability of a network by using some examples. They have also obtained several bounds and Nordhous Gaddum-type results for the rupture degree.

The rupture degree for an connected graph G is defined by

$$r(G) = \max\left\{\omega(G-S) - |S| - m(G-S) : X \subset V(G), \omega(G-S) > 1\right\},$$

where $\omega(G-S)$ is the number of components of $G-S$ and $m(G-S)$ is the order of a largest component of $G-S$. The rupture degree for a complete graph K_n is defined as $1-n$. A set $S \subseteq V(G)$ is a vertex cut of G, if either $G-S$ is disconnected or $G-S$ has only one vertex. It is shown that this parameter can be used to measure the vulnerability of networks.

Let G be an connected graph, a set $S \subset V(G)$ is called an *R-set* if it satisfies $r(G) = \omega(G-S) - |S| - m(G-S)$. The *cartesian product* of the graphs G and H, denoted by $G \square H$, has the vertex set $V(G \square H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \square H$ if (i) $u = v$ and $xy \in E(H)$ or, (ii) $x = y$ and $uv \in E(G)$. Let W_n denote wheel of order n, C_n denote cycle of order n and P_n denote path of order n. In this paper we determine the exact values of rupture degree the wheel W_n $(n \geq 5)$.

2 Rupture degree of wheel graph

Theorem 2.1. *The rupture degree of the wheel W_n $(n \geq 5)$*

$$r(W_n) = \begin{cases} \text{-3} & \text{if } n \text{ is even,} \\ \text{-2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. **Case 1:** n is even.
Let S be an arbitrary vertex cut of W_n and set $|S| = t$. If $t \leq \frac{n}{2}$, then $\omega(W_n - S) \leq \frac{n}{2} - 1 = t - 1$. Therefore we have $m(W_n - S) \geq \left\lceil \frac{n-(t-1)}{t-1} \right\rceil$. Hence

$$\begin{aligned} \omega(W_n - S) - |S| - m(W_n - S) &\leq t - 1 - t - \left\lceil \frac{(n-(t-1))}{t-1} \right\rceil \\ &\leq -1 - \left\lceil \frac{(n-t)}{t} \right\rceil \\ &\leq -3. \end{aligned}$$

If $t > \frac{n}{2}$, then $\omega(W_n - S) > n - t$. Hence

$$\omega(W_n - S) - |S| - m(W_n - S) \leq n - 2t - 1 \leq -3.$$

From the choice of S, we obtain $r(W_n) \leq -3$. It is easy to see that there is a vertex cut S^* of W_n such that $|S^*| = \frac{n}{2}$, $\omega(W_n - S^*) = \frac{n}{2} - 1$ and $m(W_n - S^*) = 2$. From the definition of rupture degree, we have

$$\begin{aligned} r(W_n) &\geq \omega(W_n - S^*) - |S^*| - m(W_n - S^*) \\ &\geq \frac{n}{2} - 1 - \frac{n}{2} - 2 \\ &\geq -3. \end{aligned}$$

This implies that $r(W_n) = -3$.
Case 2: n is odd.
Let S be an arbitrary vertex cut of W_n and set $|S| = t$. If $t \leq \frac{n+1}{2}$, then $\omega(W_n - S) \leq \frac{n-1}{2} = t - 1$. Therefore, we have $m(W_n - S) \geq \left\lceil \frac{(n-(t-1))}{t} \right\rceil = 1$. Hence

$$\begin{aligned} \omega(W_n - S) - |S| - m(W_n - S) &\leq t - 1 - t - \left\lceil \frac{(n-(t-1))}{t} \right\rceil \\ &\leq -1 - 1 = -2. \end{aligned}$$

If $t > \frac{n+1}{2}$, then $\omega(W_n - S) \leq n - t$. Hence

$$\omega(W_n - S) - |S| - m(W_n - S) \quad \leq \quad n - 2t - 1 \leq -2.$$

From the choice of S, we obtain $r(W_n) \leq -2$. It is easy to see that there is a vertex cut S^* of W_n such that $|S^*| = \frac{n+1}{2}$,
$\omega(W_n - S^*) = \frac{n-1}{2} - 1$and $m(W_n - S^*) = 1$.
From the definition of rupture degree, we have

$$\begin{aligned} r(W_n) &\geq \omega(W_n - S^*) - |S^*| - m(W_n - S^*) \\ &\geq \frac{n-1}{2} - 1 - \frac{n-1}{2} - 1 \\ &\geq -2. \end{aligned}$$

This implies that $r(W_n) = -2$. □

3 Rupture degree of Cartesian product of some graphs

3.1 Rupture degree of $P_m \Box P_n$

Proposition 3.1. [10] *Let G be an incomplete connected graph of order n. Then*

$$2\alpha(G) - n - 1 \leq r(G) \leq \frac{[\alpha(G)]^2 - \kappa(G)[\alpha(G) - 1] - n}{\alpha(G)}.$$

Theorem 3.2. *The rupture degree of path $P_m \Box P_n$, is $m, n \geq 2$.*

$$r(P_m \Box P_n) = \begin{cases} 0 & \text{if } m \text{ and } n \text{ are both odd} \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Let $G = P_m \Box P_n$ and S be a vertex cut of G and $|S| = t$. We distinguish two cases.
Case 1: m and n are odd.

If $2 \leq t \leq \frac{mn-1}{2}$, then we have $\omega\big(G - S\big) \leq \frac{mn+1}{2} = t + 1$ and $m\big(G - S\big) \geq 1$.
Thus

$$\omega\big(G - S\big) - |S| - m\big(G - S\big) \leq \frac{mn+1}{2} - \frac{mn-1}{2} - 1$$

$$r(G) \leq 0 \tag{3.1}$$

If $t \geq \frac{mn+1}{2}$, then we have $\omega(G - S) \leq mn - t$ and $m(G - S) \geq 1$.
Thus

$$\omega(G - S) - |S| - m(G - S) \leq mn - t - t - 1.$$

The function $f(t) = mn - 2t - 1$ is a decreasing function and attains its maximum value at $t = \frac{mn+1}{2}$. Thus

$$f\left(\frac{mn+1}{2}\right) = mn - 2\left(\frac{mn+1}{2}\right) - 1 = -2 \leq 0$$

we get that

$$r(G) \leq 0. \tag{3.2}$$

By (3.1) and (3.2) we have

$$r(G) \leq 0. \tag{3.3}$$

On the other hand, it is easily seen that $\alpha(G) = \dfrac{mn+1}{2}$. By Proposition 3.1 we have

$$\begin{aligned} r(G) &\geq 2\alpha(G) - n - 1 \\ &\geq 2\left(\frac{mn+1}{2}\right) - mn - 1 \\ r(G) &\geq 0. \end{aligned} \tag{3.4}$$

By (3.3) and (3.4) we have

$$r(G) = r(P_m \Box P_n) = 0.$$

Case 2: m and n are both even or m is even and n is odd or m is odd and n is even.

If $2 \leq t \leq \dfrac{mn}{2}$, then we have $\omega(G-S) \leq t$ and $m(G-S) \geq 1$. Thus

$$\omega(G-S) - |S| - m(G-S) \leq t - t - 1 = -1.$$

$$r(G) \leq -1 \tag{3.5}$$

If $t \geq \dfrac{mn+2}{2}$, then we have $\omega(G-S) \leq mn - t$ and $m(G-S) \geq 1$. Thus

$$\omega(G-S) - |S| - m(G-S) \leq mn - t - t - 1 = mn - 2t - 1.$$

The function $f(t) = mn - 2t - 1$ is a decreasing function and takes its maximum value at $t = \dfrac{mn+2}{2}$. Thus

$$f\left(\frac{mn+2}{2}\right) = mn - 2\left(\frac{mn+2}{2}\right) - 1 = -3 \leq -1.$$

This gives that

$$r(G) \leq -1. \tag{3.6}$$

By (3.5) and (3.6) we have

$$r(G) \leq -1. \tag{3.7}$$

On the other hand, it is easily seen that $\alpha(G) = \dfrac{mn}{2}$. By Proposition 3.1, we have

$$\begin{aligned} r(G) &\geq 2\alpha(G) - n - 1 \\ &\geq 2\left(\frac{mn}{2}\right) - mn - 1 \\ r(G) &\geq -1. \end{aligned} \tag{3.8}$$

By (3.7) and (3.8), if m and n are even, we have $r(G) = r(P_m \Box P_n) = -1$. □

3.2 Rupture degree of $C_m \Box C_n$

Proposition 3.3. *If S is an R-set of $C_m \Box C_n$, $m, n \geq 3$, then the components of $C_m \Box C_n - S$ must be K_1 or K_2 or both.*

Proposition 3.4. *Let S be a minimal R-set of $C_m \Box C_n$, $m, n \geq 3$. Then in $C_m \Box C_n - S$,*

(i) *if m and n are both even there exists at most $\left\lceil \dfrac{mn}{2} \right\rceil$ number of K_1. There is no K_2 in both even cases.*

(ii) *if m is even and n is odd there exists at most $\left\lceil \dfrac{m}{2} \right\rceil$ number of K_2.*

(iii) *if m is odd and n is even there exists at most* $\left\lceil \frac{n}{2} \right\rceil$ *number of* K_2.

(iv) *if m and n are both odd there exists at most* $\left\lceil \frac{m+n-2}{2} \right\rceil$ *number of* K_2.

Proof. Let $G = C_m \square C_n$, $V(C_m) = \{u_1, u_2, \cdots u_m\}$ and $V(C_n) = \{v_1, v_2, \cdots v_n\}$. Let S be a minimal R-set of G. From the definition of rupture degree, in order to get $r(G)$, S should satisfy the condition that $m(G-S)$ is small and $\omega(G-S)$ is large.

Case 1: m and n are both even. Since the graph G is a hamiltonian $\omega(G-S) \leq |S|$ for every nonempty proper subset S of $V(G)$. Since S is a minimal R-set, $|S| = \frac{mn}{2}$ and hence $\omega(G-S)$ consists of isolated vertices and $\omega(G-S) \leq \frac{mn}{2}$.

Case 2: m is even and n is odd. If all K_2 of $G-S$ are located in cycles C_m and $\omega(G-S)$ is as larger as possible, then it is easy to see that S must contain atleast 4 adjacent vertices to obtain each K_2; S does not satisfy that it is small as possible. So, in $G-S$ all K_2 are not located at cycle C_m. That is all K_2 of $G-S$ must be located on cycle C_n^r. If there exist a component K_2 which is located at the different position from that of other components K_2, we will distinguish three cases.

Subcase 2(a): If there exist two K_2 in $G-S$ which is located on the same cycle C_m^r, we denote these two K_2, as $\{(u_r, v_i), (u_r, v_{i+1})\}$ and $\{(u_r, v_j), (u_r, v_{j+1})\}$, $1 < r < n$,$1 \leq i \leq m$, $j \geq (i+3)mod(n)$. Among these two K_2, we assume that $(u_r, v_i), (u_r, v_{i+1})$ is located on the same position of that of other K_2. Now let

$$S' = S \cup \{(u_r, v_i)\} - \{(u_{r-1}, v_i), (u_r, v_{i+1}), (u_{r+1}, v_i)\},$$

or

$$S' = S \cup \{(u_r, v_{i+1})\} - \{(u_{r-1}, v_{i+1}), (u_r, v_{i+2}), (u_{r+1}, v_{i+1})\},$$

then, we have

$$\begin{aligned} |S'| &= |S| - 2, \\ m(G-S') &= m(G-S) = 2 \\ \omega(G-S') &= \omega(G-S) + 2 \end{aligned}$$

so we have

$$\begin{aligned} \omega(G-S') - |S'| - m(G-S') &= \omega(G-S) + 2 - |S| + 2 - m(G-S) \\ &> \omega(G-S) - |S| - m(G-S). \end{aligned}$$

This contradicts to the minimality of S.

Subcase 2(b): If there exist two K_2 in $G-S$ which is located on two adjacent cycles, then their position must be different from each other. We denote these two K_2 as $\{(u_r, v_i), (u_r, v_{i+1})\}$ and $\{(u_{r+1}, v_j), (u_{r+1}, v_{j+1})\}$, $0 < r < m-1$, $1 \leq i \leq n$, $j \geq (i+2)mod(n)$ or $j \leq (i-2+n)mod(n)$. Among these two K_2, we assume that $\{(u_r, v_i), (u_r, v_{i+1})\}$ is located on the same position of that other K_2. Now, let

$$S' = S \cup \{(u_{r+1}, v_j)\} - \{(u_r, v_j), (u_{r+1}, v_{j-1}), (u_{r+2}, v_j)\},$$

or

$$S' = S \cup \{(u_{r+1}, v_{i+1})\} - \{(u_r, v_{j+1}), (u_{r+1}, v_{j+1}), (u_{r+2}, v_{j+1})\},$$

then, we have

$$\begin{aligned} |S'| &= |S| - 2, \\ m(G-S') &= m(G-S) = 2 \\ \omega(G-S') &= \omega(G-S) + 2 \end{aligned}$$

so we have

$$\begin{aligned} \omega(G-S^{'})-|S^{'}|-m(G-S^{'}) &= \omega(G-S)+2-|S|+2-m(G-S) \\ &> \omega(G-S)-|S|-m(G-S). \end{aligned}$$

This contradicts to the minimality of S

Subcase 2(c): If all K_2 in $G-S$ are located on non-adjacent cycles, and there exist a K_2 whose position is different from that of others, we denote this K_2 as $\{(u_r,v_i),(u_r,v_{i+1})\}$, $0 < r < m$,$1 \leq i \leq n$. Let the other K_2 be $\{(u_h,v_j),(u_h,v_{j+1})\}$ for $1 \leq r \leq m$ and $h \neq r, 1 \leq j \leq n, j \neq i$. Now, let

$$S^{'} = S \cup \{(u_r,v_j)\} - \{(u_{r-1},v_i),(u_r,v_{i-1}),(u_r,v_j),(u_{r+1},v_i)\},$$

or

$$S^{'} = S \cup \{(u_r,v_{i+1})\} - \{(u_{r-1},v_{i+1}),(u_r,v_{i+2}),(u_r,v_j),(u_{r+1},v_{i+1})\},$$

or

$$S^{'} = S \cup \{(u_r,v_{i+1})\} - \{(u_{r-1},v_{i+1}),(u_r,v_{i+2}),(u_r,v_j),(u_{r+1},v_{i+1})\},$$

then we have,

$$\begin{aligned} |S^{'}| &= |S|-3, \\ m(G-S^{'}) &= m(G-S)=2 \\ \omega(G-S^{'}) &= \omega(G-S)+3 \end{aligned}$$

so we have

$$\begin{aligned} \omega(G-S^{'})-|S^{'}|-m(G-S^{'}) &= \omega(G-S)+6-|S|-m(G-S) \\ &> \omega(G-S)-|S|-m(G-S). \end{aligned}$$

This contradicts to the minimality of S. So, by the subcases 2(a),2(b),2(c) we know that all K_2 in $G-S$ must be located in nonadjacent cycles in same positions; hence, there exist at most $\left\lceil \frac{m}{2} \right\rceil$ number of K_2 in $G-S$.

Case 3: m is odd and n is even. The result follows as in Case 2.

Case 4: m is odd and n is odd. If all K_2 of $G-S$ are located either only in cycles C_m or only in C_n and $\omega(G-S)$ is as larger as possible, then it is easy to see that S must contain atleast 4 adjacent vertices to obtain each K_2; S does not satisfy that it is small as possible. So, all K_2 of $G-S$ must be located on cycle C_m^s and C_n^r.
If there exist a component K_2 which is located at the different position from that of other components K_2, we will distinguish three cases.

Subcase 4(a): If there exist two K_2 in $G-S$ which is located on the same cycle C_m^s,C_n^r we denote these two K_2, as $\{(u_r,v_i),(u_r,v_{i+1})\}$ and $\{(u_r,v_j),(u_r,v_{j+1})\}$, $1 < r < n$,$1 \leq i \leq m$, $j \geq (i+2)mod(m)$ and $j \geq (i+3)mod(n)$. Among these two K_2, we assume that $(u_r,v_i),(u_r,v_{i+1})$ is located on the same position of that of other K_2. Now let

$$S^{'} = S \cup \{(u_r,v_i)\} - \{(u_{r-1},v_i),(u_r,v_{i+1}),(u_{r+1},v_i)\},$$

or

$$S^{'} = S \cup \{(u_r,v_{i+1})\} - \{(u_{r-1},v_{i+1}),(u_r,v_{i+2}),(u_{r+1},v_{i+1})\},$$

then, we have

$$\begin{aligned} |S^{'}| &= |S|-2, \\ m(G-S^{'}) &= m(G-S)=2 \\ \omega(G-S^{'}) &= \omega(G-S)+2 \end{aligned}$$

so we have

$$\begin{aligned}\omega(G-S^{'})-|S^{'}|-m(G-S^{'}) &= \omega(G-S)+2-|S|+2-m(G-S)\\ &> \omega(G-S)-|S|-m(G-S).\end{aligned}$$

This contradicts the minimality of S.

Subcase 4(b): If there exist two K_2 in $G-S$ which is located on two adjacent cycles, then their position must be different from each other. We denote these two K_2 as $\{(u_r,v_i),(u_r,v_{i+1})\}$ and $\{(u_{r+1},v_j),(u_{r+1},v_{j+1})\}$, $0<r<m-1$,$1\le i\le n$, $j\ge(i+2)mod(n)$ or $j\le(i-2+n)mod(n)$.
Among these two K_2, we assume that $\{(u_r,v_i),(u_r,v_{i+1})\}$ is located on the same position of that other K_2. Now, let

$$S^{'}=S\cup\{(u_{r+1},v_j)\}-\{(u_r,v_j),(u_{r+1},v_{j-1}),(u_{r+2},v_j)\},$$

or

$$S^{'}=S\cup\{(u_{r+1},v_{i+1})\}-\{(u_r,v_{j+1}),(u_{r+1},v_{j+1}),(u_{r+2},v_{j+1})\},$$

then, we have

$$\begin{aligned}|S^{'}| &= |S|-2,\\ m(G-S^{'}) &= m(G-S)=2\\ \omega(G-S^{'}) &= \omega(G-S)+2\end{aligned}$$

so we have

$$\begin{aligned}\omega(G-S^{'})-|S^{'}|-m(G-S^{'}) &= \omega(G-S)+2-|S|+2-m(G-S)\\ &> \omega(G-S)-|S|-m(G-S).\end{aligned}$$

This contradicts the minimality of S.

Subcase 4(c): If all K_2 in $G-S$ are located on non-adjacent cycles, and there exist a K_2 whose position is different from that of others, we denote this K_2 as $\{(u_r,v_i),(u_r,v_{i+1})\}$, $0<r<m$,$1\le i\le n$. We let the other K_2 be $\{(u_h,v_j),(u_h,v_{j+1})\}$, for $1\le r\le m$, and $h\ne r, 1\le j\le n, j\ne i$.

Now, let

$$S^{'}=S\cup\{(u_r,v_j)\}-\{(u_{r-1},v_i),(u_r,v_{i-1}),(u_r,v_j),(u_{r+1},v_i)\},$$

or

$$S^{'}=S\cup\{(u_r,v_{i+1})\}-\{(u_{r-1},v_{i+1}),(u_r,v_{i+2}),(u_r,v_j),(u_{r+1},v_{i+1})\},$$

or

$$S^{'}=S\cup\{(u_r,v_{i+1})\}-\{(u_{r-1},v_{i+1}),(u_r,v_{i+2}),(u_r,v_j),(u_{r+1},v_{i+1})\},$$

then we have,

$$\begin{aligned}|S^{'}| &= |S|-3,\\ m(G-S^{'}) &= m(G-S)=2\\ \omega(G-S^{'}) &= \omega(G-S)+3\end{aligned}$$

so we have

$$\begin{aligned}\omega(G-S^{'})-|S^{'}|-m(G-S^{'}) &= \omega(G-S)+6-|S|-m(G-S)\\ &> \omega(G-S)-|S|-m(G-S).\end{aligned}$$

This contradicts the minimality of S.

So, by the subcases 4(a), 4(b), 4(c) we know that all K_2 in $G-S$ must be located on nonadjacent cycles and in same positions; hence there exist at most $\left\lceil\frac{m}{2}\right\rceil$ number of K_2 in $G-S$. □

Theorem 3.5. *The rupture degree of* $C_m \square C_n$, $m, n \geq 3$ *is*

$$r(C_m \square C_n) = \begin{cases} -1 & \textit{if } m \textit{ and } n \textit{ are both even.} \\ -\dfrac{m+4}{2} & \textit{if } m \textit{ is even and } n \textit{ is odd.} \\ -\dfrac{n+4}{2} & \textit{if } m \textit{ is odd and } n \textit{ is even.} \\ -\dfrac{(m+n+4)}{2} & \textit{if } m \textit{ and } n \textit{ are both odd.} \end{cases}$$

Proof. Let $G = C_m \square C_n$, $m, n \geq 3$.

Case 1: m and n are both even. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let

$$S = \{(u_i, v_j) \mid i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even.}\}.$$

Then $|S| = \dfrac{mn}{2}$, $\omega(G - S) = \dfrac{mn}{2}$ and $m(G - S) = 1$.
Hence

$$\begin{aligned} r(G) &\geq \omega(G - S) - |S| - m(G - S) \\ &\geq \frac{mn}{2} - \frac{mn}{2} - 1 \geq -1. \end{aligned}$$

On the other hand, by Proposition 3.4, we know that there exist at most $\left\lceil \dfrac{mn}{2} \right\rceil = \dfrac{mn}{2}$ number of K_1, and they are located at the same position of non adjacent cycles in $G - S$.
We have $|S| \geq \left\lceil mn - \dfrac{mn+1}{2} \right\rceil = \left\lceil \dfrac{mn-1}{2} \right\rceil = \dfrac{mn}{2}$,
$\omega(G - S) \leq \left\lfloor mn - \dfrac{mn-1}{2} \right\rfloor = \left\lfloor \dfrac{mn+1}{2} \right\rfloor = \dfrac{mn}{2}$ and $m(G - S) = 1$.

Therefore $r(G) \leq \dfrac{mn}{2} - \dfrac{mn}{2} - 1 = -1$ and hence $r(G)$=$r(C_m \square C_n) = -1$.

Case 2: m is even and n is odd. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let

$$S = \{(u_i, v_j) \mid i \text{ is even and } j \text{ is even or } i \text{ is odd and} j \text{is odd.}\}.$$

Then $|S| = \dfrac{mn}{2}$, $\omega(G - S) = \dfrac{m(n-1)}{2}$ and $m(G - S) = 2$. We have

$$\begin{aligned} r(G) &\geq \omega(G - S) - |S| - m(G - S) \\ &\geq \frac{m(n-1)}{2} - \frac{mn}{2} - 2. \\ &\geq -\frac{m+4}{2}. \end{aligned}$$

On the other hand, by Proposition 3.4 we know that there exist at most $\left\lceil \dfrac{m}{2} \right\rceil = \dfrac{m}{2}$ number of K_2 in $G - S$, and the other components are all K_1.
we have $|S| \geq \left\lceil \dfrac{mn - \frac{m}{2}2}{2} \right\rceil = \left\lceil \dfrac{mn-m}{2} \right\rceil$.

But, in order to get at most $\dfrac{m}{2}$ number of K_2, we must delete atleast $\dfrac{m}{2}$ number of K_2 from G and so we have $|S| \geq \left\lceil \dfrac{mn-m}{2} \right\rceil + \dfrac{m}{2} = \dfrac{mn}{2}$, and

$$\omega(G - S) \leq \frac{m}{2} + \left\lceil \frac{mn - \frac{m}{2}2}{2} \right\rceil - \frac{m}{2} = \frac{mn-m}{2}, m(G - S) = 2.$$

Therefore

$$r(G) \leq \frac{mn-m}{2} - \frac{mn}{2} - 2 = -\frac{m+4}{2}.$$

In this case, $r(G)$= $r(C_m \square C_n) = -\dfrac{m+4}{2}$

Case 3: m is odd and n is even. For $1 \le i \le m$ and $1 \le j \le n$, let

$$S = \{(u_i, v_j) \mid i \text{ is odd and } j \text{ is odd or } i \text{ is even and } j \text{ is even.}\}.$$

Then $|S| = \dfrac{mn}{2}$, $\omega(G-S) = \dfrac{(m-1)n}{2}$ and $m(G-S) = 2$. We have

$$\begin{aligned} r(G) &\ge \omega(G-S) - |S| - m(G-S) \\ &\ge \frac{(m-1)n}{2} - \frac{mn}{2} - 2. \\ &\ge -\frac{n+4}{2}. \end{aligned}$$

On the other hand, by Proposition 3.4 we know that there exist atmost $\left\lceil \dfrac{n}{2} \right\rceil = \dfrac{n}{2}$ number of K_2 in $G-S$, and the other components are all K_1. We have

$$|S| \ge \left\lceil \frac{mn - \frac{n}{2}2}{2} \right\rceil = \left\lceil \frac{mn-n}{2} \right\rceil$$

But, in order to get at most $\dfrac{n}{2}$ number of K_2, we have must delete atleast $\dfrac{n}{2}$ number of K_2 from G and so we have $|S| \ge \left\lceil \dfrac{mn-n}{2} \right\rceil + \dfrac{n}{2} = \dfrac{mn}{2}$,

$\omega(G-S) \le \dfrac{n}{2} + \left\lceil \dfrac{mn - \frac{n}{2}2}{2} \right\rceil - \dfrac{n}{2} = \dfrac{mn-n}{2}$ and $m(G-S) = 2$.

Consequently $r(G) \le \dfrac{mn-n}{2} - \dfrac{mn}{2} - 2 = -\dfrac{n+4}{2}$. In this case, $r(G)$=$r(C_m \square C_n) = -\dfrac{n+4}{2}$

Case 4: m and n are both odd. For $1 \le i \le m$ and $1 \le j \le n$, let

$$S = \{(u_i, v_j) | i \text{ is odd and} j \text{ is odd or} i \text{ is even and} j \text{is even.}\}.$$

Then $|S| = \dfrac{mn+1}{2}$, $\omega(G-S) = (m-1)\left(\dfrac{n-1}{2}\right)$ and $m(G) = 2$. We have

$$\begin{aligned} r(G) &\ge \omega(G) - |S| - m(G-S) \\ &\ge \frac{mn-m-n+1}{2} - \frac{mn+1}{2} - 2. \\ &\ge -\frac{m+n+4}{2}. \end{aligned}$$

On the other hand, by Proposition 3.4 we know that there exist at most $\left\lceil \frac{m+n-2}{2} \right\rceil = \frac{m+n-2}{2}$ number of K_2 in $G-S$, and the other components are all K_1.

We have

$$|S| \ge \left\lceil \frac{mn - \frac{m+n-2}{2}2}{2} \right\rceil = \left\lceil \frac{mn-m-n+2}{2} \right\rceil$$

But, in order to get at most $\dfrac{m+n-2}{2}$ number of K_2, we must delete atleast $\dfrac{m+n-2}{2}$ number of K_2 from G and so we have $|S| \ge \left\lceil \dfrac{mn-m-n+2}{2} \right\rceil + \dfrac{m+n-2}{2} = \dfrac{mn+1}{2}$, and

$$\omega(G) \le \frac{m+n-2}{2} + \left\lfloor \frac{mn-m-n+2}{2} \right\rfloor - \frac{m+n-2}{2}, = \frac{mn-m-n+1}{2}, \; m(G) = 2.$$

Therefore

$$r(G) \leq \frac{mn-m-n+1}{2} - \frac{mn+1}{2} - 2 = -\frac{m+n+4}{2}.$$

In this case, $r(G)$=$r(C_m \square C_n) = -\dfrac{m+n+4}{2}$. □

References

[1] A. Aytac and H. Aksu Some Results for the Rupture Degree, *Int. J. Foundations of Comp. Science* **24 (8)**,1329–1338 (2013) .

[2] A. Aytac and H. Aksu, The Rupture Degree of some Graphs, *Mathematica Balkanica* **24 (1-2)** , 85–101 (2010) .

[3] A. Aytac and Z. N. Odabas Computing the Rupture Degree in Composite Graphs*Int. J.Foundations of Comp. Science* **21 (3)**, 311–319 (2010) .

[4] C. A. Barefoot, R. Entringer and H. C. Swart, Vulnerability in Graphs - A Comparative Survey, *J. Combin. Math. Combin. Comput.* **1**, 12–22 (1987) .

[5] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications *Macmillan, London and Elsevier,* Newyork (1976).

[6] M. Hoj and E. Vumar, Some Vertex Vulnerability Parameters of Kronecker Products of Complete Graphs, *J. Xinjiang University***27**, 276–279 (2010).

[7] A. Kirlangic, The Rupture Degree and Gear Graphs, *Bull. Malays. Math. Sci. Soc.* **32(1)**, 31–36 (2009).

[8] F. Li, Q. Ye and B. Sheng, Computing Rupture Degree of some Graphs, *Wseas Transactions on Math.* **11(1)**,23–33 (2012).

[9] F. Li and X. Li Computing the Rupture Degrees of Graphs, *In: 7th International Symposium on Parallel Architectures, Algorithms and Networks; Hong Kong, China*, 368–373 (2004) .

[10] Y. K. Li, S. G. Zhang and X.L. Li, Rupture Degree of Graphs, *Int. J. Comput. Math.* **82**, 793–803 (2005) .

[11] P. Paulraja and V. Sheeba Agnes, Connectivity of Tensor Product of Graphs, *Discrete Math. Algorithm Appl.* **5**,#1350023 (10 pages) (2013) .

[12] P. Paulraja and V. Sheeba Agnes, Vulnerability Parameters of Tensor Product of Complete Equipartite Graphs, *Opuscula Math.* **33**, 741–750 (2013) .

[13] V. Sheeba Agnes, Vulnerability Parameters of Tensor Product of Cycles 2021 *J. Phys.: Conf.* Ser. 1724 012052.

[14] D.B. West *Introduction to Graph Theory,* Pearson Education, Inc (2001).

[15] Z. Wang and Z. Wang, Relationships between Rupture Degree and other Parameters, *Int. J. Comput. Math.* **83**, 839–851 (2006) .

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