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CONNECTIVITY OF THE COMPLEMENT OF GENERALIZED TOTAL GRAPH OF FINITE FIELDS

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Abstract Let R be a commutative ring with identity, $Z(R)$ be its zero-divisors, and H be a nonempty proper multiplicative prime subset of R. *The generalized total graph* of R is the simple undirected graph $GT_H(R)$ with the vertex set R and two distinct vertices x and y are adjacent if and only if $x+y \in H$. In this paper, we investigate the vertex connectivity, edge connectivity and separation of $\overline{GT(F)}$ for the finite field F. In particular, we prove that $\kappa = \kappa' = \delta$ for $\overline{GT(F)}$.

1 Introduction

Throughout this paper, R denotes a commutative ring with identity, $Z(R)$ is the set of all zero-divisors of R, $Z^*(R) = Z(R) \setminus \{0\}$ and $U(R)$ is the set of all units in R. Anderson and Livingston [5] introduced the *zero-divisor graph* of R, denoted by $\Gamma(R)$, as the undirected simple graph with vertex set $Z^*(R)$ and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Subsequently, Anderson and Badawi [3] introduced the concept of the *total graph* of a commutative ring. The *total graph* $T_\Gamma(R)$ of R is the undirected graph with vertex set R and for distinct $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Anukumar *et al.*[10, 11, 12, 13], Asir and Tamizh Chelvam [6] have extensively studied about the total graph of commutative rings.

Recently, Anderson and Badawi [3] introduced the concept of the generalized total graph of a commutative ring R. A nonempty proper subset H of R is said to be a multiplicative prime subset of R if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. For a multiplicative prime subset H of R, the *generalized total graph* $GT_H(R)$ of R is the simple undirected graph with vertex set R and two distinct vertices x and y are adjacent if and only if $x + y \in H$. One can see that every prime ideal, union of prime ideals and $H = R \backslash U(R)$ are some of the multiplicative prime subsets of R. The *unit graph* $G(R)$ of R is the simple undirected graph with vertex set R in which two distinct vertices x and y are adjacent if and only if $x+y \in U(R)$. Note that if R is finite, then $\overline{GT_{Z(R)}(R)}$ is the unit graph [9]. Tamizh Chelvam and Balamurugan [14, 15, 16, 17] have extensively studied about the generalized total graph of a finite commutative ring and its complement. The entire literature regarding graphs from rings can be found in the monograph [2].

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The complement $\overline{G}$ of the graph G is the simple graph with vertex set $V(G)$ and two distinct vertices x and y are adjacent in $\overline{G}$ if and only if they are not adjacent in G. We say that G is connected if there is a path between any two distinct vertices of G. For a vertex $v \in V(G)$, $deg(v)$ is the degree of v. For any graph G, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of vertices in G respectively. K_n denotes the complete graph of order n and $K_{m,n}$ denotes the complete bipartite graph. For basic definitions in graph theory, we refer the reader to [7] and for the terms regarding algebra one can refer [8].

In this paper, we are interested in the connectivity of the complement of the generalized total graph of fields. In section 2, we recall the structure of $GT(F)$ and its complement. In section 3, we investigate the connectivity of $\overline{GT(F)}$ and prove that $\kappa = \kappa' = \delta$ for $\overline{GT(F)}$. Throughout this paper, we assume that P is a prime ideal of R with $|P| = \lambda$ and $|R/P| = \mu$.

2 Generalized total graph of fields

In this section, we recall certain results on the generalized total graph of commutative rings.

Theorem 2.1. [3, Theorem 2.2] *Let P be a prime ideal of a finite commutative ring R, and let $|P| = \lambda$ and $|R/P| = \mu$.*

(i) *If $2 \in H$, then $GT_H(R \setminus P)$ is the union of $\mu - 1$ disjoint K_λ's;*

(ii) *If $2 \notin H$, then $GT_H(R \setminus P)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{\lambda,\lambda}$'s.*

Note that $GT(F)$ is the generalized total graph of the field F with the unique multiplicative prime subset $\{0\}$. If F is a field of characteristic 2, then $x + x = 0$ for every $x \in F$. When the characteristic of the field F is greater than 2, for any $0 \neq x \in F$, $x \neq -x$ and $x + (-x) = 0$. In view of these, one can have the following structure for $GT(F)$.

Lemma 2.2. [14, Lemma 1.1] *Let F be a finite field. Then*

$$GT(F) = \begin{cases} \underbrace{K_1 \cup \cdots \cup K_1}_{|F| \text{ copies}} & \text{if } char(F) = 2; \\ K_1 \cup \underbrace{K_{1,1} \cup \cdots \cup K_{1,1}}_{\frac{|F|-1}{2} \text{ copies}} & \text{if } char(F) > 2. \end{cases}$$

In view of the Lemma 2.2, we have the following structure for the complement of $GT(F)$.

Lemma 2.3. [14, Lemma 2.2] *Let F be a finite field. Then the following are true:*

(i) *If $char(F) = 2$, then $\overline{GT(F)} = K_{|F|}$;*

(ii) *If $char(F) > 2$, then $\overline{GT(F)}$ is a connected bi-regular graph with $\Delta = |F| - 1$ and $\delta = |F| - 2$.*

3 Properties of $\overline{GT(F)}$

In this section, we discuss about connectivity of $\overline{GT(F)}$. Note that, a *simplicial* vertex v of a graph G is a vertex whose neighbours induce a clique in G.

Lemma 3.1. *Let F be a finite field. Then the following are true :*

(i) *If $char(F) = 2$, then every vertex of $\overline{GT(F)}$ is a simplicial vertex;*

(ii) *Let $char(F) > 2$.*

  (a) *If $|F| = 3$, then the non-zero elements of F are simplicial vertices in $\overline{GT(F)}$;*

  (b) *If $|F| > 3$, then no vertex in $\overline{GT(F)}$ is a simplicial vertex.*

Proof. (i) Follows from Lemma 2.3(i).

(ii) Assume that $char(F) > 2$. If $|F| = 3$, then by Lemma 2.3(i) $\overline{GT(F)}$ is P_3. In this case, the neighbours of 0 are x, y such that $x = -y$. and hence 0 is not a simplicial vertex. Clearly 0 is the only one neigbour of x as well as y and it induces K_1 as the clique. Assume that $|F| \geq 5$. List the elements of F as $F = \{0, x_1, \ldots, x_{\frac{|F|-1}{2}}, y_1, \ldots, y_{\frac{|F|-1}{2}}\}$ where $y_i = -x_i$ for $1 \leq i \leq \frac{|F|-1}{2}$. Clearly $F \setminus \{0\}$ is the set of all neighbours of 0. For $i \leq i \leq \frac{|F|-1}{2}$, the vertices x_i and y_i are not adjacent in $\overline{GT(F)}$. Therefore the subgraph induced by the neighbours of 0 is not a clique. □

Note that, a cut vertex of a connected graph is a vertex whose deletion results in a disconnected graph. In view of following lemma, we obtain the characterization of finite fields for which $\overline{GT(F)}$ has a cut vertex.

Lemma 3.2. *Let F be a finite field. Then $\overline{GT(F)}$ has a cut vertex if and only if $F \cong \mathbb{Z}_3$.*

Proof. If $F \cong \mathbb{Z}_3$, then by Lemma 2.3, $\overline{GT(F)}$ is P_3 with $deg(0) = 2$ and $deg(1) = deg(2) = 1$. Hence 0 is the cut vertex of $\overline{GT(F)}$.

Conversely, assume that $\overline{GT(F)}$ has a cut vertex u.

If $char(F) = 2$, then by Lemma 2.3, $\overline{GT(F)}$ is $K_{|F|}$, contradiction to the assumption that $\overline{GT(F)}$ contains a cut vertex u.

Suppose that $char(F) > 2$ and $|F| \geq 5$. List the elements of F as $F = \{0, x_1, \cdots, x_{\frac{|F|-1}{2}}, y_1, \cdots y_{\frac{|F|-1}{2}}\}$ where each $y_i = -x_i$ for $1 \leq i \leq \frac{|F|-1}{2}$. Clearly $0 - x_1 - \cdots - x_{\frac{|F|-1}{2}} - y_1 - \cdots - y_{\frac{|F|-1}{2}} - 0$ is a cycle of length $|F|$ in $\overline{GT(F)}$ and so $\overline{GT(F)} \setminus \{u\}$ induces a cycle of length $|F|-1$ again a contradiction to the assumption that u is a cut vertex. Hence $F \cong \mathbb{Z}_3$. □

Note that, a cut edge of a connected graph is an edge whose deletion results in a disconnected graph. In view of following lemma, we obtain a characterization of fields for which $\overline{GT(F)}$ has a cut edge.

Lemma 3.3. *Let F be a finite field. Then $\overline{GT(F)}$ contains a cut edge e if and only if either $F \cong \mathbb{Z}_2$ or $F \cong \mathbb{Z}_3$.*

Proof. Assume that either $F \cong \mathbb{Z}_2$ or $F \cong \mathbb{Z}_3$. By Lemma 2.3 $\overline{GT(F)}$ is either K_2 or P_3. Hence $\overline{GT(F)}$ contains a cut edge e.

Conversely, assume that $\overline{GT(F)}$ has a cut edge e. If $char(F) = 2$ with $|F| > 2$, then by Lemma 2.3, $\overline{GT(F)}$ is $K_{|F|}$ and so $\overline{GT(F)} \setminus \{e\}$ is connected for every $e \in E(F)$, a contradiction. Hence $F \cong \mathbb{Z}_2$.

Suppose $char(F) > 2$ with $|F| \geq 5$. Consider the partition, $F = \{0\} \bigcup_{i=1}^{\frac{|F|-1}{2}} \{x_i\} \bigcup_{i=1}^{\frac{|F|-1}{2}} \{y_i\}$, where $y_i = -x_i$ for $1 \leq i \leq \frac{|F|-1}{2}$. Note that $deg(0) = |F| - 1$ and $\langle \bigcup_{i=1}^{\frac{|F|-1}{2}} \{x_i\} \rangle = \langle \bigcup_{i=1}^{\frac{|F|-1}{2}} \{y_i\} \rangle = K_{\frac{|F|-1}{2}}$ is a subgraph of $\overline{GT(F)}$. Also x_i, y_i are not adjacent in $\overline{GT(F)}$. Therefore $\overline{GT(F)} \setminus \{e\}$ induces a connected subgraph, a contradiction. This gives that $F \cong \mathbb{Z}_3$. □

Lemma 3.4. *Let F be a finite field with $|F| > 4$ and $S \subset V(\overline{GT(F)})$ with $|S| = 3$. Then the subgraph induced by S is either K_3 or $K_{1,2}$.*

Proof. If $char(F) = 2$, then by Lemma 2.3, $\langle S \rangle = K_3 \subset \overline{GT(F)}$.
Assume that $char(F) > 2$. Let $S = \{0, x_1, x_2\}$. If $x_2 = -x_1$, then $\langle S \rangle = K_{1,2}$. If $x_2 \neq -x_1$, then $\langle S \rangle = K_3$. If $0 \notin S$ and no two of them are additive inverses, then $\langle S \rangle = K_3$. □

Recall that, the *vertex connectivity* of a graph G is the minimum number of vertices whose deletion disconnects G, which is denoted by $\kappa(G)$. The *edge connectivity* of a graph G is the minimum number of edges whose deletion disconnects G, which is denoted by $\kappa'(G)$. The following theorem shows that $\kappa(\overline{GT(F)}) = \kappa'(\overline{GT(F)}) = \delta(\overline{GT(F)})$.

Theorem 3.5. *Let F be a finite field. Then $\kappa(\overline{GT(F)}) = \kappa'(\overline{GT(F)}) = \delta(\overline{GT(F)})$.*

Proof. Assume that $char(F) = 2$. Then $\overline{GT(F)}$ is complete and hence

$$\kappa(\overline{GT(F)}) = \kappa'(\overline{GT(F)}) = |F| - 1 = \delta(\overline{GT(F)}).$$

Assume that $char(F) \neq 2$. By Lemma 2.3, $deg_{\overline{GT(F)}}(0) = |F| - 1 = \Delta$ and $deg_{\overline{GT(F)}}(x) = |F| - 2 = \delta$. Let $0 \neq x \in F$ and $E_x = \{e = xy : \ y \neq -x\}$. Clearly E_x is the set of edges which are incident at x in $\overline{GT(F)}$. Therefore $\overline{GT(F)} \setminus E_x$ is disconnected and so $\kappa'(\overline{GT(F)}) = |F| - 2 = \delta(\overline{GT(F)})$.

Assume that $char(F) > 2$. If $|F| = 3$, by Lemma 3.3, $\kappa(\overline{GT(F)}) = |F| - 2 = 1 = \delta(\overline{GT(F)})$.

If $|F| \geq 5$, by Lemma 2.3, $\delta(\overline{GT(F)}) = |F| - 2$ and so $\kappa \leq |F| - 2$.

Claim: $\kappa = |F| - 2$.
Suppose the set of vertices $W = \{v_1, \ldots, v_{|F|-3}\} \subset V(\overline{GT(F)})$ is the vertex cut of $\overline{GT(F)}$. Then by Lemma 3.4, the subgraph induced by the set $\langle V(\overline{GT(F)}) \setminus W \rangle$ either K_3 or $K_{1,2}$, which is a contradiction to W is a vertex cut. Hence $\kappa = |F| - 2 = \delta(\overline{GT(F)})$. □

Corollary 3.6. *Let F be a finite field with $char(F) > 2$ and $x \in F \setminus \{0\}$. Then any vertex cut of $\overline{GT(F)}$ is of the form $F \setminus \{x, -x\}$.*

Proof. Proof follows from Lemma 3.3 and Theorem 3.5. □

Corollary 3.7. *Let F be a finite field. Then the following are true:*

(i) *$\overline{GT(F)}$ is not a 2-connected graph;*

(ii) *$\overline{GT(F)}$ is not 2-edge connected graph;*

(iii) *$\overline{GT(F)}$ is 3-edge connected if and only if either $F \cong F_4$ or $F \cong F_5$.*

Proof. Proof follows from Lemma 2.3 and Theorem 3.5. □

Recall that, a *separation* of a connected graph G is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. This common vertex is called a *separating* vertex of G.

A graph G is said to be *non-separable* if it is connected and has no separating vertices; otherwise, it is separable. Note that any complete graph is non-separable.

Lemma 3.8. *Let F be a finite field and $F \ncong \mathbb{Z}_3$. Then $\overline{GT(F)}$ is non-separable.*

Proof. If $char(F) = 2$, then by Lemma 2.3, $\overline{GT(F)}$ is $K_{|F|}$ and so non-separable.

Assume that $char(F) > 2$. By the assumption that $F \ncong \mathbb{Z}_3$, we have $|F| \geq 5$. Suppose $\overline{GT(F)}$ is separable. Then $\overline{GT(F)}$ may be decomposed into two nonempty connected subgraphs H_1 and H_2, with just one vertex u in common. Let $e_i = uu_i$ be an edge of H_i incident with u, $i = 1, 2$.

Case(i). Suppose $u = 0$ and $u_2 = -u_1$. Since $|F| \geq 5$, there exists a non-zero element u_3 in $F \setminus \{u, u_1, u_2\}$ such that u_3 is adjacent with u, u_1 and u_2 in $\overline{GT(F)} \setminus \{0\}$. Therefore $\overline{GT(F)} \setminus \{0\} \neq H_1 \cup H_2$, a contradiction.

Case(ii). Suppose $u = 0$ and $u_2 \neq -u_1$. In this case u_1 is adjacent with u_2 in $\overline{GT(F)} \setminus \{0\}$, which is a contradiction to H_1 and H_2 is a separation of $\overline{GT(F)}$.

Case(iii). Suppose $u \neq 0$. Suppose either $u = -u_1$, or $u = -u_2$.
Then $\langle \{0, u_1, u_2\} \rangle = K_3$ in $\overline{GT(F)} \setminus \{0\}$ and so $\overline{GT(F)} \setminus \{0\} \neq H_1 \cup H_2$, which is a contradiction to $\overline{GT(F)}$ is separable.

If $u_1 = -u_2$, then $\langle \{0, u, u_1, u_2\} \rangle$ induces $K_{1,3}$ in $\overline{GT(F)}$ and so $\overline{GT(F)} \setminus \{0\} \neq H_1 \cup H_2$, which is also a contradiction to our assumption. Hence $\overline{GT(F)}$ is non-separable. □

An *atom* of a graph G is a minimal subset X of $V(G)$ such that $d(X) = \kappa'$ and $|X| \leq \frac{n}{2}$. Thus if $\kappa' = \delta$, then any vertex of minimum degree is a *singleton atom*. On the other hand, if $\kappa' < \delta$, then G has no singleton atom.

From Theorem 3.5, we have the following lemma.

Lemma 3.9. *Let F be a finite field.*

(i) *If $char(F) = 2$, then every element of F is an atom in $\overline{GT(F)}$;*

(ii) *If $char(F) > 2$, then every non-zero element of F is an atom in $\overline{GT(F)}$.*

The following proposition is a known one.

Proposition 3.10. [7, Proposition 9.13] *The atoms of a graph are pairwise disjoint.*

From Lemma 3.9 and Proposition 3.10, we have the following.

Lemma 3.11. *Let F be a finite field. Then the following are true:*

(i) *If $char(F) = 2$, then any two elements of F are pairwise disjoint in $\overline{GT(F)}$;*

(ii) *If $char(F) > 2$, then any two non-zero elements of F are pairwise disjoint in $\overline{GT(F)}$.*

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