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ON GENERAL RANDIĆ ENERGY OF GRAPHS

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Abstract Let d_i be the degree of vertex v_i of G then general Randić matrix $GR(G) = [r_{ij}]$ is defined as $r_{ij} = (d_i d_j)^\alpha$, $\alpha \in \mathbb{R}$ if the vertices v_i and v_j are adjacent in G and $r_{ij} = 0$, otherwise. The general Randić energy $E_{GR}(G)$ of G is the sum of the absolute values of the eigenvalues of $GR(G)$. In this paper, we compute the general Randić energy of the graphs obtained by graph operations like m-Splitting, m-Shadow and duplication graph.

1 Introduction

All graphs considered here are simple connected graphs without multiple, directed or weighted edges. Let G be a graph with $V(G) = \{v_1, v_2, ..., v_n\}$ as a vertex set and $E(G)$ as a edge set. Let d_i be the degree of a vertex v_i, for each $i = 1, 2, ..., n$. The adjacency matrix $A(G) = [a_{ij}]$ of a graph G is a square matrix of order n, where

$$a_{ij} = \begin{cases} 1 & ; \text{ if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & ; \text{ otherwise} \end{cases}$$

Let $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of $A(G)$ then they all real numbers with their sum is zero as $A(G)$ is a symmetric matrix. The set of eigenvalues with their multiplicity is known as spectrum of a graph and it is denoted by $Spec(G)$. In 1978, Gutman[7] have introduced the concept of energy of graph. According to him energy of graph $\mathcal{E}(G)$ is defined as sum of absolute values of eigenvalues of graph G. That is,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$$

A brief account of spectra of graph and graph energy can be found in Balakrishnan [2], Li *et al.* [17] and Cvetković *et al.* [4]. Topological indices are numerical values of a graph which are invariant under isomorphism of graphs. Various type of topological indices are mainly used in qualitative structure-property relationship(QSPR) and qualitative structure-activity relationship(QSAR)[9]. In 1975, Randić [21] has introduced one such topological index and termed it as Randić index which is defined by

$$R = \sum_{i\sim j} (d_i d_j)^{-\frac{1}{2}}$$

where the summation is taken over all pairs of adjacent vertices v_i and v_j. Randić index is useful in chemistry and pharmacology, in particular it is very useful in designing quantitative structure property and structure activity relations. A brief account on Randić index can be found in [8, 15, 16, 22].

In 1998, Bollobás and Erdös[3] have generalized the concept of Randić index by replacing $-\frac{1}{2}$ power with any real number and named it as general Randić index which is denoted and defined as

$$R_\alpha = R_\alpha(G) = \sum_{i\sim j} (d_i d_j)^\alpha, \alpha \in \mathbb{R}$$

In 2010, Bozkurt *et al.* [11, 12] pointed out that the Randić index is purposeful to produce a graph matrix of order n which is known as Randić matrix $R(G) = [r_{ij}]$, where

$$r_{ij} = \begin{cases} (d_i d_j)^{-\frac{1}{2}} & ;\ \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & ;\text{otherwise} \end{cases}$$

The connection between the Randić matrix and the Randić index is obvious: The sum of all elements of $R(G)$ is equal to $2R$. Let $\rho_1, \rho_2, ..., \rho_n$ are eigenvalues of matrix $R(G)$ then the Randić energy [11, 12] is defined as the sum of absolute values of Randić eigenvalues of graph G which is denoted as $RE(G)$. That is,

$$RE = RE(G) = \sum_{i=1}^{n} |\rho_i|$$

Alikhani and Ghanbari[1] have found the Randić energy of some standard graph families. Rojo and Medina[23] have constructed a bipartite graphs having same Randić energy. Some more results related to Randić energy can be found in [10, 26]. In [6], Gu *et al.* have introduced the concept of general Randić matrix and general Randić energy. The general Randić matrix $GR(G)$ of a graph G is a square matrix which is defined by $GR(G) = [g_{ij}]$, where

$$g_{ij} = \begin{cases} (d_i d_j)^{\alpha} & ;\ \text{if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & ;\ \text{otherwise} \end{cases} \quad \text{, where } \alpha \in \mathbb{R}$$

For $\alpha = -\frac{1}{2}$, the above matrix reduces to Randić matrix and for $\alpha = 0$, it reduces to adjacency matrix. The general Randić energy is defined as the sum of absolute values of eigenvalues of $GR(G)$.

$$E_{GR}(G) = \sum_{i=1}^{n} |\mu_i|$$

where μ_i's are eigen values of the general Randić matrix of graph G. In [6], they have also established some bounds on general Randić energy in terms of general Randić index, general Randić eigenvalues and smallest degree of a graph. They have established a lower bounds for general Randić spectral radius of a connected graph. Liu and Shiu[18] have characterized the class of connected graphs with distinct eigenvalues and also obtained general Randić polynomial of subdivision of a graph.

Ramane and Gudodagi [19] have given general Randić polynomial and general Randić energy of some standard graph families like the path P_n, the cycle C_n, the complete graph K_n, complete bipartite graph $K_{m,n}$, star graph S_n, friendship graph F_n, Dutch Windmill graph D_4^n and D_5^n, K_4-Windmill graph K_4^n and double star $S(p, q)$. Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic graphs if $\mathcal{E}(G_1) = \mathcal{E}(G_2)$. Ramane *et al.*[20] have constructed infinitely many pairs of equienergetic graphs. In the context of equienergetic graphs, we define general Randić equienergetic graphs in which two non-isomorphic graphs are said to be general Randić equienergetic if they have same general Randić energy.

In the present work, our focus is to explore the concept of general Randić energy in the context of various graph operations. Also some results are also proved for general Randić equienergetic graphs. In addition graphs having different order but of equal general Randić energy are investigated.

2 General Randić Energy of m-Splitting Graph

Definition 2.1. [25] The *m-Splitting graph* $Spl_m(G)$ of a graph G is obtained by adding to each vertex v a new m vertex $v_1, v_2, ..., v_m$, such that v_i, $1 \leqslant i \leqslant m$ is adjacent to every vertex that is adjacent to v in G.

Definition 2.2. [13] For the matrices $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{p\times q}$ the Kronecker product of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Proposition 2.3. *[13] If λ is an eigenvalue of matrix $A = [a_{ij}]_{m\times m}$ with corresponding eigenvector x, and μ is an eigenvalue of matrix $B = [b_{ij}]_{n\times n}$ with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

Proposition 2.4. *[5]* Let

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric block matrix. Then the spectrum of A is the union of spectra of $A_0 - A_1$ and $A_0 + A_1$.

The following result relates the general Randić energy of graph G and its m-Splitting.

Theorem 2.5. *For any graph G, $E_{GR}(Spl_m(G)) = \sqrt{(m+1)^{4\alpha} + 4m(m+1)^{2\alpha}}E_{GR}(G)$.*

Proof. Let G be a graph with $v_1, v_2, ..., v_n$ as vertices of then its general Randić matrix $R(G)$ is given by

$$GR(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & 0 & r_{23} & \cdots & r_{2n} \\ r_{31} & r_{32} & 0 & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

Now, consider m-copies of vertex v_i for $1 \leq i \leq n$, say $v_i^1, v_i^2, ..., v_i^m$ and then join each vertex v_i^k, for $1 \leq k \leq m$ to neighbors of vertex v_i to obtain $m-$Splitting of given graph G. Then the matrix $GR(Spl_m(G))$ can be written as follows

$$GR(Spl_m(G)) = \begin{bmatrix} (m+1)^{2\alpha}GR(G) & (m+1)^{\alpha}GR(G) & \cdots & (m+1)^{\alpha}GR(G) \\ (m+1)^{\alpha}GR(G) & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ (m+1)^{\alpha}GR(G) & O & \cdots & O \end{bmatrix}$$

That is,

$$GR(Spl_m(G)) = \begin{bmatrix} (m+1)^{2\alpha} & (m+1)^{\alpha} & \cdots & (m+1)^{\alpha} \\ (m+1)^{\alpha} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (m+1)^{\alpha} & 0 & \cdots & 0 \end{bmatrix} \otimes GR(G)$$

$$= A \otimes R(G)$$

where $A = \begin{bmatrix} (m+1)^{2\alpha} & (m+1)^{\alpha} & \cdots & (m+1)^{\alpha} \\ (m+1)^{\alpha} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (m+1)^{\alpha} & 0 & \cdots & 0 \end{bmatrix}$. Since, A is a matrix of rank 2 so, it means that matrix A has only two non-zero eigenvalues, say ρ_1 and ρ_2.

Also, we know that

$$\rho_1 + \rho_2 = tr(A) = (m+1)^{2\alpha} \tag{2.1}$$

Now, consider the matrix

$$A^2 = \begin{bmatrix} (m+1)^{4\alpha} + m(m+1)^{2\alpha} & (m+1)^{3\alpha} & \cdots & (m+1)^{3\alpha} \\ (m+1)^{3\alpha} & (m+1)^{2\alpha} & \cdots & (m+1)^{2\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ (m+1)^{3\alpha} & (m+1)^{2\alpha} & \cdots & (m+1)^{2\alpha} \end{bmatrix}_{(m+1)\times(m+1)}$$

Here,

$$\rho_1^2 + \rho_2^2 = tr(A^2) = (m+1)^{4\alpha} + 2m(m+1)^{2\alpha} \tag{2.2}$$

by solving equations (2.1) and (2.2), we have

$$\rho_1 = \frac{(m+1)^{2\alpha} + \sqrt{(m+1)^{4\alpha} + 4m(m+1)^{2\alpha}}}{2} \quad \text{and}$$

$$\rho_2 = \frac{(m+1)^{2\alpha} - \sqrt{(m+1)^{4\alpha} + 4m(m+1)^{2\alpha}}}{2}$$

Hence, $Spec(A) = \begin{pmatrix} 0 & \rho_1 & \rho_2 \\ m-1 & 1 & 1 \end{pmatrix}$

Since, $GR(Spl_m(G)) = A \otimes GR(G)$, it follows that if $\mu_1, \mu_2, ..., \mu_n$ are eigenvalues of $GR(G)$, then by Proposition 2.3, we have $Spec(GR(Spl_m(G))) =$

$$\begin{pmatrix} 0 & \rho_1\mu_1 & \cdots & \rho_1\mu_n & \rho_2\mu_1 & \cdots & \rho_2\mu_n \\ n(m-1) & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Hence,

$$E_{GR}(Spl_m(G)) = \sum_{i=1}^{n} \left| \frac{(m+1)^{2\alpha} \pm \sqrt{(m+1)^{4\alpha} + 4m(m+1)^{2\alpha}}}{2} \mu_i \right|$$

$$= \sum_{i=1}^{n} |\mu_i| \left(\frac{(m+1)^{2\alpha} + \sqrt{(m+1)^{4\alpha} + 4m(m+1)^{2\alpha}}}{2} + \frac{-(m+1)^{2\alpha} + \sqrt{(m+1)^{4\alpha} + 4m(m+1)^{2\alpha}}}{2} \right)$$

Therefore,

$$E_{GR}(Spl_m(G)) = \sqrt{(m+1)^{4\alpha} + 4m(m+1)^{2\alpha}}\, E_{GR}(G)$$

□

Corollary 2.6. *Let G_1 and G_2 be general Randić equienergetic graphs then $Spl_m(G_1)$ and $Spl_m(G_2)$ are also general Randić equienergetic graphs.*

Proof. Proof of this corollary follows from Theorem 2.5. □

3 General Randić Energy of m-Shadow Graph

Definition 3.1. [25] The *m-Shadow graph* $D_m(G)$ of a connected graph G is constructed by taking m copies of G say $G_1, G_2, ..., G_m$. Then Join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leqslant i, j \leqslant m$.

The following result relates the general Randić energy of graph G and its m-Shadow.

Theorem 3.2. *For any graph G, $E_{GR}(D_m(G)) = m^{2\alpha+1}E_{GR}(G)$.*

Proof. Let G be a graph with $v_1, v_2, ..., v_n$ as vertices of then its general Randić matrix $R(G)$ is given by

$$GR(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & 0 & r_{23} & \cdots & r_{2n} \\ r_{31} & r_{32} & 0 & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

Now, consider m-copies $G_1, G_2, ..., G_m$ of graph G and then join each vertex of u of graph G_i to the neighbors of the corresponding vertex v in graph G_j, $1 \leqslant i, j \leqslant m$ to obtain $m-$Shadow $D_m(G)$. Then the general Randić matrix of graph $D_m(G)$ can be written as

$$GR(D_m(G)) = \begin{bmatrix} m^{2\alpha}GR(G) & m^{2\alpha}GR(G) & \cdots & m^{2\alpha}GR(G) \\ m^{2\alpha}GR(G) & m^{2\alpha}GR(G) & \cdots & m^{2\alpha}GR(G) \\ \vdots & \vdots & \ddots & \vdots \\ m^{2\alpha}GR(G) & m^{2\alpha}GR(G) & \cdots & m^{2\alpha}GR(G) \end{bmatrix}$$

That is,

$$GR(D_m(G)) = \begin{bmatrix} m^{2\alpha} & m^{2\alpha} & \cdots & m^{2\alpha} \\ m^{2\alpha} & m^{2\alpha} & \cdots & m^{2\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ m^{2\alpha} & m^{2\alpha} & \cdots & m^{2\alpha} \end{bmatrix} \otimes GR(G)$$

Therefore, $GR(D_m(G)) = B \otimes R(G)$

Since, we know that the spectrum of B is $\begin{pmatrix} 0 & m^{2\alpha+1} \\ m-1 & 1 \end{pmatrix}$

Hence, by Proposition 2.3

$$Spec((D_m(G)) = \begin{pmatrix} 0 & m^{2\alpha+1}\mu_1 & m^{2\alpha+1}1\mu_2 & ... & m^{2\alpha+1}\mu_n \\ n(m-1) & 1 & 1 & ... & 1 \end{pmatrix}$$

where μ_i, $i = 1, 2, ..., n$ are eigenvalues of $GR(G)$.
Therefore,

$$E_{GR}(D_m(G)) = \sum_{i=1}^{n} \left| m^{2\alpha+1}\mu_i \right| = m^{2\alpha+1}E_{GR}(G)$$

□

Corollary 3.3. *If G_1 and G_2 are two general Randić equienergetic graphs, then $D_m(G_1)$ and $D_m(G_2)$ are also general Randić equienergetic graphs.*

Proof. Proof of this corollary follows from Theorem 3.2. □

4 General Randić Energy of Duplication Graph

Definition 4.1. [24] Let G be a graph on n vertices labelled as $V = \{v_1, v_2, v_3, ..., v_n\}$. Then take another set $U = \{u_1, u_2, u_3, ..., u_n\}$ of n vertices. Now define a graph H with $V(H) = V \cup U$ and edge set of H consisting only of those edges joining u_i to neighbors of v_i in G, for each i. The resultant graph H is called the duplication graph of G denoted by DG. Also, define $D^2(G) = D(D(G))$ and $D^n(G) = D(D^{n-1}(G))$.

Proposition 4.2. *[14] For any graph G, $E(DG) = 2E(G)$.*

Proposition 4.3. *[6] Let $G = G_1 \cup G_2 \cup ... \cup G_p$ then, $E_{GR}(G) = E_{GR}(G_1) + E_{GR}(G_2) + ... + E_{GR}(G_p)$.*

Theorem 4.4. *Let G be any graph of order n with $\mu_1, \mu_2, ..., \mu_n$ are as general Randić eigenvalues of G then, $E_{GR}(DG) = 2E_{GR}(G)$.*

Proof. Let G be any graph on n vertices then construct a duplication graph DG of graph G by considering two copies of vertex set of G as $V = \{v_1, v_2, v_3, ..., v_n\}$ and $U = \{u_1, u_2, u_3, ..., u_n\}$ and then join vertex u_i to neighbors of v_i, for each i.
Then the general Randić matrix of duplication graph DG of graph G can be written as

$$GR(DG) = \begin{bmatrix} O & GR(G) \\ GR(G) & O \end{bmatrix}$$

So, from Proposition 2.4 spectra of $R(DG)$ can be given by

$$Spec(GR(DG)) = Spec(GR(G)) \cup Spec(-GR(G))$$

$$\text{So, } Spec(GR(DG)) = \begin{pmatrix} \mu_i & -\mu_i \\ 1 & 1 \end{pmatrix}, \text{for each } i = 1, 2, ..., n.$$

Hence,

$$\begin{aligned} E_{GR}(DG) &= \sum_{i=1}^{n} |\mu_i| + \sum_{i=1}^{n} |-\mu_i| \\ &= 2\sum_{i=1}^{n} |\mu_i| \\ &= 2E_{GR}(G) \end{aligned}$$

□

Corollary 4.5. *For any graph G, DG and $G \cup G$ are general Randić equienergetic graphs.*

Proof. Proof of this corollary follows from Therem 4.4 and Proposition 4.3. □

Theorem 4.6. *Let G be any graph of order n with $\mu_1, \mu_2, ..., \mu_n$ are as general Randić eigenvalues of G then, $E_{GR}(D^nG) = 2^nE_{GR}(G)$.*

Proof. Let G be any graph on n vertices then construct a graph D^nG of graph G by considering duplication graph of $D^{n-1}G$.
Then the general Randić matrix of graph D^nG of graph G can be written as

$$R(D^nG) = \begin{bmatrix} O & GR(D^{n-1}G) \\ GR(D^{n-1}G) & O \end{bmatrix}$$

So, from Proposition 2.4 spectra of $GR(D^nG)$ can be given by

$$Spec(GR(D^nG)) = Spec(GR(D^{n-1}G)) \cup Spec(-GR(D^{n-1}G))$$

So, $Spec(GR(D^nG)) = \begin{pmatrix} \mu_i & -\mu_i \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$, for each $i = 1, 2, ..., n$.

Hence,

$$\begin{aligned} E_{GR}(D^nG) &= 2^{n-1}\sum_{i=1}^{n}|\mu_i| + 2^{n-1}\sum_{i=1}^{n}|-\mu_i| \\ &= 2^n \sum_{i=1}^{n}|\mu_i| \\ &= 2^n E_{GR}(G) \end{aligned}$$

□

Definition 4.7. [26] Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. Then take another m sets $U_1, U_2, ..., U_m$ of n vertices, where $U_i = \{u_1^i, u_2^i, ..., u_n^i\}$, for each $i = 1, 2, ..., m$. Now define a new graph H with vertex set $V(H) = V(G) \cup U_1 \cup ... \cup U_m$ and each vertex u_j^i is adjacent to neighbors of vertex v_j of graph G. The resultant graph is known as $m-$duplication graph of G denoted by $m - DG$.

Theorem 4.8. *Let G be any graph of order n with $\mu_1, \mu_2, ..., \mu_n$ are as general Randić eigenvalues of G then, $E_{GR}(m - DG) = 2m^{\alpha+\frac{1}{2}} E_{GR}(G)$.*

Proof. Let G be any graph of order n then, the general Randić matrix of graph $m - DG$ can be written as

$$GR(m - DG) = \begin{bmatrix} O & m^\alpha R(G) & \cdots & m^\alpha R(G) \\ m^\alpha R(G) & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ m^\alpha R(G) & O & \cdots & O \end{bmatrix}$$

That is,

$$\begin{aligned} GR(m - DG) &= \begin{bmatrix} 0 & m^\alpha & \cdots & m^\alpha \\ m^\alpha & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m^\alpha & 0 & \cdots & 0 \end{bmatrix} \otimes GR(G) \\ &= A \otimes GR(G) \end{aligned}$$

wher,e $A = \begin{bmatrix} 0 & m^\alpha & \cdots & m^\alpha \\ m^\alpha & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m^\alpha & 0 & \cdots & 0 \end{bmatrix}_{(m+1)}$

Since, A is a matrix of rank 2 so, it means that matrix A has only two non-zero eigenvalues, say ρ_1 and ρ_2.
Also, we know that

$$\rho_1 + \rho_2 = tr(A) = 0 \tag{4.1}$$

Now, consider the matrix

$$A^2 = \begin{bmatrix} m^{2\alpha+1} & 0 & \cdots & 0 \\ 0 & m^{2\alpha} & \cdots & m^{2\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & m^{2\alpha} & \cdots & m^{2\alpha} \end{bmatrix}_{(m+1)}$$

Here,

$$\rho_1^2 + \rho_2^2 = tr(A^2) = 2m^{2\alpha+1} \tag{4.2}$$

by solving equations (4.1) and (4.2), we have $\rho_1 = m^{\alpha+\frac{1}{2}}$ and $\rho_2 = -m^{\alpha+\frac{1}{2}}$.

Hence, $Spec(A) = \begin{pmatrix} 0 & m^{\alpha+\frac{1}{2}} & -m^{\alpha+\frac{1}{2}} \\ m-1 & 1 & 1 \end{pmatrix}$

Since, $GR(m-DG) = A \otimes GR(G)$, it follows that if $\mu_1, \mu_2, ..., \mu_n$ are eigenvalues of $GR(G)$, then by Proposition 2.3, we have

$$Spec(m-DG) = \begin{pmatrix} 0 & m^{\alpha+\frac{1}{2}}\mu_i & -m^{\alpha+\frac{1}{2}}\mu_i \\ n(m-1) & 1 & 1 \end{pmatrix}$$

Hence,

$$\begin{aligned} E_{GR}(m-DG) &= \sum_{i=1}^{n} \left| m^{\alpha+\frac{1}{2}}\mu_i \right| + \sum_{i=1}^{n} \left| -m^{\alpha+\frac{1}{2}}\mu_i \right| \\ &= 2m^{\alpha+\frac{1}{2}} \sum_{i=1}^{n} |\mu_i| = 2m^{\alpha+\frac{1}{2}} E_{GR}(G) \end{aligned}$$

□

Corollary 4.9. *Let G_1 and G_2 be general Randić equienergetic graphs then $m - DG_1$ and $m - DG_2$ are also general Randić equienergetic graphs.*

Proof. Proof of this corollary follows from Theorem 4.8. □

Corollary 4.10. *For any graph G, $E_{GR}(m-DG) = E_{GR}(DG)$ iff $\alpha = -\frac{1}{2}$.*

Proof. Proof of this corollary follows from Theorem 4.4 and Theorem 4.8. □

Corollary 4.11. *For any graph G, $E_{GR}(D_m(G)) = E_{GR}(DG)$ iff $m = 2$ and $\alpha = 0$.*

Proof. Proof of this corollary follows from Theorem 3.2 and Theorem 4.4. □

Corollary 4.12. *For any graph G, $E_{GR}(D_m(G)) = E_{GR}(D^nG)$ iff $m = 2$ and $\alpha = \dfrac{n-1}{2}$.*

Proof. Proof of this corollary follows from Theorem 3.2 and Theorem 4.6. □

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