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Restrained Domination Polynomial of Cycles

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Abstract Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is said to be a restrained dominating set if each vertex in $V - S$ is adjacent to a vertex in S and to a vertex in $V - S$. Let $\mathcal{D}_r(G, i)$ be the collection of restrained dominating sets of G of cardinality i and $d_r(G, i) = |\mathcal{D}_r(G, i)|$. The restrained domination polynomial of G is denoted by $D_r(G, x)$ and is defined as $D_r(G, x) = \sum\limits_{i=\gamma_r(G)}^{|V(G)|} d_r(G, i)x^i$. In this paper we construct $\mathcal{D}_r(C_n, i)$ and obtain a formula for $d_r(C_n, i)$.

1 Introduction

All graphs considered here are simple and undirected. The terms not defined here are taken as in [4]. Let $G = (V, E)$ be a graph of order $|V| = n$. A set $S \subseteq V$ is said to be a *dominating set* [5] of G if each vertex in $V - S$ is adjacent to a vertex in S. The *domination number* of G is denoted by $\gamma(G)$ and is defined as the minimum cardinality of a dominating set of G. Let $\mathcal{D}(G, i)$ be the collection of dominating sets of G of cardinality i and $d(G, i) = |\mathcal{D}(G, i)|$. The *domination polynomial* of a graph G is defined as $D(G, x) = \sum\limits_{i=\gamma(G)}^{n} d(G, i)x^i$. The concept of domination polynomial was introduced by Arocha and further developed by S.Alikhani [1, 2]. A set $S \subseteq V$ is said to be a *restrained dominating set* [3] of G if each vertex in $V - S$ is adjacent to a vertex in S and to a vertex in $V - S$. The *restrained domination number* of a graph G is denoted by $\gamma_r(G)$ and is defined as the minimum cardinality of a restrained dominating set of G. A restrained dominating set of G of minimum cardinality is called γ_r-set of G. Let $\mathcal{D}_r(G, i)$ be the collection of restrained dominating sets of G of cardinality i and let $d_r(G, i) = |\mathcal{D}_r(G, i)|$. We call the polynomial $D_r(G, x) = \sum\limits_{i=\gamma_r(G)}^{n} d_r(G, i)x^i$, the *restrained domination polynomial* of a graph G. The concept of restrained domination polynomial was introduced by K.Kayathri and G.Kokilambal in 2019. In [6] K.Kayathri and G.Kokilambal gave a recurrence relation for finding the restrained domination polynomial of cycles which was given by $D_r(C_n, x) = 3D_r(P_{n-2}, x) + D_r(P_n, x)$ for $n \geq 3$. They constructed the familes of restrained dominating sets of C_n of cardinality i by the families of restrained dominating sets of P_n and P_{n-2} of cardinality i. In this paper we construct the families of restrained dominating sets of C_n with cardinality i by the families of restrained dominating sets of C_{n-1} and C_{n-3} with cardinality $i - 1$.

As usual we use $\lfloor x \rfloor$, for the greatest integer less than or equal to x. In this paper we use the notation $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

2 Restrained Dominating Sets of Cycles

Let $C_n,\ n \geq 3$ be the cycle with n vertices. In this paper we denote the set of all vertices and edges of C_n by $V(C_n) = \{1, 2, \ldots, n\}$ and $E(C_n) = \{(i, i + 1)/1 \leq i \leq n - 1\} \cup \{(1, n)\}$ respectively. Let $\mathcal{D}_r(C_n, i)$ be the collection of restrained dominating sets of C_n of cardinality i and $|\mathcal{D}_r(C_n, i)| = d_r(C_n, i)$. We need the following lemmas to prove our main results in the section:

Lemma 2.1. *For any cycle C_n with $n \geq 3$, the following results hold:*

(i) *[3]* $\gamma_r(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$.

(ii) $\mathcal{D}_r(C_n, i) = \phi \Leftrightarrow$ *Any one of the following hold*
(a) $n - i \equiv 1 (mod\ 2)$
(b) $i > n$
(c) $i < n - 2\lfloor \frac{n}{3} \rfloor$.

(iii) $\mathcal{D}_r(C_n, i) \neq \phi \Leftrightarrow n - 2\lfloor \frac{n}{3} \rfloor \leq i \leq n$ *and* $n - i \equiv 0(mod\ 2)$.

Proof. As (ii) and (iii) are contra positive statements, it is enough to prove one among them. We shall prove (iii).
Assume that $\mathcal{D}_r(C_n, i) \neq \phi$. Then there exists a restrained dominating set S of C_n of cardinality i. It is clear that $\gamma_r(C_n) \leq i \leq n$. First we notice that if n is odd(even), then $\gamma_r(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$ is odd(even). It is enough to prove that $n - i \equiv 0\ (mod\ 2)$. Suppose that the induced subgraph of $V - S$ has a component H of order ≥ 3. Then H contains an induced $P_3 : v_j v_{j+1} v_{j+2}$ and the vertex v_{j+1} is not dominated by any other vertex of S which is a contradiction to S is a restrained dominating set of C_n. Thus order of each component of $\langle V - S \rangle$ is at most 2. Also $\langle V - S \rangle$ has no isolated vertices. It follows that $\langle V - S \rangle$ is isomorphic to ϕ or mP_2 for $m \geq 1$. Hence $n - i \equiv 0\ (mod\ 2)$.

Conversely suppose that $n - 2\lfloor \frac{n}{3} \rfloor \leq i \leq n$ and $n - i \equiv 0\ (mod\ 2)$. It is enough to show that there exists a restrained dominating set S of C_n of cardinality i. If $|S| = n - 2\lfloor \frac{n}{3} \rfloor$, then any γ_r- set of C_n satisfies our requirement. Also if $|S| = n$, then $S = V(C_n)$. Suppose that $n - 2\lfloor \frac{n}{3} \rfloor < i < n$. Let T be a γ_r- set of C_n. Then each component of $\langle V - T \rangle$ is isomorphic to P_2. Let $S = T \bigcup H$ where H is the union of some components of $\langle V - T \rangle$ chosen in such a way that $|S| = i$. Then S is also a restrained dominating set of C_n. Thus $n - i = n - (n - 2\lfloor \frac{n}{3} \rfloor + |H|)$ is even. Hence $\mathcal{D}_r(C_n, i) \neq \phi$. □

To find the collection of restrained dominating sets of C_n of cardinality i, it is enough to consider $\mathcal{D}_r(C_{n-1}, i - 1)$ and $\mathcal{D}_r(C_{n-3}, i - 1)$ and it is not necessary to consider restrained dominating sets of C_{n-5} of cardinality $i - 1$. This is proved in Lemma 2.2 and we do not consider $\mathcal{D}_r(C_{n-7}, i - 1)$ because it is impossible to find $Y \in \mathcal{D}_r(C_{n-7}, i - 1)$ such that $Y \bigcup \{x\} \in \mathcal{D}_r(C_n, i)$ for any $x \in [n]$.

Lemma 2.2. *Suppose that* $Y \in \mathcal{D}_r(C_{n-5}, i - 1)$ *with* $Y \bigcup \{x\} \in \mathcal{D}_r(C_n, i)$ *for some* $x \in [n]$. *Then* $Y \in \mathcal{D}_r(C_{n-3}, i - 1)$.

Proof. Suppose that $Y \in \mathcal{D}_r(C_{n-5}, i - 1)$ and $Y \bigcup \{x\} \in \mathcal{D}_r(C_n, i)$ for some $x \in [n]$. It is clear that $\{1, n - 5\}$ is a subset of Y. Otherwise $Y \bigcup \{x\} \notin \mathcal{D}_r(C_n, i)$ for any $x \in [n]$. Hence $Y \in \mathcal{D}_r(C_{n-3}, i - 1)$. □

Lemma 2.3. *If* $\mathcal{D}_r(C_n, i) \neq \phi$, *then we have*

(i) $\mathcal{D}_r(C_{n-1}, i - 1) \neq \phi$ *and* $\mathcal{D}_r(C_{n-3}, i - 1) \neq \phi \Leftrightarrow n - 2\lfloor \frac{n-1}{3} \rfloor \leq i \leq n - 2$.

(ii) $\mathcal{D}_r(C_{n-3}, i - 1) = \phi$ *and* $\mathcal{D}_r(C_{n-1}, i - 1) \neq \phi \Leftrightarrow i = n$.

(iii) $\mathcal{D}_r(C_{n-1}, i - 1) = \phi$ *and* $\mathcal{D}_r(C_{n-3}, i - 1) \neq \phi \Leftrightarrow i = q$ *and* $n = 3q$ *for some positive integer* q.

Proof. It is given that $\mathcal{D}_r(C_n, i) \neq \phi$. Then by applying (iii) of Lemma 2.1, we have $n - i \equiv 0(mod\ 2)$ and $n - 2\lfloor \frac{n}{3} \rfloor \leq i \leq n$.
(i) Assume that $\mathcal{D}_r(C_{n-1}, i - 1) \neq \phi$. Then by applying (iii) of Lemma 2.1, we have $(n - 1) - (i - 1) \equiv 0\ (mod\ 2)$ and $n - 2\lfloor \frac{n-1}{3} \rfloor \leq i \leq n$. Also $\mathcal{D}_r(C_{n-3}, i - 1) \neq \phi$. Again by applying (iii) of Lemma 2.1, we have $(n - 3) - (i - 1) \equiv 0\ (mod\ 2)$ and $n - 2 - 2\lfloor \frac{n-3}{3} \rfloor \leq i \leq n - 2$. From these we can conclude that $n - 2\lfloor \frac{n-1}{3} \rfloor \leq i \leq n - 2$.

Conversely assume that $n - 2\lfloor \frac{n-1}{3} \rfloor \leq i \leq n - 2$ and $n - i \equiv 0\ (mod\ 2)$. Then by applying (iii) of Lemma 2.1, we have $\mathcal{D}_r(C_{n-1}, i - 1) \neq \phi$ and $\mathcal{D}_r(C_{n-3}, i - 1) \neq \phi$.
(ii) Assume that $\mathcal{D}_r(C_{n-3}, i - 1) = \phi$. Then by Lemma 2.1 (ii), we have $i - 1 > n - 3$ or $i - 1 < (n - 3) - 2\lfloor \frac{n-3}{3} \rfloor$ and the condition $(n - 3) - (i - 1) \equiv 1\ (mod\ 2)$ is not possible . Since $\mathcal{D}_r(C_{n-1}, i - 1) \neq \phi$, by applying (iii) of Lemma 2.1, we have $(n - 1) - (i - 1) \equiv 0\ (mod\ 2)$ and $(n - 1) - 2\lfloor \frac{n-1}{3} \rfloor \leq i - 1 \leq n - 1$. If $(i - 1) < (n - 3) - 2\lfloor \frac{n-3}{3} \rfloor$, then $D_r(C_{n-1}, i - 1) = \phi$

which is a contradiction. Hence we have $n-3 < i-1$ and also the possible conditions are $i-1 \leq (n-1)$ and $(n-1)-(i-1) \equiv 0 \ (mod\ 2)$. Other conditions do not hold due to $(n-2) < i$. Thus we conclude that $n-2 < i \leq n$. Since $n \equiv i \ (mod\ 2)$, $i = n-1$ is impossible. Hence $i = n$.

Conversely assume that $i = n$. Then by applying (ii) of Lemma 2.1, we have $\mathcal{D}_r(C_{n-3}, i) = \phi$ and $\mathcal{D}_r(C_{n-1}, i) \neq \phi$.

(iii) Assume that $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$. Then by applying (ii) of Lemma 2.1, we have $n-3-2\lfloor \frac{n-3}{3} \rfloor \leq i-1 \leq n-3$ and $(n-3)-(i-1) \equiv 0 \ (mod\ 2)$. Also $\mathcal{D}_r(C_{n-1}, i-1) = \phi$, then by applying (ii) of Lemma 2.1, we have $i-1 > n-1$ or $n-1-2\lfloor \frac{n-1}{3} \rfloor > i-1$ and the condition $(n-1)-(i-1) \equiv 1 (mod\ 2)$ is not possible. Since $i-1 \leq n-3$, the condition $i-1 > n-1$ is not possible. Hence the only possible condition is $i-1 < n-1-2\lfloor \frac{n-1}{3} \rfloor$. Now $n-3-2\lfloor \frac{n-3}{3} \rfloor \leq i < n-2\lfloor \frac{n-1}{3} \rfloor$ which gives us $n = 3q$ and $i = q$ for some $q \in \mathbb{N}$.

Conversely assume that $n = 3q$ and $i = q$ for some $q \in \mathbb{N}$. Then by applying (ii) of Lemma 2.1, we have $\mathcal{D}_r(C_{n-1}, i-1) = \phi$ and $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$. □

Theorem 2.4. *For every $n \geq 3$ and i is a positive integer satisfying the condition that $n-2\lfloor \frac{n}{3} \rfloor \leq i \leq n$ and $n-i \equiv 0 \ (mod\ 2)$, the following are true:*

(i) *If $\mathcal{D}_r(C_{n-1}, i-1) = \phi$ and $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$, then* $\mathcal{D}_r(C_n, i) = \mathcal{D}_r(C_n, \frac{n}{3}) = \{\{1,4,7,\ldots,n-2\}, \{2,5,8,\ldots,n-1\}, \{3,6,9,\ldots,n\}\}$.

(ii) *If $\mathcal{D}_r(C_{n-3}, i-1) = \phi$ and $\mathcal{D}_r(C_{n-1}, i-1) \neq \phi$, then $\mathcal{D}_r(C_n, i) = \mathcal{D}_r(C_n, n) = \{[n]\}$.*

(iii) *If $\mathcal{D}_r(C_{n-1}, i-1) \neq \phi$ and $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$, then $\mathcal{D}_r(C_n, i) = \bigcup\limits_{k=1}^{6} \mathcal{A}_k$*

where $\mathcal{A}_1 = \{Z \bigcup \{n\}, Z \bigcup \{n-2\} / Z \in \mathcal{D}_r(C_{n-1}, i-1) \bigcap \mathcal{D}_r(C_{n-3}, i-1)\}$

$\mathcal{A}_2 = \{Z \bigcup \{n\} / Z \in \mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1) \text{ and } n-1 \in Z\}$

$\mathcal{A}_3 = \{Z \bigcup \{n-1\} / Z \in D_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1) \text{ and } 1, n-1 \notin Z\}$

$\mathcal{A}_4 = \{Z \bigcup \{n-2\} / Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1) \text{ and } 1 \in Z,\ n-3 \notin Z\}$

$\mathcal{A}_5 = \{Z \bigcup \{n\} / Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1) \text{ and } 1 \notin Z,\ n-3 \in Z\}$

$\mathcal{A}_6 = \{Z \bigcup \{n-1\} / Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1) \text{ and } 1, n-3 \notin Z\}$.

Proof. (i) Assume that $\mathcal{D}_r(C_{n-1}, i-1) = \phi$ and $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$. Then by Lemma 2.3 (iii), $n = 3q$ and $i = q$ for some $q \in \mathbb{N}$. So $\mathcal{D}_r(C_n, i) = \mathcal{D}_r(C_n, \frac{n}{3}) = \{\{1,4,\ldots,n-2\}, \{2,5,\ldots,n-1\}, \{3,6,\ldots,n\}\}$.

(ii) Assume that $\mathcal{D}_r(C_{n-3}, i-1) = \phi$ and $\mathcal{D}_r(C_{n-1}, i-1) \neq \phi$. Then by Lemma 2.3 (ii), $i = n$. We have $\mathcal{D}_r(C_n, i) = \mathcal{D}_r(C_n, n) = \{[n]\}$.

(iii) Suppose that $\mathcal{D}_r(C_{n-1}, i-1) \neq \phi$ and $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$. Let $Y \in \mathcal{A}_1$. Then either $Y = Z \bigcup \{n\}$ or $Y = Z \bigcup \{n-2\}$ for some $Z \in \mathcal{D}_r(C_{n-1}, i-1) \bigcap \mathcal{D}_r(C_{n-3}, i-1)$. In both cases the vertices labeled $n-3$ and 1 must belong to Z. Otherwise $Z \notin \mathcal{D}_r(C_{n-1}, i-1) \bigcap \mathcal{D}_r(C_{n-3}, i-1)$. Hence $Y \in \mathcal{D}_r(C_n, i)$. It follows that $\mathcal{A}_1 \subset \mathcal{D}_r(C_n, i)$. Let $Y \in \mathcal{A}_2$. Then there exists $Z \in \mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1)$ such that $Y = Z \bigcup \{n\}$ and $n-1 \in Z$. Clearly $Y = Z \bigcup \{n\} \in \mathcal{D}_r(C_n, i)$. Thus $\mathcal{A}_2 \subset \mathcal{D}_r(C_n, i)$. Suppose $Y \in \mathcal{A}_3$, then $Y = Z \cup \{n-1\}$ for some $Z \in \mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1)$ with $1, n-1 \notin Z$. In this case $Y = Z \bigcup \{n-1\} \in \mathcal{D}_r(C_n, i)$. Hence $\mathcal{A}_3 \subset \mathcal{D}_r(C_n, i)$. Suppose $Y \in \mathcal{A}_4$, then there exists $Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1)$ with $1 \in Z, n-3 \notin Z$ and $Y = Z \bigcup \{n-2\} \in \mathcal{D}_r(C_n, i)$. Hence $\mathcal{A}_4 \subset \mathcal{D}_r(C_n, i)$. Similarly we can show that $\mathcal{A}_5 \subset \mathcal{D}_r(C_n, i)$ and $\mathcal{A}_6 \subset \mathcal{D}_r(C_n, i)$. Thus $\bigcup\limits_{i=1}^{6} \mathcal{A}_i \subseteq \mathcal{D}_r(C_n, i)$.

Let $Y \in \mathcal{D}_r(C_n, i)$. Then at least one of the vertices labeled with $1, n, n-1, n-2$ must belong to Y since otherwise $Y \notin \mathcal{D}_r(C_n, i)$. It is enough to show that for each $Y \in \mathcal{D}_r(C_n, i)$, there exists k with $1 \leq k \leq 6$ such that $Y \in \mathcal{A}_k$. Here we consider six cases. Other cases are similar.

Case 1. $n \in Y$ and $1, n-1, n-2 \notin Y$.

Since $Y \in \mathcal{D}_r(C_n, i)$, the vertex $n-2$ is dominated by $n-3$ and so $n-3 \in Y$. Then there is an element $Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1)$ with $1 \notin Z, n-3 \in Z$ such that $Y = Z \bigcup \{n\}$ and $1, n-1, n-2 \notin Z$ .Thus $Y \in \mathcal{A}_5$.

Case 2. $1, n \in Y$ and $n-1, n-2 \notin Y$.
The vertex $n-2$ is dominated by $n-3$ and $n-3 \in Y$. In this case we can find $Z \in \mathcal{D}_r(C_{n-1}, i-1) \bigcap \mathcal{D}_r(C_{n-3}, i-1)$ such that $Y = Z \bigcup \{n\}$. In this case $Y \in \mathcal{A}_1$.
Case 3. $n, n-1 \in Y$ and $1, n-2 \notin Y$.
In this case there is an element $Z \in D_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1)$ with $n-1 \in Z$ and $Y = Z \bigcup \{n\}$. Thus $Y \in \mathcal{A}_2$.
Case 4. $1, n-2 \in Y$ and $n, n-1 \notin Y$.
Then there exists $Z \in D_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1)$ with $1 \in Z$ such that $Y = Z \bigcup \{n-2\}$. Hence $Y \in \mathcal{A}_4$.
Case 5. $n-1 \in Y$ and $1, n, n-2 \notin Y$.
Since $Y \in \mathcal{D}_r(C_n, i)$, the vertex $n-3 \notin Y$. In this case there is an element $Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1)$ such that $Y = Z \bigcup \{n-1\}$ and $1, n-3 \notin Z$. Also $1, n, n-2 \notin Z$. Hence $Y \in \mathcal{A}_6$.
Case 6. $n-2, n-1 \in Y$ and $1, n \notin Y$.
In this case there exists $Z \in \mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1)$ with $1, n-1 \notin Z$ such that $Y = Z \cup \{n-1\}$. Hence $Y \in \mathcal{A}_3$.
The following cases are not possible:

(a) $1 \in Y$ and $n, n-1, n-2 \notin Y$
(b) $n-2 \in Y$ and $1, n, n-1 \notin Y$
(c) $1, n-1 \in Y$ and $n, n-2 \notin Y$
(d) $n, n-2 \in Y$ and $1, n-1 \notin Y$
(e) $1, n-1, n-2 \in Y$ and $n \notin Y$
(f) $1, n, n-2 \in Y$ and $n-1 \notin Y$

In this case $Y \notin \mathcal{D}_r(C_n, i)$. From the above argument we can see that if $Y \in \mathcal{D}_r(C_n, i)$, there exists a positive integer k $(1 \leq k \leq 6)$ such that $Y \in \mathcal{A}_k$. Hence $\mathcal{D}_r(C_n, i) = \bigcup\limits_{k=1}^{6} \mathcal{A}_k$. □

Theorem 2.5. *If $\mathcal{D}_r(C_n, i)$ is the collection of restrained dominating sets of cycle C_n of cardinality i, then $|\mathcal{D}_r(C_n, i)| = |\mathcal{D}_r(C_{n-1}, i-1)| + |\mathcal{D}_r(C_{n-3}, i-1)|$.*

Proof. We consider the three cases in Theorem 2.4 and we rewrite in the following form

(i) If $\mathcal{D}_r(C_{n-1}, i-1) = \phi$ and $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$, then $\mathcal{D}_r(C_n, i) = \{Z_1 \bigcup \{n-2\}, Z_2 \bigcup \{n-1\}, Z_3 \bigcup \{n\} / 1 \in Z_1, 2 \in Z_2, 3 \in Z_3 \textit{ and } Z_1, Z_2, Z_3 \in \mathcal{D}_r(C_{n-3}, i-1)\}$.

(ii) If $\mathcal{D}_r(C_{n-3}, i-1) = \phi$ and $\mathcal{D}_r(C_{n-1}, i-1) \neq \phi$, then $\mathcal{D}_r(C_n, i) = \{Z \cup \{n\} / Z \in \mathcal{D}_r(C_{n-1}, i-1)\}$.

(iii) If $\mathcal{D}_r(C_{n-1}, i-1) \neq \phi$ and $\mathcal{D}_r(C_{n-3}, i-1) \neq \phi$, then $\mathcal{D}_r(C_n, i) = \bigcup\limits_{k=1}^{6} \mathcal{A}_k$.

Where $\mathcal{A}_1 = \{Z \bigcup \{n\}, Z \bigcup \{n-2\} / Z \in \mathcal{D}_r(C_{n-1}, i-1) \bigcap \mathcal{D}_r(C_{n-3}, i-1)\}$

$\mathcal{A}_2 = \{Z \bigcup \{n\} / Z \in \mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1) \textit{ and } n-1 \in Z\}$

$\mathcal{A}_3 = \{Z \bigcup \{n-1\} / Z \in D_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1) \textit{ and } 1, n-1 \notin Z\}$

$\mathcal{A}_4 = \{Z \bigcup \{n-2\} / Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1) \textit{ and } 1 \in Z,\ n-3 \notin Z\}$

$\mathcal{A}_5 = \{Z \bigcup \{n\} / Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1) \textit{ and } 1 \notin Z,\ n-3 \in Z\}$

$\mathcal{A}_6 = \{Z \bigcup \{n-1\} / Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1) \textit{ and } 1, n-3 \notin Z\}$.

We consider here three cases.
Case 1. For $n = 3q$ and $i = q$, from (i) we have $|\mathcal{D}_r(C_n, i)| = 3 = |\mathcal{D}_r(C_{n-3}, i-1)|$ and $|\mathcal{D}_r(C_{n-1}, i-1)| = 0$. Hence $|\mathcal{D}_r(C_n, i)| = |\mathcal{D}_r(C_{n-1}, i-1)| + |\mathcal{D}_r(C_{n-3}, i-1)|$.
Case 2. For $i = n$, from (ii) we have $|\mathcal{D}_r(C_n, i)| = 1 = |\mathcal{D}_r(C_{n-1}, i-1)|$ and $|\mathcal{D}_r(C_{n-3}, i-1)| = 0$. Hence $|\mathcal{D}_r(C_n, i)| = |\mathcal{D}_r(C_{n-1}, i-1)| + |\mathcal{D}_r(C_{n-3}, i-1)|$.
Case 3. For $n - 2\lfloor \frac{n-1}{3} \rfloor \leq i \leq n-2$, since $\mathcal{A}_k^{'}s$ are pairwise disjoint for $1 \leq k \leq 6$, from (iii) we have $|\mathcal{D}_r(C_n, i)| = \sum\limits_{k=1}^{6} |\mathcal{A}_k|$.
Claim 1. $|\mathcal{A}_1| = 2|\mathcal{D}_r(C_{n-1}, i-1) \bigcap \mathcal{D}_r(C_{n-3}, i-1)|$.
For any $Z \in \mathcal{D}_r(C_{n-3}, i-1) \bigcap \mathcal{D}_r(C_{n-1}, i-1)$, we have two sets $Z \bigcup \{n\}$ and $Z \bigcup \{n-2\}$ which are restrained dominating sets of cardinality i. Thus each element in $\mathcal{D}_r(C_{n-3}, i-1) \bigcap \mathcal{D}_r(C_{n-1}, i-1)$ is counted twice in $|\mathcal{A}_1|$. Hence $|\mathcal{A}_1| = 2|\mathcal{D}_r(C_{n-3}, i-1) \bigcap \mathcal{D}_r(C_{n-1}, i-1)|$.

Claim 2. $|\mathcal{A}_2| + |\mathcal{A}_3| = |\mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1)|$.
For any element $Z \in \mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1)$ we consider two possibilities:

(i) $1, n-1 \notin Z$.

(ii) $n-1 \in Z$.

For (i), we have the set $Z \bigcup \{n-1\}$ which is in $\mathcal{A}_3$. For (ii), we have the set $Z \bigcup \{n\}$ which is in $\mathcal{A}_2$. Thus in each of the possibilties, the element $Z \in \mathcal{D}_r(C_{n-1}, i-1)$ is counted exactly once, either in $\mathcal{A}_2$ or in $\mathcal{A}_3$ and $\mathcal{A}_k$'s are disjoint. Hence we have $|\mathcal{A}_2| + |\mathcal{A}_3| = |\mathcal{D}_r(C_{n-1}, i-1) - \mathcal{D}_r(C_{n-3}, i-1)|$.
Claim 3. $|\mathcal{A}_4| + |\mathcal{A}_5| + |\mathcal{A}_6| = |\mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1)|$.
For $Z \in \mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1)$, we consider the following possiblities:

(i) $1 \in Z$ but $n-3 \notin Z$.

(ii) $1 \notin Z$ but $n-3 \in Z$.

(iii) $1, n-3 \notin Z$.

The case when $1, n-3 \in Z$ is included as a part of claim 1. For each of the possibility we have the unique set $Z \bigcup \{n-2\}$ (for (i)), $Z \bigcup \{n\}$(for (ii)) and $Z \bigcup \{n-1\}$ (for(iii)). Thus each element in $\mathcal{D}_r(C_{n-3}, i-1) - \mathcal{D}_r(C_{n-1}, i-1)$ is counted exactly once in one of $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathcal{A}_6$. Hence we have the result. □

Theorem 2.6. *For any cycle C_n with $n \geq 6$, $D_r(C_n, x) = x[D_r(C_{n-1}, x) + D_r(C_{n-3}, x)]$ where $D_r(C_3, x) = 3x + x^3$, $D_r(C_4, x) = 4x^2 + x^4$ and $D_r(C_5, x) = 5x^3 + x^5$.*

Proof. Follows from Theorem 2.5. □

Theorem 2.7. *If $D_r(C_n, x)$ is the restrained domination polynomial of C_n, then the following holds:*

(i) *For every $n \in \mathbb{N}$, $d_r(C_n, n) = 1$.*

(ii) *For every $n \in \mathbb{N}$, $d_r(C_n, n-2) = n$.*

(iii) *For every $n \in \mathbb{N}$, $d_r(C_{3n}, n) = 3$.*

(iv) *For every $n \in \mathbb{N}$, $d_r(C_{3n+1}, n+1) = 3n+1$.*

(v) *For every natural number $n \geq 5$, $d_r(C_n, n-4) = \frac{n(n-5)}{2}$.*

Proof. (i) For any graph G, the only restrained dominating set of G of cardinality n is $V(G)$. It follows that $d_r(C_n, n) = 1$.
(ii) Since $\delta(C_n) = 2$, for any edge $e = uv$ the set $V(G) - \{u, v\}$ is a restrained dominating set of cardinality $n-2$. Hence $d_r(C_n, n-2) = |E(C_n)| = n$.
(iii) By Theorem 2.4, $\mathcal{D}_r(C_{3n}, n) = \{\{1, 4, 7, \ldots, 3n-2\}, \{2, 5, 8, \ldots, 3n-1\}, \{3, 6, 9, \ldots, 3n\}\}$. Hence $d_r(C_{3n}, n) = 3$.
(iv) We shall prove this by mathematical induction on n.
For $n = 1$, $\mathcal{D}_r(C_4, 2) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$ and hence $d_r(C_4, 2) = 4$. Thus the result is true for $n = 1$. Assume that the result is true for all natural numbers less than n. To prove that the result is true for n. By applying induction hypothesis, (iii) and Theorem 2.5, we have

$$\begin{aligned} d_r(C_{3n+1}, n+1) &= d_r(C_{3n}, n) + d_r(C_{3n-2}, n) \\ &= 3 + d_r(C_{3(n-1)+1}, (n-1) + 1) \\ &= 3 + 3(n-1) + 1 \\ &= 3n + 1. \end{aligned}$$

(v) We shall prove this result by mathematical induction on n.
For $n = 5$, $\mathcal{D}_r(C_5, 4) = \phi$ and $d_r(C_5, 4) = |\mathcal{D}_r(C_5, 4)| = 0$. Also $\frac{5(5-5)}{2} = 0$. The result is true for $n = 5$. For $n = 6$, $\mathcal{D}_r(C_6, 2) = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ and $d_r(C_6, 2) = 3$. Also $\frac{6(6-5)}{2} = 3$.

Thus the result is true for $n = 5, 6$. Assume that the reult is true for all natural numbers less than n. By induction hypothesis, (ii) and Theorem 2.5, we have

$$\begin{aligned} d_r(C_n, n-4) &= d_r(C_{n-1}, n-5) + d_r(C_{n-3}, n-5) \\ &= \frac{(n-1)(n-6)}{2} + (n-3) \\ &= \frac{n(n-5)}{2}. \end{aligned}$$

□

By using Theorem 2.5 we obtain $d_r(C_n, i)$ for $1 \leq n \leq 10$ in Table 1.

n \ i	1	2	3	4	5	6	7	8	9	10
3	3	0	1							
4	0	4	0	1						
5	0	0	5	0	1					
6	0	3	0	6	0	1				
7	0	0	7	0	7	0	1			
8	0	0	0	12	0	8	0	1		
9	0	0	3	0	18	0	9	0	1	
10	0	0	0	10	0	25	0	10	0	1

Table 1

Theorem 2.8. *Suppose that $n \geq 3$ and i is a positive integer satisfying the condition that $n - 2\lfloor \frac{n}{3} \rfloor \leq i \leq n$. Then the coefficient of $u^n v^i$ in the expansion of the function $f(u, v) = \dfrac{u^4v^2(4 + 3u^2 + v^2 + uv + u^2v^2)}{1 - uv - u^3v}$ is equal to $d_r(C_n, i)$.*

Proof. Set $f(u, v) = \sum_{n=4}^{\infty}\sum_{i=2}^{\infty} d_r(C_n, i)u^n v^i$. By the recursive formula for $d_r(C_n, i)$ in Theorem 2.5 we can write $f(u, v)$ in the following form

$$f(u, v) = \sum_{n=4}^{\infty}\sum_{i=2}^{\infty}(d_r(C_{n-1}, i-1) + d_r(C_{n-3}, i-1))u^n v^i.$$

$$= uv\sum_{n=4}^{\infty}\sum_{i=2}^{\infty} d_r(C_{n-1}, i-1)u^{n-1}v^{i-1} + u^3v\sum_{n=4}^{\infty}\sum_{i=2}^{\infty} d_r(C_{n-3}, i-1)u^{n-3}v^{i-1}$$

$$= uv(d_r(C_3, 1)u^3v + d_r(C_3, 3)u^3v^3) + uvf(u, v) + u^3v(d_r(C_1, 1)uv + d_r(C_2, 2)u^2v^2 + d_r(C_3, 1)u^3v + d_r(C_3, 3)u^3v^3) + u^3vf(u, v)$$

By substituting the values from Table 1, we have

$$f(u, v)(1 - uv - u^3v) = uv(3u^3v + u^3v^3) + u^3v(uv + u^2v^2 + 3u^3v + u^3v^3)$$

$$f(u, v) = \frac{u^4v^2(4 + 3u^2 + v^2 + uv + u^2v^2)}{1 - uv - u^3v}.$$

□

3 Conclusion

In this paper we have found a recurrence relation for the restrained domination polynomial of C_n. We have also found some properties of the coefficients of the restrained domination polynomial of cycles. In future we plan to investigate the polynomial for several other graphs.

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