Palestine Journal of Mathematics

Vol. 12(Special Issue II)(2023) , 65–72

FORK-DECOMPOSITION OF SOME TOTAL GRAPHS

A. Samuel Issacraj and J. Paulraj Joseph

Communicated by R. Kala

MSC 2010 Classifications: 05C70, 05C51, 05C76.

Keywords and phrases: Decomposition, Fork, Total graph

Abstract Let $G = (V, E)$ be a graph. *Fork* is a tree obtained by subdividing any edge of a star of size three exactly once. In this paper, we investigate a necessary and sufficient condition for the existence of fork-decomposition of some total graphs.

1 Introduction

We consider only simple, finite and undirected graphs. Let K_n denote the complete graph on n vertices and $K_{m,n}$ denote the complete bipartite graph with parts of sizes m and n. Let P_k denote the path of length $k-1$ and S_k denote the star of size $k-1$. A vertex of degree 1 is called a *pendant vertex* and the vertex adjacent to it is called a *support*. Terms not defined here are used in the sense of Bondy and Murty [4]. A decomposition of a graph G is a collection $\mathcal{C} = \{H_1, H_2, \dots, H_r\}$ of subgraphs of G such that the set $\{E(H_1), E(H_2), \dots, E(H_r)\}$ forms a partition of $E(G)$. If each H_i is isomorphic to a graph H, then $\mathcal{C}$ is called a $H-$decomposition of G. If a graph G admits a $H-$decomposition, then $|E(H)|$ divides $|E(G)|$.

Decomposition of arbitrary graphs into subgraphs of small size are assuming importance in the literature. There are several studies on the isomorphic decomposition of graphs into paths [8, 11], cycles [2], trees [3], stars [12], sunlet [1] etc. Also there are studies on the isomorphic decomposition of total graphs into P_4 [6]. The general problem of H-decompositions was proved to be NP-complete for any H of size greater than 2 by Dor and Tarsi [7].

A tree F obtained from the claw $K_{1,3}$ by subdividing one edge exactly once is called a *fork*. Since it resembles the graph model of human body in the stand-at-ease position, the vertices and edges are named as follows.
a - Head, ab - Neck, b - Throat, bc - Body, c - Hip, cd & ce - Legs, d & e - Feet as given in the Figure 1.

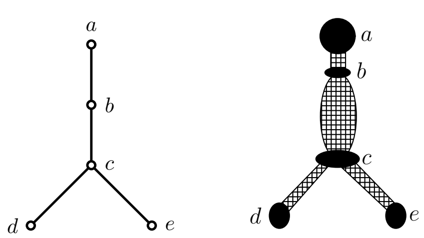

Figure 1. Fork

This graph was defined by Simone and Sassano in the name of *chair graph* in 1993, when they studied the stability number of bull and chair-free graphs [5]. In 2014, Barat and Gerbner [3] studied decomposition of 191-edge connected graphs which can be decomposed into unique trees of size 4 as a possible attempt to solve the following conjecture.
Conjecture 1 For each tree T, there exists a natural number k_T such that the following holds: if G is a k_T -edge-connected simple graph such that $|E(T)|$ divides $|E(G)|$, then G has a T-decomposition.

The edge-connectivity constants in the solved cases of Conjecture 1 are seemingly far from best possible. Very little is known about lower bounds. A tree is a fork if and only if its degree sequence is $(1, 1, 1, 2, 3)$. In this paper, we investigate the existence of fork-decomposition for some total graphs.

Since $|E(F)| = 4$, it follows that if G admits a fork-decomposition, then

$$|E(G)| \equiv 0 \ (mod\ 4) \tag{1.1}$$

Definition 1.1. The total graph of G, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds;

(i) x, y are in $V(G)$ and x is adjacent to y in G.

(ii) x, y are in $E(G)$ and x, y are adjacent in G.

(iii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

Remark 1.2. The number of edges in the total graph is $2|E(G)| + \frac{1}{2} \sum_{v \in V(G)} (d(v))^2$.

The following results are used in the subsequent sections.

Theorem 1.3. *[9] The complete bipartite graph $K_{m,n}$ is fork-decomposable if and only if $mn \equiv 0 \ (mod\ 4)$ except $K_{2,4i+2}$, $(i = 1, 2, ...)$.*

Theorem 1.4. *[9] The Complete graph K_n can be decomposed into forks if and only if $n = 8k$ or $n = 8k + 1$, for all $k \geq 1$*

Theorem 1.5. *[9]*

(i) For $m \geq 3$, $K_m \circ K_1$ is fork-decomposable if and only if $m \equiv 0, \ 7 \ (mod\ 8)$

(ii) For $m \geq 3$, $K_m \circ \overline{K_2}$ is fork-decomposable if and only if $m \equiv 0, \ 5 \ (mod\ 8)$

2 Total graph of paths, cycles and wheels

In this section, we investigate a necessary and sufficient condition for the existence of decomposition of Total graph of paths, cycles and wheel into forks.

Example 2.1. The fork-decomposition of $T(K_{1,3})$ is given in Figure 2.

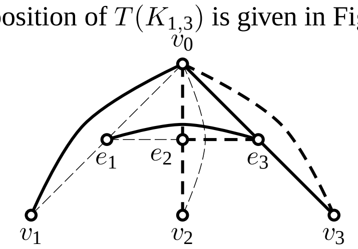

Figure 2. Fork-decomposition of $T(K_{1,3})$

Observation 2.2. $T(P_n)$ is not fork-decomposable, since the number of edges in $T(P_n)$ is odd.

Theorem 2.3. *$T(C_n)$ is fork-decomposable for all values of $n \geq 3$.*

Proof. Let the vertices of C_n be $v_1, v_2, \ldots, v_n$ and let the edges of C_n be $e_1, e_2, \ldots, e_n$ where $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_n v_1$. Then the vertices of $T(C_n)$ is given by $\{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n\}$ and $E(T(C_n)) = \{e_i e_{i+1}, v_i v_{i+1}, v_i e_i, v_i e_{i-1}\}$ where $1 \leq i \leq n$ and the subscripts are taken modulo n.

The number of edges in $T(C_n)$ is $4n$ and it satisfies the equation (1.1) for all values of n. Then a fork-decomposition of $T(C_n)$ is given by $\{e_i v_i, e_i v_{i+1}, e_i e_{i+1}, v_{i+1} v_{i+2}\}$ where $1 \leq i \leq n$ and the subscripts are taken modulo n. □

Theorem 2.4. *The graph $T(W_n)$ is fork-decomposable if and only if $n \equiv 0 \ (mod\ 8)$ or $n \equiv 7 \ (mod\ 8)$.*

Proof. The number of edges in $T(W_n)$ is $2(2n) + \frac{1}{2}(n.3^2 + 1.n^2) = \frac{n^2+17n}{2}$. If $T(W_n)$ is fork-decomposable, $\frac{n^2+17n}{2}$ is a multiple of 4. Then $n(n+17) \equiv 0\ (mod\ 8)$ which implies $n \equiv 0\ (mod\ 8)$ or $n \equiv -17\ (mod\ 8)$. Hence $n \equiv 0\ (mod\ 8)$ or $n \equiv 7\ (mod\ 8)$.

Now let us prove the converse part. Let the vertices of W_n be $\{u, v_1, v_2, \ldots, v_n\}$ where u is the central vertex and v_i's are the vertices of C_n in W_n. Let the edges of W_n be $\{e_i, f_i\}$ where $e_i = v_iv_{i+1}$, $f_i = v_iu$ for $1 \leq i \leq n$ and the subscripts are taken modulo n.

Assume that $n \equiv 0\ (mod\ 8)$. Consider the set of forks $F_1 = \{\{v_iv_{i+1}, v_ie_i, v_if_i, f_iu\}/1 \leq i \leq n\}$ and $F_2 = \{\{e_iv_{i+1}, e_ie_{i+1}, e_if_{i+1}, v_{i+1}u\}/1 \leq i \leq n\}$. Here the subscripts are taken modulo n. The induced subgraph $\langle\{f_1, f_2, \ldots, f_n, e_1, e_2, \ldots, e_n\}\rangle$ obtained after removing F_1 and F_2 from $T(W_n)$ is isomorphic to $K_n \circ K_1$ which is fork-decomposable by Theorem 1.5.

Assume that $n \equiv 7\ (mod\ 8)$. Consider the set of forks $F_3 = \{\{v_iv_{i+1}, v_ie_i, v_if_i, f_iu\}/1 \leq i \leq n\}$ and $F_4 = \{\{e_iv_{i+1}, e_ie_{i+1}, e_if_{i+1}, v_{i+1}u\}/1 \leq i \leq n\}$. Here the subscripts are taken modulo n. The induced subgraph $\langle\{f_1, f_2, \ldots, f_n, e_1, e_2, \ldots, e_n\}\rangle$ obtained after removing F_3 and F_4 from $T(W_n)$ is isomorphic to $K_n \circ K_1$ which is fork-decomposable by Theorem 1.5. □

3 Total graphs of Star and its subdivision graph

In this section, we investigate a necessary and sufficient condition for the existence of fork-decomposition of total graphs of star and its subdivision graph.

Theorem 3.1. *$T(K_{1,n})$ is fork-decomposable if and only if $n \equiv 0\ (mod\ 8)$ or $n \equiv 3\ (mod\ 8)$.*

Proof. The number of edges in $T(K_{1,n})$ is $2n + \frac{1}{2}(1.n^2 + n.1^2) = \frac{n(n+5)}{2}$. If $T(K_{1,n})$ is fork-decomposable, then $n(n+5) \equiv 0\ (mod\ 8)$ which implies $n \equiv 0\ (mod\ 8)$ or $n+5 \equiv 0\ (mod\ 8)$. Hence $n \equiv 0\ (mod\ 8)$ or $n \equiv 3\ (mod\ 8)$.

Conversely, assume that $n \equiv 0\ (mod\ 8)$ or $n \equiv 3\ (mod\ 8)$. Let the vertices of $K_{1,n}$ be $\{v_0, v_1, v_2, \ldots, v_n\}$ and let v_0 be the vertex of degree n. Let the edges of $K_{1,n}$ be $\{e_1, e_2, \ldots, e_n\}$. Then the vertex set of $T(K_{1,n})$ is given by $\{v_0, v_1, \ldots, v_n, e_1, e_2, \ldots, e_n\}$ and the edge set of $T(K_{1,n})$ is given by $\{v_iv_0, v_ie_i, e_iv_0, e_ie_j\}$ where $1 \leq i, j \leq n$ and $i \neq j$.

For $n = 3$, the fork decomposition of $T(K_{1,3})$ is given in Figure 2. For $n = 8$, the induced subgraph obtained by removing the cycle $e_1e_2 \ldots e_8e_1$ from the induced subgraph $\langle e_1, e_2, \ldots, e_8\rangle$ is isomorphic to $\overline{C_8}$ which is fork-decomposable by Figure 3.

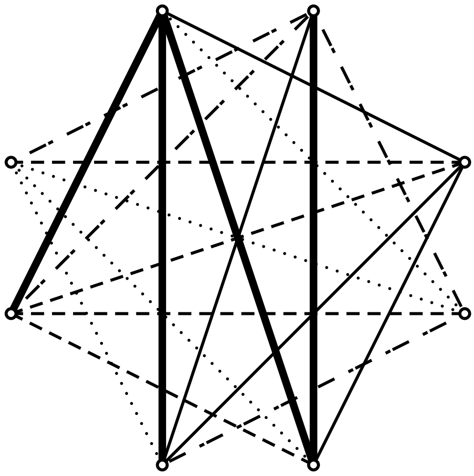

Figure 3. Fork-decomposition of $\overline{C_8}$

The fork-decomposition of the subgraph obtained after removing $\overline{C_8}$ from $T(K_{1,8})$ is given by $\{e_iv_0, e_iv_i, e_ie_{e+1}, v_0v_{i+1}\}$ for $1 \leq i \leq 8$ and the subscripts are taken modulo 8.

Assume that $n \equiv 0\ (mod\ 8)$. Then the induced subgraph $\langle\{v_0, e_i, e_{i+1}, e_{i+1}, \ldots, e_{i+7}, v_i, v_{i+1}, \ldots, v_{i+8}\}\rangle$ where $i \equiv 1\ (mod\ 8)$ is isomorphic to $\frac{n}{8}$ copies of $T(K_{1,8})$ which is fork-decomposable. After removing $\frac{n}{8}$ copies of $T(K_{1,8})$, the induced subgraph $\langle\{e_i/1 \leq i \leq n\}\rangle$ is decomposable into $\binom{n}{2}$ copies of $K_{8,8}$ which is fork-decomposable by Theorem 1.3. Thus,

$E(T(K_{1,n})) = \underbrace{E(T(K_{1,8})) \cup \ldots \cup E(T(K_{1,8}))}_{\frac{n}{8}\ times} \cup \underbrace{E(K_{8,8}) \cup \ldots \cup E(K_{8,8})}_{\binom{n}{2}\ times}$. Hence $T(K_{1,n})$ is fork-decomposable.

Assume that $n \equiv 3\ (mod\ 8)$. Then the induced subgraph $\langle\{v_0, v_1, v_2, v_3, e_1, e_2, e_3\}\rangle$ is isomorphic to $T(K_{1,3})$ and the induced subgraph $\langle\{v_0, v_4, v_5, \ldots, v_n, e_4, e_5, \ldots, e_n\}\rangle$ is isomorphic to $T(K_{1,n-3})$ which is fork-decomposable. After removing $T(K_1, 3)$ and $T(K_{1,n-3})$, the induced subgraph $\langle\{e_1, e_2, \ldots, e_n\}\rangle$ is isomorphic to $K_{3,n-3}$ which is fork-decomposable by Theorem 1.3. Thus,

$E(T(K_{1,n})) = E(T(K_{1,3})) \cup E(T(K_{1,n-3})) \cup K_{3,n-3}$.

Hence $T(K_{1,n})$ is fork-decomposable. ◻

Definition 3.2. A subdivision graph of a graph G, denoted by $S(G)$ is the graph obtained from G by subdividing each edge exactly once.

Theorem 3.3. *$T(S(K_{1,n}))$ is fork-decomposable if and only if $n \equiv 0\ (mod\ 8)$ or $n \equiv 3\ (mod\ 8)$.*

Proof. Let G_n be the graph $S(K_{1,n})$. The number of edges in $T(G_n)$ is $2(2n) + \frac{1}{2}\,(1.n^2 + n.2^2 + n.1^2)$. If $T(G_n)$ is fork-decomposable, then $\frac{n^2+13n}{2}$ is a multiple of 4. This implies $n(n+13) \equiv 0\ (mod\ 8)$. Hence $n \equiv 0\ (mod\ 8)$ or $n \equiv 3\ (mod\ 8)$.

Now let us prove the converse part. Let the vertices of G_n be $v_0, v_1, \ldots, v_n, u_1, u_2, \ldots, u_n$, where u_i's are the pendant vertices and v_i $(i \neq 0)$ are the support vertices to the corresponding u_i's. Here $deg(v_0) = n$. Let the edges of G_n be e_i, f_i where $e_i = v_0v_i$, $f_i = u_iv_i$ and $1 \le i \le n$. Then the vertex set of $T(G_n)$ is $\{v_o, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ and the edge set of $T(G_n)$ is $\{v_0e_i, v_0v_i, e_iv_i, e_if_i, e_ie_j, v_if_i, v_iu_i, f_iu_i\}$ where $1 \le i, j \le n$ and $i \neq j$.

For $n = 3$, a fork-decomposition of $T(G_3)$ is given by $\{v_iv_0, v_iu_i, v_ie_i, e_ie_{i+1}\}$ and $\{f_iu_i, f_iv_i, f_ie_i, e_iv_0\}$ for $1 \le i \le n$ and the subscripts are taken modulo n. For $n = 8$, the induced subgraph $\langle\{e_1, e_2, \ldots, e_8\}\rangle$ after removing the cycle $e_1, e_2, \ldots, e_8, e_1$ is isomorphic to $\overline{C_8}$ which is fork-decomposable by Figure 3. Then a fork-decomposition of $T(G_8) - \overline{C_8}$ is given by $\{v_ie_i, v_iv_0, v_iu_i, u_if_i\}$ and $\{e_iv_0, e_ie_{i+1}, e_if_i, f_iv_i\}$ for $i = 1, 2\ldots, 8$ and the subscripts are taken modulo 8.

Assume that $n \equiv 0\ (mod\ 8)$. The induced subgraph $\langle\{v_0, v_i, v_{i+1}, \ldots v_{i+7}, u_i, u_{i+1}, \ldots, u_{i+7}, e_i, e_{i+1}, \ldots, e_{i+7}, f_i, f_{i+1}, \ldots, f_{i+7}\}\rangle$ is isomorphic to $\frac{n}{8}$ copies of $T(G_8)$ for $i \equiv 1\ (mod\ 8)$. After removing $\frac{n}{8}$ copies of $T(G_8)$, the induced subgraph $\langle\{e_i/1 \le i \le n\}\rangle$ is decomposable into $\binom{n}{2}$ copies of $K_{8,8}$ which is fork-decomposable by Theorem 1.3. Thus,

$E(T(K_{1,n})) = \underbrace{E(T(S(K_{1,8}))) \cup \ldots \cup E(T(S(K_{1,8})))}_{\frac{n}{8}\ times} \cup \underbrace{E(K_{8,8}) \cup \ldots \cup E(K_{8,8})}_{\binom{n}{2}\ times}$. Hence $T(S(K_{1,n}))$ is fork-decomposable.

Assume that $n \equiv 3\ (mod\ 8)$. The induced subgraph $\langle\{v_0, v_1, v_2, v_3, u_1, u_2, u_3, e_1, e_2, e_3, f_1, f_2, f_3\}\rangle$ is isomorphic to $T(S(K_{1,3}))$ and the induced subgraph $\langle\{v_0, v_4, v_5, \ldots, v_n, u_4, u_5, \ldots, u_n, e_4, e_5, \ldots, e_n, f_4, f_5, \ldots, f_n\}\rangle$ is isomorphic to $T(S(K_{1,n-3}))$. Hence $T(S(K_{1,3}))$ and $T(S(K_{1,n-3}))$ are fork-decomposable, since $n \equiv 3\ (mod\ 8)$. After removing $T(S(K_1, 3))$ and $T(S(K_{1,n-3}))$, the induced subgraph $\langle\{e_1, e_2, \ldots, e_n\}\rangle$ is isomorphic to $K_{3,n-3}$ which is fork-decomposable by Theorem 1.3. Thus $E(T(S(K_{1,n}))) = E(T(S(K_{1,3}))) \cup E(T(K_{1,n-3})) \cup K_{3,n-3}$. Hence $T(S(K_{1,n}))$ is fork-decomposable. ◻

4 Total graphs of Bi-stars

In this section, we investigate a necessary and sufficient condition for the existence of fork-decomposition of Total graph of Bi-stars.

Definition 4.1. Bi-star $B_{m,n}$ is the graph obtained by joining the center vertices of $K_{1,m}$ and $K_{1,n}$ by an edge.

Remark 4.2. Number of edges in Total graph of $B_{m,n}$ is $2(m+n+1) + \frac{1}{2}(m + m^2 + 1 + 2m + n + n^2 + 1 + 2n) = \frac{1}{2}(4m + 4n + 4 + m^2 + n^2 + 3m + 3n + 2) = \frac{1}{2}(m(m+7) + n(n+7) + 6)$.

Theorem 4.3. *$T(B_{m,n})$ is fork-decomposable if and only if it satisfies any one of the following conditions:*

(i) $m \equiv 2, 7\ (mod\ 8)$ *and* $n \equiv 0, 1\ (mod\ 8)$

(ii) $m \equiv 3, 6\ (mod\ 8)$ *and* $n \equiv 4, 5\ (mod\ 8)$

(iii) $m \equiv 0, 1\ (mod\ 8)$ *and* $n \equiv 2, 7\ (mod\ 8)$

(iv) $m \equiv 4, 5\ (mod\ 8)$ *and* $n \equiv 3, 6\ (mod\ 8)$.

Proof. Let $V(B_{m,n}) = \{v_0, v_1, v_2, \ldots, v_n, u_0, u_1, u_2, \ldots, u_m\}$ where $v_i's(1 \leq i \leq n)$ are pendant vertices wit support v_0 and $u_j's(1 \leq j \leq m)$ are pendant vertices with support u_0. Let $E(B_{m,n}) = \{e_i, f_j, g_1\ /\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$ where $e_i = v_0 v_i; 1 \leq i \leq n,\ f_j = u_0 u_j$ where $1 \leq j \leq m$ and $g_1 = v_0 u_0$.

Then $V(T(B_{m,n})) = \{v_0, v_i, u_0, u_j, e_i, f_j, g_1\ /\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and $E(T(B_{m,n})) = \{v_0 v_i, v_0 e_i, v_i e_i\ /\ 1 \leq i \leq n\} \cup \{e_i e_j\ /\ i \neq j, 1 \leq i, j \leq n\} \cup \{f_i f_j\ /\ i \neq j, 1 \leq i, j \leq m\}$ $\cup \{v_0 g_1, u_0 g_1, v_0 u_0\} \cup \{u_0 u_j, u_0 f_j, f_j u_j\ /\ 1 \leq j \leq m\} \cup \{g_1 e_i, g_1 f_j\ /\ 1 \leq i \leq n, 1 \leq j \leq m\}$.

The number of edges in $T(B_{m,n})$ is $\frac{1}{2}[m(m+7) + n(n+7) + 6]$ If the graph $T(B_{m,n})$ is fork-decomposable, then $m(m+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$.

If $m \equiv 0\ (mod\ 8)$, then $m = 8a$, where a is any arbitrary integer. Hence $8a(8a+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$. Since $8a(8a+7) \equiv 0\ (mod\ 8)$, $n^2 + 7n + 6 \equiv 0\ (mod\ 8)$. Thus $(n+1)(n+6) \equiv 0\ (mod\ 8)$ which implies that $n \equiv 7\ (mod\ 8)$ or $n \equiv 2\ (mod\ 8)$.

If $m \equiv 1\ (mod\ 8)$, then $m = 8a+1$, where a is any arbitrary integer. Then $(8a+1)(8a+1+7)+n(n+7)+6 \equiv 0\ (mod\ 8)$ which implies that $(8a)(8a+8)+8a+8+n^2+7n+6 \equiv 0\ (mod\ 8)$. Thus $(n+1)(n+6) \equiv 0\ (mod\ 8)$, which implies that $n \equiv 7\ (mod\ 8)$ or $n \equiv 2\ (mod\ 8)$.

If $m \equiv 2\ (mod\ 8)$, then $m = 8a+2$, where a is any arbitrary integer. Then $(8a+2)(8a+2+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$ which implies that $(8a)(8a+9) + 2(8a) + 18 + n^2 + 7n + 6 \equiv 0\ (mod\ 8)$. Thus $n(n+7) \equiv 0\ (mod\ 8)$, which implies that $n \equiv 0\ (mod\ 8)$ or $n \equiv 1\ (mod\ 8)$.

If $m \equiv 3\ (mod\ 8)$, then $m = 8a+3$, where a is any arbitrary integer. Then $(8a+3)(8a+3+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$ which implies that $(8a)(8a+10) + 3(8a) + 30 + n^2 + 7n + 6 \equiv 0\ (mod\ 8)$. Thus $n^2 + 7n + 12 + 24 \equiv 0\ (mod\ 8)$ which implies that $(n+3)(n+4) \equiv 0\ (mod\ 8)$. Hence $n \equiv 5\ (mod\ 8)$ or $n \equiv 4\ (mod\ 8)$.

If $m \equiv 4\ (mod\ 8)$, then $m = 8a+4$, where a is any arbitrary integer. Then $(8a+4)(8a+4+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$ which implies that $(8a)(8a+11) + 4(8a) + 44 + n^2 + 7n + 6 \equiv 0\ (mod\ 8)$. Thus $n^2 + 7n + 10 \equiv 0\ (mod\ 8)$ which implies that $(n+2)(n+5) \equiv 0\ (mod\ 8)$. Hence $n \equiv 6\ (mod\ 8)$ or $n \equiv 3\ (mod\ 8)$.

If $m \equiv 5\ (mod\ 8)$, then $m = 8a+5$, where a is any arbitrary integer. Then $(8a+5)(8a+5+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$ which implies that $(8a)(8a+12) + 5(8a) + 60 + n^2 + 7n + 6 \equiv 0\ (mod\ 8)$. Thus $n^2 + 7n + 10 \equiv 0\ (mod\ 8)$ which implies that $(n+2)(n+5) \equiv 0\ (mod\ 8)$. Hence $n \equiv 6\ (mod\ 8)$ or $n \equiv 3\ (mod\ 8)$.

If $m \equiv 6\ (mod\ 8)$, then $m = 8a+6$, where a is any arbitrary integer. Then $(8a+6)(8a+6+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$ which implies that $(8a)(8a+13) + 6(8a) + 78 + n^2 + 7n + 6 \equiv 0\ (mod\ 8)$. Thus $n^2 + 7n + 12 \equiv 0\ (mod\ 8)$ which implies that $(n+3)(n+4) \equiv 0\ (mod\ 8)$. Hence $n \equiv 5\ (mod\ 8)$ or $n \equiv 4\ (mod\ 8)$.

If $m \equiv 7\ (mod\ 8)$, then $m = 8a+7$, where a is any arbitrary integer. Then $(8a+7)(8a+7+7) + n(n+7) + 6 \equiv 0\ (mod\ 8)$ which implies that $(8a)(8a+14) + 7(8a) + 98 + n^2 + 7n + 6 \equiv 0\ (mod\ 8)$. Thus $n^2 + 7n \equiv 0\ (mod\ 8)$ which implies that $n(n+7) \equiv 0\ (mod\ 8)$. Hence $n \equiv 0\ (mod\ 8)$ or $n \equiv 1\ (mod\ 8)$.

Now let us prove the converse part. Since the graph $T(B_{m,n})$ is same as the graph $T(B_{n,m})$, it is enough to prove the result for the first two cases alone.

Case 1. $m \equiv 2, 7\ (mod\ 8)$ and $n \equiv 0, 1\ (mod\ 8)$.

Sub case 1a. $m \equiv 2\ (mod\ 8)$ and $n \equiv 0\ (mod\ 8)$.

First let us prove the result for $m = 2$ and $n = 8$. The induced subgraph $\langle\{e_i/i = 1, 2, \ldots, 8\}\rangle$ is isomorphic to K_8 which is fork-decomposable by Theorem 1.4. The fork-decomposition of the subgraph obtained after removing the above induced subgraph is given by $\{f_2 g_1,\ f_2 u_0,\ f_2 f_1,\ u_0 u_1\}$, $\{f_1 u_1,\ f_1 u_0,\ f_1 g_1,\ g_1 v_0\}$, $\{u_0 v_0,\ u_0 g_1,\ u_0 u_2,\ u_2 f_2\}$, $\{e_i v_0,\ e_i v_i,\ e_i g_1,\ v_0 v_{i+1}\}$ where $1 \leq i \leq 8$ and the subscripts are taken modulo 8.

For $m > 2$ and $n > 8$, the induced subgraph $\langle\{u_0, f_m, f_{m-1}, v_m, v_{m-1}, g_1, v_0, e_n, e_{n-1}, \ldots, e_{n-7},\ v_n, v_{n-1}, \ldots, v_{n-7}\}\rangle$ is isomorphic to $T(B_{2,8})$ which is fork-decomposable. The edge induced subgraph $\langle\{e_ie_k, e_iv_i\}\rangle$ where $i \neq k$, and $i, k = 1, 2, \ldots, n-8$ and the edge induced subgraph $\langle\{f_jf_l, f_ju_j\}\rangle$ where $j \neq l$, and $j, l = 1, 2, \ldots, m-2$ is isomorphic to $K_{n-8} \circ K_1$ and $K_{m-2} \circ K_1$ respectively which are fork-decomposable by Theorem 1.5. Consider the induced subgraph obtained after removing the above forks. The induced subgraphs $\langle\{e_i/i = 1, 2, \ldots, n-8\}\rangle$ and $\langle\{f_j/j = 1, 2, \ldots, m-2\}\rangle$ are respectively isomorphic to $K_{8,n-8}$ and $K_{2,m-2}$ which are fork-decomposable by Theorem 1.3. The remaining subgraph can be decomposed into $\frac{n-8}{8} + \frac{m-2}{8} = \frac{m+n-10}{8}$ copies of H_1 given in figure 4.

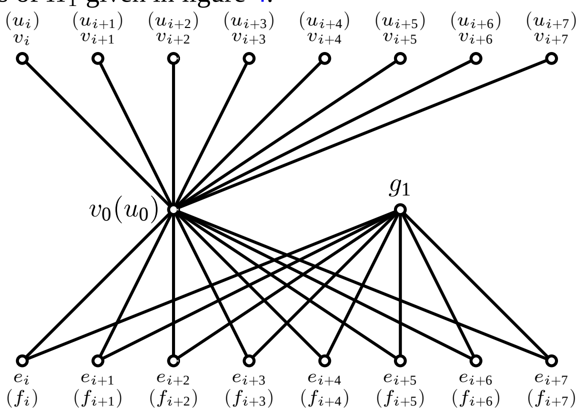

Figure 4. H_1

The fork-decomposition of the graph H_1 is given by $\{v_0v_i,\ v_0v_{i+5},\ v_0e_i,\ e_ig_1\}$ where $i = 1, 2, 3$, $\{v_0v_i,\ v_0e_i,\ v_0e_{i+2},\ e_ig_1\}$ where $i = 4, 5$, $\{g_1e_6,\ g_1e_7,\ g_1e_8,\ e_8v_0\}$ $\{u_0u_i,\ u_0u_{i+5},\ u_0f_i,\ f_ig_1\}$ where $i = 1, 2, 3$, $\{u_0u_i,\ u_0f_i,\ u_0f_{i+2},\ f_ig_1\}$ where $i = 4, 5$, $\{g_1f_6,\ g_1f_7,\ g_1f_8,\ f_8u_0\}$. Hence $T(B_{m,n})$ is fork-decomposable.

Sub case 1b. $m \equiv 2\ (mod\ 8)$ and $n \equiv 1\ (mod\ 8)$.

Let us prove the result for $m = 2$ and $n = 9$. The induced subgraph $\langle\{e_i\}\rangle$ is isomorphic to K_9 which is fork-decomposable by Theorem 1.4. The fork-decomposition of the subgraph obtained after removing the above induced subgraph is given by $\{f_2g_1,\ f_2u_0,\ f_2f_1,\ u_0u_1\}$, $\{f_1u_1,\ f_1u_0,\ f_1g_1,\ g_1v_0\}$, $\{u_0v_0,\ u_0g_1,\ u_0u_2,\ u_2f_2\}$, $\{e_iv_0,\ e_iv_i,\ e_ig_1,\ v_0v_{i+1}\}$ where $1 \leq i \leq 9$ and the subscripts are taken modulo 9.

Now assume that $m \equiv 2\ (mod\ 8)$ and $n \equiv 1\ (mod\ 8)$. The induced subgraph $\langle\{u_0, f_m, f_{m-1},\ v_m, v_{m-1},\ g_1, v_0,\ e_n, e_{n-1}, \ldots, e_{n-8},\ v_n, v_{n-1}, \ldots, v_{n-8}\}\rangle$ is isomorphic to $T(B_{2,9})$ which is fork-decomposable. The edge induced subgraph $\langle\{e_ie_k, e_iv_i\}\rangle$ where $i \neq k$, and $i, k = 1, 2, \ldots, n-9$ and the edge induced subgraph $\langle\{f_jf_l, f_ju_j\}\rangle$ where $j \neq l$, and $j, l = 1, 2, \ldots, m-2$ are respecitvely isomorphic to $K_{n-9} \circ K_1$ and $K_{m-2} \circ K_1$ which are fork - decomposable by Theorem 1.5. Consider the induced subgraph obtained after removing the above forks. The induced subgraph $\langle\{e_i/i = 1, 2, \ldots, n-9\}\rangle$ and $\langle\{f_j/j = 1, 2, \ldots, m-2\}\rangle$ are respectively isomorphic to $K_{9,n-9}$ and $K_{2,m-2}$ which are fork-decomposable by Theorem 1.3. The remaining subgraph can be decomposed into $\frac{n-9}{8} + \frac{m-2}{8} = \frac{m+n-11}{8}$ copies of H_1 given in figure 4, which is fork-decomposable. Hence $T(B_{m,n})$ is fork-decomposable.

Sub case 1c. $m \equiv 7\ (mod\ 8)$ and $n \equiv 0, 1\ (mod\ 8)$.

First let us prove the result for $m = 7$ and $n \equiv 0, 1\ (mod\ 8)$. The induced subgraph $\langle\{u_0, f_7, f_6, v_7, v_6, g_1, v_0, e_1, e_2, \ldots, e_n, v_1, v_2, \ldots, v_n\}\rangle$ is isomorphic to $T(B_{2,n})$ which is fork-decomposable. The induced subgraph $\langle\{u_0, f_3, f_4, f_5, u_3, u_4, u_5\}\rangle$ is isomorphic to $T(K_{1,3})$ which is fork-decomposable by Theorem 3.1. Consider the induced subgraph obtained after removing the above subgraphs $T(B_{2,n})$ and $T(K_{1,3})$. Consider the collection of forks $\{f_1g_1,\ f_1f_6,\ f_1f_7,\ g_1f_5\}$, $\{g_1f_2,\ g_1f_3,\ g_1f_4,\ f_2f_1\}$, $\{f_2f_4,\ f_2f_5,\ f_2u_2,\ u_2u_0\}$, $\{u_0u_1,\ u_0f_2,\ u_0f_1,\ f_1f_4\}$, $\{f_1u_1,\ f_1f_3,\ f_1f_5,\ f_3f_2\}$. Consider the induced subgraph obtained after removing above subgraphs and collection of forks. The induced subgraph thus obtained $\{f_2, f_3, \ldots, f_7\}$ is isomorphic to $K_{2,4}$ which is fork-decomposable by Theorem 1.3. Hence $T(B_{7,n})$ is fork-decomposable.

For $m > 7$, the induced subgraph $\langle\{u_0,\ f_m, f_{m-1}, \ldots, f_{m-6}\ v_m, v_{m-1}, \ldots, v_{m-6}\ g_1, v_0,\ e_1, e_2, \ldots, e_n,\ v_1, v_2, \ldots, v_n\}\rangle$ is isomorphic to $T(B_{7,n})$ which is fork-decomposable. The edge induced subgraph $\langle\{f_jf_l,\ f_ju_j\}\rangle$ where $j \neq l$, and $j, l = 1, 2, \ldots, m-7$ is isomorphic to $K_{m-7} \circ K_1$ which is fork-decomposable by Theorem 1.5. Consider the induced subgraph obtained after removing the above forks. The induced subgraph $\langle\{f_j/j = 1, 2, \ldots, m-7\}\rangle$ is isomorphic to $K_{7,m-7}$ which is fork-decomposable by Theorem 1.3. The remaining subgraph can be decomposed into $\frac{m-7}{8}$ copies of H_1 given in figure 4, which is fork-decomposable. Hence $T(B_{m,n})$ is fork-decomposable.

Case 2. $m \equiv 3, 6\ (mod\ 8)$ and $n \equiv 4, 5\ (mod\ 8)$.

First we shall prove the result for $T(B_{3,4})$, $T(B_{3,5})$, $T(B_{6,4})$, $T(B_{6,5})$. After proving these cases, the general case can be proved by repeating the above process. Consider $T(B_{3,4})$. The induced subgraphs $\langle\{u_0, u_1, u_2, u_3, f_1, f_2, f_3\}\rangle$ and $\langle\{v_0, v_1, v_2, v_3, e_1, e_2, e_3\}\rangle$ are isomorphic to 2 copies of $T(K_{1,3})$ which is fork-decomposable by Theorem 3.1. The remaining subgraph is fork-decomposable as follows: $\{g_1f_1,\ g_1f_2,\ g_1u_0,\ u_0v_0\}$, $\{g_1e_1,\ g_1e_2,\ g_1e_3,\ e_1e_4\}$, $\{v_0g_1,\ v_0e_4,\ v_0v_4,\ e_4e_3\}$, $\{e_4v_4,\ e_4e_3,\ e_4g_1,\ g_1f_3\}$.

Consider $T(B_{3,5})$. The induced subgraphs $\langle\{u_0, u_1, u_2, u_3, f_1, f_2, f_3\}\rangle$ and $\langle\{v_0, v_1, v_2, v_3, e_1, e_2, e_3\}\rangle$ are isomorphic to 2 copies of $T(K_{1,3})$ which is fork-decomposable by Theorem 3.1. The remaining subgraph is fork-decomposable as follows: $\{e_ie_4,\ e_ie_5,\ e_ig_1,\ g_1f_i\}$, where $i = 1, 2, 3$, $\{e_4v_4,\ e_4v_0,\ e_4g_1,\ v_0u_0\}$, $\{e_5e_4,\ e_5v_5,\ e_5v_0,\ v_0v_4\}$, $\{g_1u_0,\ g_1e_5,\ g_1v_0,\ v_0v_5\}$.

Consider $T(B_{6,4})$. The induced subgraph $\langle\{g_1, u_0, u_1, u_2, u_3, f_1, f_2, f_3, v_0, v_1, v_2, v_3, v_4, e_1, e_2, e_3, e_4\}\rangle$ is isomorphic to $T(B_{3,4})$ which is fork-decomposable. The induced subgraph $\langle\{u_0, f_4, f_5, f_6, u_4, u_5, u_6\}\rangle$ is isomorphic to $T(K_{1,3})$ which is fork-decomposable by Theorem 3.1. The induced subgraph $\langle\{g_1, f_1, f_2, \ldots, f_6\}\rangle$ obtained after removing above subgraphs $T(B_{3,4})$ and $T(K_{1,3})$ is isomorphic to $K_{3,4}$ which is fork-decomposable by Theorem 1.3.

Consider $T(B_{6,5})$. The induced subgraph $\langle\{g_1,\ u_0, u_1, u_2, u_3,\ f_1, f_2, f_3,\ v_0, v_1, v_2,\ v_3, v_4, v_5\ e_1, e_2, e_3, e_4, e_5\}\rangle$ is isomorphic to $T(B_{3,5})$ which is fork-decomposable. The induced subgraph $\langle\{u_0, f_4, f_5, f_6, u_4, u_5, u_6\}\rangle$ is isomorphic to $T(K_{1,3})$ which is fork-decomposable by Theorem 3.1. The induced subgraph $\langle\{g_1, f_1, f_2, \ldots, f_6\}\rangle$ obtained after removing above subgraphs $T(B_{3,5})$ and $T(K_{1,3})$ is isomorphic to $K_{3,4}$ which is fork-decomposable by Theorem 1.3. □

5 Conclusion

In this paper, we have investigated the existence of fork-decomposition of some total graphs. A study on the fork-decomposition of product graphs and some more total graphs is finalized and will appear as a separate paper.

References

[1] Abolape D. Akwu, Deborah O.A. Ajayi, *Sunlet Decomposition of Certain Equipartite Graphs,* Int. J. Comb., (2013,) Article ID 907249, 4 pages.

[2] Alspach, B., Gavlas, H., *Cycle decomposition of K_n and $K_n - I$,* J .Combin. Theory (B), **81** (2001) 77-99.

[3] Barat, J., Gerbner, D., *Edge decomposition of graph into copies of tree with four edges,* Electron. J. Combin., **21 (1)** (2014).

[4] Bondy, J.A., Murty, U.S.R. , *Graph Theory,* Springer International Edition, 2008.

[5] Caterina De Simone and Antonio Sassano, *Stability number of bull and chair-free graphs,* Discrete Appl. Math., **41** (1993) 121-129.

[6] Chitra Devi P., Paulraj Joseph J., *P_4 Decomposition of Total Graphs,* J. Discrete Math. Sci. Cryptogr., **17 (5-6),** (2014) 473-498.

[7] Dor, D., Tarsi, M., *Graph-decomposition is NP-complete: a complete proof of Holyer's conjecture,* SIAM J. Comput., **26** (1997) 1166–1187.

[8] Ming-qing Zhai, Chang-hong Lu, *Path Decomposition of Graph with given path length,* Acta Math. Appl. Sin. Engl. Ser., **22, (4)** (2006) 633-638.

[9] Paulraj Joseph, J., Samuel Issacraj, A., *Fork-decomposition of graphs,* Pre-Conference Proceedings of the International Conference on Discrete Mathematics, ISBN:978-93-91077-53-2, (2022) 426-431.

[10] Richard Hammack, Wilfied Imrich, Samndi Klavzar, *Handbook of Product Graphs, Second Edition,* Taylor and Francis Group, (2011).

[11] Sunil Kumar, C., *On P_4-decomposition of Graphs,* Taiwanese J. Math., **7(4)** (2003) 657-664.

[12] Yang Zhao, Baoyindureng Wu, *Star decomposition of graphs,* Discrete Math. Algorithms Appl., **7(2)** (2015) 1550016 (9 pages). Resource. http://mathworld.wolfram.com/ForkGraph.html

Author information

A. Samuel Issacraj, Research Scholar, Reg. No. 17214012091092, Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamilnadu, India.
E-mail: samuelissacraj@gmail.com

J. Paulraj Joseph, Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamilnadu, India.
E-mail: prof.jpaulraj@gmail.com