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# STRONG SPLIT DOMINATION POLYNOMIAL OF GRAPHS

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**Abstract** Let $G = (V, E)$ be a simple graph with $n$ vertices. The strong split domination polynomial of $G$ of order $n$ is the polynomial $D_{ss}(G, x) = \sum_{i=\gamma_{ss}(G)}^{n-2} d_{ss}(G, i)x^i$, where $d_{ss}(G, i)$ is the number of strong split dominating sets of $G$ of cardinality $i$. In this paper, we obtain the strong split domination polynomial of some graphs.

## 1 Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set $V$ and edge set $E$. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. A dominating set with cardinality $\gamma(G)$ is called a $\gamma$-set. For a detailed treatment of this parameter, the reader is referred to [8]. A dominating set $D \subseteq V(G)$ is a strong split dominating set if the induced subgraph $\langle V - D\rangle$ is totally disconnected with at least two vertices. The strong split domination number is the minimum size of a strong split dominating set of $G$ and is denoted by $\gamma_{ss}(G)$. Strong split domination in graph was introduced by Kulli and Janakiraman [9]. For more details on strong split domination, we refer [10].

A graph $G$ is said to be complete, if every pair of distinct vertices are adjacent. A complete graph on $n$ vertices is denoted by $K_n$. The union $G = G_1 \cup G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The corona of two graphs $G_1$ and $G_2$ was defined by Frucht and Harary in [5] as the graph $G = G_1 \circ G_2$ obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i$th vertex of $G_1$ is adjacent to every vertex in the $i$th copy of $G_2$. The join of two graphs $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv/u \in V(G_1) \text{ and } v \in V(G_2)\}$. So the join is obtained from $G_1$ and $G_2$ by joining every vertex of $G_1$ to each vertex of $G_2$ by an edge[7]. The Jelly fish graph [6] $J(m, n)$ is obtained from a 4-cycle $u, v, x, y$ by joining $x$ and $y$ with an edge and appending $m$ pendent edges to $u$ and $n$ pendent edges to $v$.

Graph polynomials are powerful and well-developed tools to express graph parameters. Domination polynomial in graphs was first introduced by Arocha and Llano [4] in 2000. Later in 2014, Alikhani and Peng made a slight modification of that definition and investigated its properties in [3]. The definition that we follow for domination polynomial is that of Alikhani and Peng.

**Definition 1.1.** [3] Given a graph $G$, the domination polynomial of $G$ of order $n$ is the polynomial

$$D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$$

where $d(G, i)$ is the number of dominating sets of $G$ of cardinality $i$.

**Definition 1.2.** [11] For a graph $G$, a strong split domination polynomial of $G$ of order $n$ is the polynomial

$$D_{ss}(G, x) = \sum_{i=\gamma_{ss}(G)}^{n-2} d_{ss}(G, i)x^i$$

where $d_{ss}(G,i)$ is the number of strong split dominating sets of $G$ of cardinality $i$.

Alikhani and others have contributed a lot to domination polynomial by constructing dominating sets for paths and cycles using the recursive relation. Using this recursive formula, the domination polynomial of paths and cycles and their properties were obtained[1],[2].

Motivated by the concept of domination polynomial by Alikhani and Peng, we introduced the concept of Strong split domination polynomial in [11] for some special classes of graphs. Also by using the recursive relation, the strong split domination polynomial of paths and cycles were obtained in[12]. In this paper, strong split domination polynomial of various graphs are given.

## 2 Main Result

In this section, we obtain the strong split domination polynomial for certain graphs.

**Lemma 2.1.** *For any graph $G$ with $n$ vertices, $\gamma_{ss}(G \circ K_1) = n$.*

*Proof.* Let $G$ be any graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $u_i (1 \leq i \leq n)$ be the corresponding pendant vertices adjacent to $v_i$ in $G \circ K_1$. The set $\{v_1, v_2, \ldots, v_n\}$ is a strong split dominating set as they dominate $\{u_1, u_2, \ldots, u_n\}$. Hence $\gamma_{ss}(G \circ K_1) \leq n$. Also $\gamma_{ss}(G \circ K_1) \geq n$ and so $\gamma_{ss}(G \circ K_1) = n$. □ □

**Theorem 2.2.** *For a complete graph $K_n$ with $n \geq 2$ verices,*

$$D_{ss}(K_n \circ K_1, x) = x^n \left((1+x)^n + n(1+x)^{n-1} - 2nx^{n-1} - x^n\right).$$

*Proof.* Let $K_n$ be a complete graph with vertex set $V_1 = \{v_1, v_2, \ldots, v_n\}$. Let $u_i$ be the pendant vertices adjacent to vertex $v_i (1 \leq i \leq n)$. Then $K_n \circ K_1$ has the vertex set $V = V_1 \cup V_2$ with $2n$ vertices. The set $V_1 = \{v_1, v_2, \ldots, v_n\}$ dominates the vertices of $K_n$ and the vertices $\{u_1, u_2, \ldots, u_n\}$ whereas the set $V_2 = \{u_1, u_2, \ldots, u_n\}$ is not a strong split dominating set as $< V - V_2 >$ is a complete graph. Consider the set $V_1$. For every $v_i$,$(V_1 - v_i) \cup u_i$ is a strong split dominating set. Thus the number of minimum strong split dominating sets is $n+1$. The strong split dominating sets of cardinality $n+1$ can be of the following two types.

(i) $V_1 \cup \{u_i\}$

(ii) $(V_1 - v_i) \cup \{u_i, u_j\} (i \neq j)$

The number of strong split dominating sets of the first type is $\binom{n}{1}$ and the number of strong split dominating sets of the second type is $n\binom{n-1}{1}$. Proceeding like this, we get $d_{ss}(K_n \circ K_1, i) = \binom{n}{i-n} + n\binom{n-1}{i-n}$ for $n+1 \leq i \leq 2n-2$ and $d_{ss}(K_n \circ K_1, n) = n+1$.

$$\begin{aligned} D_{ss}(K_n \circ K_1, x) &= (n+1)x^n + \sum_{i=n+1}^{2n-2} \left( \binom{n}{i-n} + n\binom{n-1}{i-n} \right) x^i. \\ &= x^n \left( n + 1 + \sum_{i=1}^{n-2} \left( \binom{n}{i} + n\binom{n-1}{i} \right) x^i \right). \end{aligned}$$

Thus $D_{ss}(K_n \circ K_1, x) = x^n \left((1+x)^n + n(1+x)^{n-1} - 2nx^{n-1} - x^n\right)$. □

**Example 2.3.** Consider the graph $G \cong K_4 \circ K_1$ and is shown in Figure 1. Here $G$ has 8 vertices and $\gamma_{ss}(G) = 4$. The $\gamma_{ss}$ sets are $\{v_1, v_2, v_3, v_4\}, \{u_1, v_2, v_3, v_4\}, \{v_1, u_2, v_3, v_4\}, \{v_1, v_2, u_3, v_4\}, \{v_1, v_2, v_3, u_4\}$.

There are 16 strong split dominating sets of cardinality 5 for $G$ and they are $\{v_1, v_2, v_3, v_4, u_1\}, \{v_1, v_2, v_3, v_4, u_2\}, \{v_1, v_2, v_3, v_4, u_3\}, \{v_1, v_2, v_3, v_4, u_4\}, \{u_1, v_2, v_3, v_4, u_2\}, \{u_1, v_2, v_3, v_4, u_3\}, \{u_1, v_2, v_3, v_4, u_4\}, \{v_1, u_2, v_3, v_4, u_1\}, \{v_1, u_2, v_3, v_4, u_3\}, \{v_1, u_2, v_3, v_4, u_4\}, \{v_1, v_2, u_3, v_4, u_1\}, \{v_1, v_2, u_3, v_4, u_2\}, \{v_1, v_2, u_3, v_4, u_4\}, \{v_1, v_2, v_3, u_4, u_1\}, \{v_1, v_2, v_3, u_4, u_2\}, \{v_1, v_2, v_3, u_4, u_3\}$.

Similarly, one can check that there are 18 strong split dominating sets with cardinality 6. Thus $D_{ss}(G, x) = 18x^6 + 16x^5 + 5x^4$

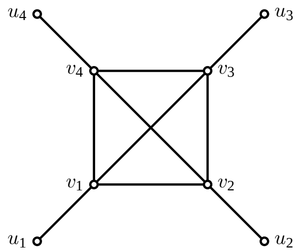


**Figure 1.**

**Theorem 2.4.** *Let $G_1$ and $G_2$ be two graphs with $n_1$ and $n_2$ vertices respectively. Then*

$$D_{ss}(G_1 \vee G_2, x) = D_{ss}(G_1, x)x^{n_2} + D_{ss}(G_2, x)x^{n_1}.$$

*Proof.* Let $(G_1, V_1)$ and $(G_2, V_2)$ be two graphs with $|V_1| = n_1$ and $|V_2| = n_2$. Now by the definition of $G_1 \vee G_2$, every vertex in the set $V_1$ is adjacent to every vertex in the set $V_2$ and every vertex in the set $V_2$ is adjacent to every vertex in the set $V_1$. Any strong split dominating set of $G_1 \vee G_2$ should contain either $V_1$ or $V_2$. For any strong split dominating set of $G_1$ of cardinality $i$ say $S$, $S \cup V_2$ will be a strong split dominating set of $G_1 \vee G_2$ of cardinality $n_2 + i$. Similarly, for any strong split dominating set of $G_2$ of cardinality $i$ say $T$, $T \cup V_1$ will be a strong split dominating set of $G_1 \vee G_2$ of cardinality $n_1 + i$. Hence $D_{ss}(G_1 \vee G_2, x) = D_{ss}(G_1, x)x^{n_2} + D_{ss}(G_2, x)x^{n_1}$. □

**Example 2.5.** The illustration for the join graph $G \cong P_5 \vee P_4$ is shown in Figure 2.

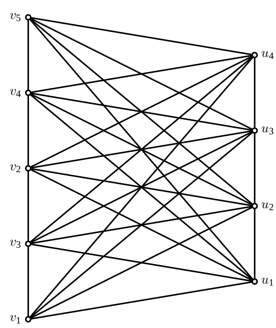


**Figure 2.**

We observe that $\gamma_{ss}(G) = 6$ and $\{u_1, u_2, u_3, u_4, v_2, v_4\}$ is the only $\gamma_{ss}$ set. There are 9 strong split dominating sets of cardinality 7 namely,
$\{u_1, u_2, u_3, u_4, v_2, v_4, v_1\}, \{u_1, u_2, u_3, u_4, v_2, v_4, v_3\}, \{u_1, u_2, u_3, u_4, v_2, v_4, v_5\},$
$\{u_1, u_2, u_3, u_4, v_1, v_3, v_5\}, \{u_1, u_2, u_3, u_4, v_2, v_3, v_5\}, \{u_1, u_2, u_3, u_4, v_1, v_2, v_5\},$
$\{v_1, v_2, v_3, v_4, v_5, u_1, u_3\}, \{v_1, v_2, v_3, v_4, v_5, u_2, u_3\}, \{v_1, v_2, v_3, v_4, v_5, u_2, u_4\}.$
Thus $D_{ss}(G, x) = 9x^7 + x^6$

**Theorem 2.6.** *Let $G_1$ and $G_2$ be any two graphs with $n_1$ and $n_2$ vertices respectively and let $G = G_1 \cup G_2$. Then*

$$D_{ss}(G, x) = D_{ss}(G_1, x)D_{ss}(G_2, x) + x^{n_1}(D_{ss}(G_2, x)) + x^{n_2}(D_{ss}(G_1, x)) \quad (2.1)$$
$$+ n_1 x^{n_1 - 1}(D_{ss}(G_2, x)) + n_2 x^{n_2 - 1}(D_{ss}(G_2, x)) + (n_1 n_2)\, x^{n_1 + n_2 - 2}. \quad (2.2)$$

*Proof.* Let $G = G_1 \cup G_2$ and $|V_1| = n_1$, $|V_2| = n_2$. If $D$ is any strong split dominating set $G$ with cardinality $k$,then $k \leq n_1 + n_2 - 2$. Let $k = k_1 + k_2$, where $k_1 = |D \cap V(G_1)|$ and $k_2 = |D \cap V(G_2)|$.
**Case 1:** $k_1 \leq n_1 - 2$ and $k_2 \leq n_2 - 2$
Any strong split dominating set of $k$ vertices in $G$ arises by choosing a strong split dominating set of $j$ vertices in $G_1$(for some $\gamma_{ss}(G_1) \leq n_1 - 2\}$) and a strong split dominating set of $k - j$ vertices

in $G_2$(for some $\gamma_{ss}(G_2) \leq n_2 - 2)$. This accounts for the expression $D_{ss}(G_1, x)D_{ss}(G_2, x)$ in (2.1).
**Case 2:** $k_1 = n_1$ and $k_2 \leq n_2 - 2$
Any strong split dominating set of $k$ vertices in $G$ arises by choosing $n_1$ vertices in $G_1$ and a strong split dominating set of $k - n_1$ vertices in $G_2$(for some $\gamma_{ss}(G_2) \leq n_2 - 2)$. This accounts for the expression $x^{n_1}(D_{ss}(G_2, x))$ in (2.1).
**Case 3:** $k_1 \leq n_1 - 2$ and $k_2 = n_2$
This case is analogous to case 2 and this accounts for the expression $x^{n_2}(D_{ss}(G_1, x))$ in (2.1).
**Case 4:** $k_1 = n_1 - 1$ and $k \leq n_2 - 2$
Any strong split dominating set of $k$ vertices in $G$ arises by choosing $n_1 - 1$ vertices in $G_1$ and a strong split dominating set of $k - (n_1 - 1)$ vertices in $G_2$(for some $\gamma_{ss}(G_2) \leq n_2 - 2)$. This accounts for the expression $n_1 x^{n_1 - 1}(D_{ss}(G_2, x))$ in (2.2).
**Case 5:** $k_1 \leq n_1 - 2$ and $k_2 = n_2 - 1$
This case is analogous to case 4 and this accounts for the expression $n_2 x^{n_2 - 1}(D_{ss}(G_1, x))$ in (2.2).
**Case 6:** $k_1 = n_1 - 1$ and $k_2 = n_2 - 1$
Any strong split dominating set of $k$ vertices in $G$ arises by choosing $n_1 - 1$ vertices in $G_1$ and a strong split dominating set of $n_2 - 1$ vertices in $G_2$. This accounts for the expression $n_1 n_2 x^{n_1 + n_2 - 2}$ in (2.2).
Hence

$$\begin{aligned} D_{ss}(G, x) &= D_{ss}(G_1, x)D_{ss}(G_2, x) + x^{n_1}(D_{ss}(G_2, x)) + x^{n_2}(D_{ss}(G_1, x)) \\ &\quad + n_1 x^{n_1 - 1}(D_{ss}(G_2, x)) + n_2 x^{n_2 - 1}(D_{ss}(G_2, x)) + (n_1 n_2)\, x^{n_1 + n_2 - 2}. \end{aligned}$$

□

**Example 2.7.** The illustration for the graph $G \cong C_4 \cup K_{1,3}$ is shown in Figure 3. Here $G$ has 8 vertices and $\gamma_{ss}(G) = 3$. The $\gamma_{ss}$ sets are $\{v_1, v_3, u_1\}$ and $\{v_2, v_4, u_1\}$.

The following are all the 10 strong split dominating sets of cardinality 4 namely,
$\{v_1, v_3, u_1, v_2\}, \{v_1, v_3, u_1, v_4\}, \{v_1, v_3, u_1, u_2\}, \{v_1, v_3, u_1, u_3\}, \{v_1, v_3, u_1, u_4\},$
$\{v_2, v_4, u_1, v_1\}, \{v_2, v_4, u_1, v_3\}, \{v_2, v_4, u_1, u_2\}, \{v_2, v_4, u_1, u_3\}, \{v_2, v_4, u_1, u_4\}.$

Similarly, one can find that there are 21 strong split dominating sets of cardinality 5 and another 21 strong split dominating sets of cardinality6. Thus $D_{ss}(G, x) = 21x^6 + 21x^5 + 10x^4 + 2x^3$.

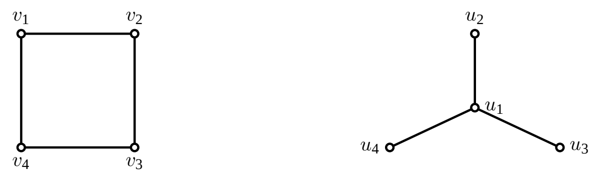


**Figure 3.**

**Theorem 2.8.** *The strong split domination polynomial of the Jelly fish graph is given by*

$$D_{ss}(J(m,n), x) = x^3(1+x)^{m+n}(x+2) + x^3(x^m(1+x)^n + x^n(1+x)^m) - x^{m+n+3}(x+m+n+4).$$

*Proof.* The Jelly fish graph $J(m, n)$ has $m + n + 4$ vertices. The minimum strong split dominating sets are $D_1 = \{u, v, x\}$ and $D_2 = \{u, v, y\}$ and so the strong split domination number $\gamma_{ss}(J(m, n)) = 3$. Any strong split dominating set $D$ will be of the following types:

(i) $u \in D$ and $v \in D$.

(ii) $u \in D$ and $v \notin D$ (or) $u \notin D$ and $v \in D$.

(iii) $u \notin D$ and $v \notin D$.

The strong split dominating sets $D$ of the first type can be the following three cases.
**Case 1.1:** $\{u, v, x\} \subseteq D, y \notin D$

Every subset $D \subset V(J(m,n)) - \{y\}$ with $|D| \geq 3$ containing $\{u,v,x\}$ is a strong split dominating set. Any strong split dominating set of cardinality $i$ is obtained by choosing $i-3$ elements from the remaining $m+n$ elements. Let $i$ be a natural number such that $3 \leq i \leq m+n+2$. Hence for $3 \leq i \leq m+n+2$, $d_{ss}(J(m,n),i) = \binom{m+n}{i-3}$
**Case 1.2:** $\{u,v,y\} \subseteq D, x \notin D$
As in case 1.1, $d_{ss}(J(m,n),i) = \binom{m+n}{i-3}$ for $3 \leq i \leq m+n+2$
**Case 1.3:** $\{u,v,x,y\} \subseteq D$
Let $i$ be a natural number, $4 \leq i \leq m+n+2$. Every subset $D \subseteq V(J(m,n))$ with $|D| \geq 4$ containing $\{u,v,x,y\}$ is a strong split dominating set. Any strong split dominating set of cardinality $i$ is obtained by choosing $i-4$ elements from the remaining $m+n$ elements. Hence for $4 \leq i \leq m+n+2$, $d_{ss}(J(m,n),i) = \binom{m+n}{i-4}$.

The strong split dominating sets $D$ of the second type will be the following two cases.
**Case 2.1:** $u \in D$ and $v \notin D$
Let $i$ be a natural number, $n+3 \leq i \leq m+n+2$. Every subset $D \subseteq V(J(m,n)) - \{v\}$ with $|D| \geq n+3$ containing $\{v_1,v_2,\ldots,v_n,u,x,y\}$ is a strong split dominating set. Any set $S$ such that $D \subseteq S, |S| = |D| + i(n+3 \leq i \leq m+n+2)$ is a strong split dominating set. Hence for $n+3 \leq i \leq m+n+2$, $d_{ss}(J(m,n),i) = \binom{m}{i-(n+3)}$
**Case 2.2:** $u \notin D$ and $v \in D$
As in case 2.1, $d_{ss}(J(m,n),i) = \binom{n}{i-(m+3)}$, for $m+3 \leq i \leq m+n+2$. Among the subsets of the third type, $V(J(m,n)) - \{u,v\}$ there is only one strong split dominating set namely, $\{u_1,u_2,\ldots,u_m,v_1,v_2,\ldots,v_n,x,y\}$.
Hence $d_{ss}(J(m,n),m+n+2) = 1$.

$$
\begin{aligned}
D_{ss}(J(m,n),x) &= 2\sum_{i=3}^{m+n+2}\binom{m+n}{i-3}x^i + \sum_{i=4}^{m+n+2}\binom{m+n}{i-4}x^i \\
&\quad + \sum_{i=n+3}^{m+n+2}\binom{m}{i-(n+3)}x^i + \sum_{i=m+3}^{m+n+2}\binom{n}{i-(m+3)}x^i + x^{m+n+2} \\
&= 2x^3\sum_{i=0}^{m+n-1}\binom{m+n}{i}x^i + x^4\sum_{i=0}^{m+n-2}\binom{m+n}{i}x^i \\
&\quad + x^{n+3}\sum_{i=0}^{m-1}\binom{m}{i}x^i + x^{m+3}\sum_{i=0}^{n-1}\binom{n}{i}x^i + x^{m+n+2} \\
&= 2x^3\left((1+x)^{m+n} - x^{m+n}\right) + x^4\left((1+x)^{m+n} - (m+n)x^{m+n-1} - x^{m+n}\right) \\
&\quad + x^{m+3}\left((1+x)^n - x^n\right) + x^{n+3}\left((1+x)^m - x^m\right) + x^{m+n+2} \\
D_{ss}(J(m,n),x) &= x^3(1+x)^{m+n}(x+2) + x^3\left(x^m(1+x)^n + x^n(1+x)^m\right) + x^{m+n+2} \\
&\quad - (m+n+4)x^{m+n+3} - x^{m+n+4}.
\end{aligned}
$$

□

**Example 2.9.** The illustration for the Jelly fish graph $G \cong J(3,2)$ is shown in Figure 4. Here $G$ has 9 vertices, $\gamma_{ss}(G) = 3$ and the $\gamma_{ss}$sets are $\{u,v,x\},\{u,v,y\}$.

The following are all the 11 strong split dominating sets of cardinality 4 namely,
$\{u,v,x,u_1\},\{u,v,x,u_2\},\{u,v,x,u_3\},\{u,v,x,v_1\},\{u,v,x,v_2\},\{u,v,y,u_1\},\{u,v,y,u_2\},$
$\{u,v,y,u_3\},\{u,v,y,v_1\},\{u,v,y,v_2\},\{u,v,x,y\}$.

Similarly, one can find that there are 26 strong split dominating sets are of cardinality 5, 34 strong split dominating sets with cardinality 6 and 26 strong split dominating sets are of cardinality 7. Thus $D_{ss}(G,x) = 26x^7 + 34x^6 + 26x^5 + 11x^4 + 2x^3$.

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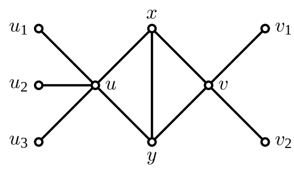


**Figure 4.**

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