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Some classes of Heronian mean anti-magic graphs

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Abstract Let $G = (V, E)$ be a finite, simple, connected and undirected graph with p vertices and q edges. Consider a labeling function $f : V \to \{1, 2, \ldots, q + 1\}$ and the induced edge labeling $f^* : E \to \{1, 2, \ldots, q + 1\}$ is defined by $f^*(e = uv) = \left\lceil \frac{f(u)+\sqrt{f(u)f(v)}+f(v)}{3} \right\rceil$ or $\left\lfloor \frac{f(u)+\sqrt{f(u)f(v)}+f(v)}{3} \right\rfloor$ for $e \in E$. Then f is said to be a Heronian mean labeling if induced edge labels $f^*(e)$ are distinct. An anti-magic labeling is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, q\}$ such that the weights are pairwise distinct, where the weight at one vertex is the sum of all labels of the edges incident to such vertex. A Heronian mean labeling f is said to be Heronian mean anti-magic if $w(v_i) \neq w(v_j)$ for all distinct vertices $v_i, v_j \in V(G)$, where $w(v) = \sum\limits_{u \in N(v)} f^*(uv)$. A graph is called Heronian mean anti-magic graph if it admits a Heronian mean anti-magic labeling. In this paper, we investigate certain Heronian mean anti-magic graphs.

1 Introduction

Throughout this paper, by G we mean a simple, connected and undirected graph with vertex set V, edge set E, p vertices and q edges. An assignment of integers to the vertices or edges or both, subject to certain conditions is called a graph labeling. The mean labeling of graphs was introduced by Somasundaram and Ponraj[7] in 2004. The concept of Heronian mean was introduced and briefly studied by Sandhya[8]. Further Power-3 Heronian mean labeling was studied by Kaaviya Shree and Sharmilaa[5]. Hartsfield and Ringel[3] introduced the concept of anti-magic labeling of graphs. Further, several variations of anti-magic labeling such as vertex anti-magic and edge anti-magic were also defined and studied by Martin Baca et al.[4],[6]. In combining the latter two concepts, we introduced the concept of Heronian mean anti-magic labeling of graphs[9], and studied the existence of Heronian mean anti-magic graphs. In this paper, we continue our study on Heronian mean anti-magic graphs. In particular, we prove that certain special classes of snakes and their generalizations are Heronian mean anti-magic. A useful survey of graph labeling can be found in Gallian[1]. We follow Harary[2] for other notations and standard terminologies.

2 Preliminaries

In this section, we recall certain basic definitions which are needed for future reference. An assignment of integers to the vertices or edges or both, subject to certain conditions is called graph labeling. Recall that, there are so many labelings studied on various classes of graphs. An anti-magic labeling is a bijection from the set of edges to the set of integers $\{1, 2, 3, \ldots, q\}$ such that the weights are pairwise distinct, where the weight at one vertex is the sum of all labels of the edges incident to such vertex. A function $f : V \to \{1, 2, 3, \ldots, q+1\}$ is said to be a Heronian mean labeling if the induced edge labeling $f^* : E(G) \to \{1, 2, 3, \ldots, q+1\}$ defined by $f^*(e = uv) = \left\lceil \frac{f(u)+\sqrt{f(u)f(v)}+f(v)}{3} \right\rceil$ or $\left\lfloor \frac{f(u)+\sqrt{f(u)f(v)}+f(v)}{3} \right\rfloor$ gives distinct labels for distinct edges.

A Heronian mean labeling f is said to be Heronian mean anti-magic if $w(v_i) \neq w(v_j)$ for all distinct vertices $v_i, v_j \in V(G)$, where $w(v) = \sum_{u \in N(v)} f^*(uv)$. A graph is called Heronian mean anti-magic graph if it admits Heronian mean anti-magic labeling. Let us see an example for Heronian mean anti-magic graph.

Example 2.1. Consider the graph G given in Figure 1. The vertices of G are $u_1, u_2, u_3, u_4, u_5, u_6$ and it contains 5 edges. Define $f : V \to \{1, 2, 3, \ldots, 6\}$ by $f(u_1) = 1,\ f(u_2) = 2,\ f(u_3) = 3,\ f(u_4) = 4,\ f(u_5) = 5,\ f(u_6) = 6$. Then the induced edge labels are $f^*(u_1u_2) = 1,\ f^*(u_2u_3) = 2,\ f^*(u_3u_4) = 3,\ f^*(u_2u_5) = 4,\ f^*(u_5u_6) = 5$. Note that the induced edge labels are all distinct and so the labeling f satisfies the Heronian mean labeling condition.

Now, we have to show that $w(v) = \sum_{u \in N(v)} f^*(uv)$ are distinct for all the vertices in G. In fact, $w(u_1) = 1, w(u_2) = 7, w(u_3) = 2, w(u_4) = 3, w(u_5) = 9, w(u_6) = 5$ and therefore, for every vertex v in $V(G)$ the sum of the resulting edge labels incident with v are distinct and so the labeling f graph satisfies the Heronian mean anti-magic labeling condition. Hence, the graph G is a Heronian mean anti-magic graph.

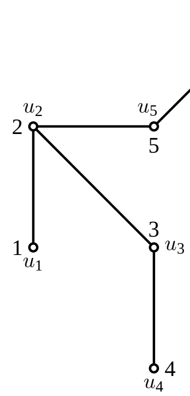

Figure 1.

In order to substantiate our concept of Heronian mean anti-magic graphs, let us see below that a complete bipartite graph $K_{1,n},\ n > 5$ is not a Heronian mean anti-magic graph.

Theorem 2.2. *For any positive integer* $n > 5$, *the complete bipartite graph* $K_{1,n}$ *is not a Heronian mean anti-magic graph.*

Proof. let $G = K_{1,n},\ n > 5$ with $V(G) = \{u, u_i/1 \leq i \leq n\}$ and $E(G) = \{uu_i/1 \leq i \leq n\}$. Here $|V(G)| = p = n + 1$ and $|E(G)| = q = n$. Let the apex vertex be u. Since $p = q + 1$, 1 must be label for some vertex in any Heronian mean labeling of G. Without loss of generality, one can take the vertex labeling $f : V \to \{1, 2, 3, \ldots, q + 1\}$ as,

Case 1: $f(u) = 1$.

Since $p = q + 1$, the remaining n vertices will receive labels $2, 3, \ldots, n + 1$. Let $f(u_1) = 2, f(u_2) = 3, f(u_3) = 4, \ldots, f(u_n) = n + 1$. Then the induced Heronian mean edge label for uu_1 will be 1 or 2. Similarly, the edge label for uu_2 will be either 1 or 2, the edge label for uu_3 will be either 2 or 3 and the edge label for uu_4 will be either 2 or 3. Thus, at least two of these four edges will receive the same edge labels. Therefore, in this case, the graph does not satisfy the Heronian mean condition.

Case 2: $f(u) = 2$.

Since $p = q + 1$, the labels for remaining n vertices will be $1, 3, 4, \ldots, n + 1$. Without loss of generality, one can take $f(u_1) = 1,\ f(u_2) = 3,\ f(u_3) = 4, \ldots, f(u_n) = n + 1$. The induced Heronian mean edge labels for uu_1 will be either 1 or 2. Similarly, the edge uu_2 will receive 2 or 3, the edge uu_3 will receive 2 or 3, the edge uu_4 will receive 3 or 4 and the edge uu_5 will receive 3 or 4 as Heronian mean labels. Thus we note that at least two of these five edges will receive the same edge labels. Therefore, in this case the graph does not satisfy the Heronian mean condition.

Case 3: $f(u) = a,\ 3 \leq a \leq 5$.

Similar to cases 1 and 2, two edges will receive the same Heronian mean edge labels and hence the Heronian mean condition is not satisfied.

Case 4: $f(u) > 5$.

By calculating Heronian mean edge labels, one can see that no edge shall receive 1 or 2 as their edge labels. Since we can never label n distinct edges with $n-1$ values $\{3, 4, \ldots, n+1\}$, the graph does not satisfy the Heronian mean condition.

Thus, there is no Heronian mean labeling for $K_{1,n}, n > 5$. Hence $K_{1,n}, n > 5$ is not a Heronian mean anti-magic graph. □

Now, let us see the structure of certain classes of graphs, which are proved as Heronian mean anti-magic.

For every positive integer $n \geq 2$, a triple triangular snake $T(T_n)$ consists of three triangular snakes that have a common path. Thus, a triple triangular snake is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} to three new vertices x_i, y_i and z_i for $i = 1, 2, \ldots, n-1$. Assume that for $1 \leq i \leq n-1$, the vertices lying below the path are y_i and the vertices above the path are x_i and z_i, where in addition x_i lies below z_i. One can refer Figure 2 for $T(T_6)$.

The double quadrilateral snake $D(QS_n)$ is obtained from two quadrilateral snakes with a common path P_n, where the vertices of the path are u_i for $1 \leq i \leq n$. The vertices of the quadrilaterals lying above the common path are denoted by x_i, y_i where i is odd in order and the quadrilaterals lying below the common path are denoted by x_i, y_i where i is even in order. We have given $D(QS_7)$ in Figure 3.

An irregular triangular snake IT_n is obtained from a path P_n with consecutive vertices $u_1, u_2, \ldots, u_n$ whose vertex set $V = V(P_n) \cup \{v_i / 1 \leq i \leq n-2\}$ and the edge set is $E = E(P_n) \cup \{u_i v_i, u_{i+2} v_i / 1 \leq i \leq n-2\}$. The vertices v_i, i is odd lie above the path and the vertices v_i lie below the path. We have given IT_9 in Figure 4.

An alternate double triangular snake $AD(T_n)$ is obtained from two alternate triangular snakes with a common path and the vertices of the path are given by u_i. The vertices of the triangles lying above the path are denoted by x_i and the vertices of the triangles lying below the path are denoted by y_i. We have given $AD(T_9)$ in Figure 5.

3 Main Results

In this section, we prove that graphs $T(T_n)$, $D(QS_n)$, IT_n, $AD(T_n)$ are Heronian mean anti-magic.

Theorem 3.1. *For every integer $n \geq 2$, the triple triangular snake graph $T(T_n)$ is a Heronian mean anti-magic graph.*

Proof. Let $G = T(T_n)$ for $n \geq 2$. As described above, the vertex set of $G = T(T_n)$ is $V(G) = \{u_i, x_j, y_j, z_j \ / \ 1 \leq i \leq n, \ 1 \leq j \leq n-1\}$ and the edge set is $E(G) = \{u_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i x_i, u_i y_i, u_i z_i, u_{i+1} x_i, u_{i+1} y_i, u_{i+1} z_i / 1 \leq i \leq n-1\}$. Here $|V(G)| = 4n-3$ and $|E(G)| = 7n-7$. Define $f : V(G) \to \{1, 2, 3, \ldots, 7n-6\}$ by

$$f(u_i) = \begin{cases} 7i-5 & \text{for } i = 1; \\ 7i-6 & \text{for } 2 \leq i \leq n. \end{cases}$$

$$f(x_i) = \begin{cases} 7i-4 & \text{for } i = 1; \\ 7i-2 & \text{for } 2 \leq i \leq n-1. \end{cases}$$

$$f(y_i) = \begin{cases} 7i-2 & \text{for } i = 1; \\ 7i & \text{for } 2 \leq i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 7i-6 & \text{for } i = 1; \\ 7i-4 & \text{for } 2 \leq i \leq n-1. \end{cases}$$

Then the induced edge labels are

$$f^*(u_i u_{i+1}) = \begin{cases} 7i-2 & \text{for } i = 1; \\ 7i-3 & \text{for } 2 \leq i \leq n-1. \end{cases}$$

$f^*(u_i x_i) = 7i-5, \ 1 \leq i \leq n-1,$

$f^*(u_i z_i) = 7i-6, \ 1 \leq i \leq n-1,$

$f^*(u_iy_i) = 7i - 4,\ 1 \le i \le n-1,$
$f^*(u_{i+1}x_i) = 7i - 1, 1 \le i \le n-1.$

$$f^*(u_{i+1}z_i) = \begin{cases} 7i-3 & \text{for } i=1; \\ 7i-2 & \text{for } 2 \le i \le n-1. \end{cases}$$

$f^*(u_{i+1}y_i) = 7i, 1 \le i \le n-1.$

Thus, the obtained edge labels are all distinct and so the labeling f satisfies the Heronian mean condition.

Now, we have to show that $w(v) = \sum_{u \in N(v)} f^*(uv)$ are distinct for all the vertices in $T(T_n)$.

Note that
$w(u_1) = 11.$
$w(u_i) = 4(14i - 13),\ 2 \le i \le n-1.$
$w(u_n) = 2(14n - 17).$
$w(x_i) = 14i - 6,\ 1 \le i \le n-1.$
$w(y_i) = 14i - 4,\ 1 \le i \le n-1.$

$$w(z_i) = \begin{cases} 14i-9 & \text{for } i=1; \\ 14i-8 & \text{for } 2 \le i \le n-1. \end{cases}$$

Therefore, for every vertex v in $V(G)$, the sum of the resulting edge labels incident with v are distinct and the labeling f satisfies the Heronian mean anti-magic condition. Hence, the graph $T(T_n)$ admits Heronian mean anti-magic labeling. □

As described in the proof of Theorem 3.1, a Heronian mean anti-magic label is obtained for $T(T_6)$ and the same is exhibited in Figure 2 below.

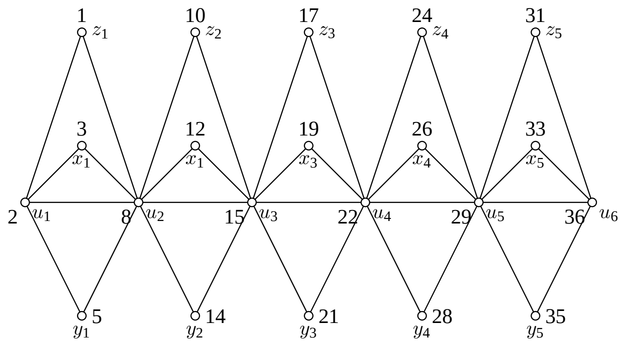

Figure 2. $T(T_6)$

Theorem 3.2. *For every integer $n \ge 2$, the double quadrilateral snake graph $D(QS_n)$ is a Heronian mean anti-magic graph.*

Proof. Let $G = D(QS_n)$ for $n \ge 2$. Let $V(G) = \{u_i, x_j, y_j / 1 \le i \le n, 1 \le j \le 2n-2\}$ be the vertex set and $E(G) = \{u_iu_{i+1}/1 \le i \le n-1\} \cup \{u_ix_{2i-1}, u_ix_{2i}, u_{i+1}y_{2i-1}, u_{i+1}y_{2i}/1 \le i \le n-1\} \cup \{x_iy_i\}/1 \le i \le 2n-2\}$ be the edge set. Here $|V(G)| = 5n-4$ and $|E(G)| = 7n-7$. Define $f : V(G) \to \{1, 2, 3, 7n-6\}$ by,
$f(u_1) = 1;$
$f(u_i) = 7i - 8,\ 2 \le i \le n.$
$f(x_1) = 2;$
$f(y_1) = 3;$
For $2 \le i \le 2n-2,$

$$f(x_i) = \begin{cases} \frac{7i}{2} & \text{for } i \equiv 0\ (mod\ 2); \\ \frac{7i-1}{2} & \text{for } i \equiv 1\ (mod\ 2). \end{cases}$$

For $2 \le i \le 2n-2,$

$$f(y_i) = \begin{cases} \frac{7i+2}{2} & \text{for } i \equiv 0\ (mod\ 2); \\ \frac{7i+1}{2} & \text{for } i \equiv 1\ (mod\ 2). \end{cases}$$

Then the induced edge labels are,

$f^*(u_1u_2) = 4$.
$f^*(u_iu_{i+1}) = 7i - 5,\ 2 \le i \le n-1$.
$f^*(u_ix_{2i-1}) = 7i - 6,\ 1 \le i \le n-1$.
$f^*(u_ix_{2i}) = 7i - 4,\ 1 \le i \le n-1$.
$f^*(u_{i+1}y_{2i-1}) = 7i - 2,\ 1 \le i \le n-1$.
$f^*(u_{i+1}y_{2i}) = 7i - 1,\ 1 \le i \le n-1$.
$f^*(x_1y_1) = 2$.
For $2 \le i \le 2n-2$,

$$f^*(x_iy_i) = \begin{cases} \frac{7i}{2} & \text{for } i \equiv 0\ (mod\ 2); \\ \frac{7i+1}{2} & \text{for } i \equiv 1\ (mod\ 2). \end{cases}$$

Thus, the obtained edge labels are all distinct and so the labeling f satisfies the Heronian mean condition. Now, we have to show that $w(v) = \sum_{u \in N(v)} f^*(uv)$ are distinct for all the vertices in $D(QS_n)$.
$w(u_1) = 8$;
$w(u_2) = 42$;
$w(u_i) = 2(21i - 22),\ 3 \le i \le n-1$;

$$w(u_n) = \begin{cases} 13\ for\ n = 2; \\ 21n - 29,\ otherwise. \end{cases}$$

$w(x_1) = 3$;
$w(y_1) = 7$;
For $2 \le i \le 2n-2$,

$$w(x_i) = \begin{cases} 7i - 4 & \text{for } i \equiv 0\ (mod\ 2); \\ 7i - 2 & \text{for } i \equiv 1\ (mod\ 2). \end{cases}$$

For $2 \le i \le 2n-2$,

$$w(y_i) = \begin{cases} 7i - 1 & \text{for } i \equiv 0\ (mod\ 2); \\ 7i + 2 & \text{for } i \equiv 1\ (mod\ 2). \end{cases}$$

Therefore, for every vertex v in $V(G)$, the sum of the resulting edge labels incident with v are distinct and the labeling f satisfies the Heronian mean anti-magic condition. Hence, the graph $D(QS_n)$ admits Heronian mean anti-magic labeling. □

As described in the proof of Theorem 3.2, a Heronian mean anti-magic label is obtained for $D(QS_7)$ and the same is exhibited in Figure 3 below.

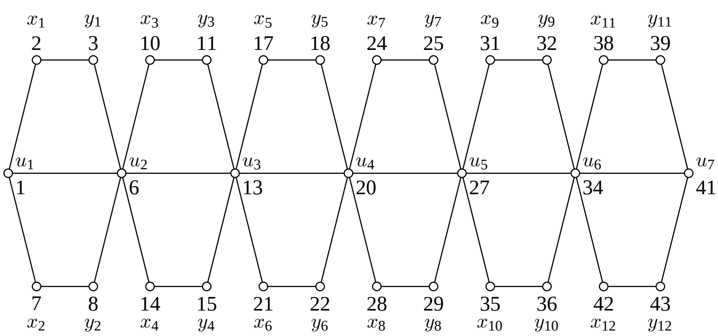

Figure 3. $D(QS_7)$

Theorem 3.3. *For every integer $n \ge 3$, the irregular triangular snake graph $G = IT_n$ is a Heronian mean anti-magic graph.*

Proof. Let $G = IT_n, n \ge 3$ and let $V(G) = \{u_i, v_j / 1 \le i \le n, 1 \le j \le n-2\}$ be the vertex set and $E(G) = \{u_iu_{i+1}/1 \le i \le n-1\} \cup \{u_iv_i, u_{i+2}v_i/1 \le i \le n-2\}$ be the edge set. Here $|V(G)| = 2n-2$ and $|E(G)| = 3n-5$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n-4\}$ by,
$f(u_1) = 1$;

$f(u_i) = \begin{cases} 3i - 3 & \text{for } 2 \le i \le 3; \\ 3i - 4 & \text{for } 4 \le i \le n. \end{cases}$
$f(v_1) = 2$, $f(v_2) = 7$, $f(v_3) = 9$;
$f(v_i) = 3i + 1,\ 4 \le i \le n - 2.$
Then the induced edge labels are,
$f^*(u_iu_{i+1}) = \begin{cases} 3i - 1 & \text{for } 1 \le i \le 2; \\ 3i - 3 & \text{for } 3 \le i \le n - 1. \end{cases}$
$f^*(u_iv_i) = \begin{cases} 3i - 2 & \text{for } 1 \le i \le 2; \\ 3i - 1 & \text{for } 3 \le i \le n - 2. \end{cases}$
$f^*(u_3v_1) = 3$;
$f^*(u_{i+2}v_i) = 3i + 1,\ 2 \le i \le n - 2.$
Thus, the obtained edge labels are all distinct and so the labeling f satisfies the Heronian mean condition.
Now, we have to show that $w(v) = \sum_{u \in N(v)} f^*(uv)$ are distinct for all the vertices in $G = IT_n, n \ge 3$.
$w(u_1) = 3$;
$w(u_i) = \begin{cases} 11i - 11 & \text{for } 2 \le i \le 4; \\ 12i - 15 & \text{for } 5 \le i \le n - 2. \end{cases}$
$w(u_{n-1}) = \begin{cases} 7 & \text{for } n = 3; \\ 9n - 22 & \text{for } n = 4, 5; \\ 9n - 23, & otherwise \end{cases}$
$w(u_n) = \begin{cases} 3n - 1 & \text{for } n = 3; \\ 6n - 11, & otherwise \end{cases}$
$w(v_1) = 4$, $w(v_2) = 11$;
$w(v_i) = 6i,\ 3 \le i \le n - 2.$
Therefore, for every vertex v in $V(G)$, the sum of the resulting edge labels incident with v are distinct and the labeling f satisfies the Heronian mean anti-magic condition. Hence, the graph IT_n admits Heronian mean anti-magic labeling. □

As described in the proof of Theorem 3.3, a Heronian mean anti-magic label is obtained for IT_9 and the same is exhibited in Figure 4 below.

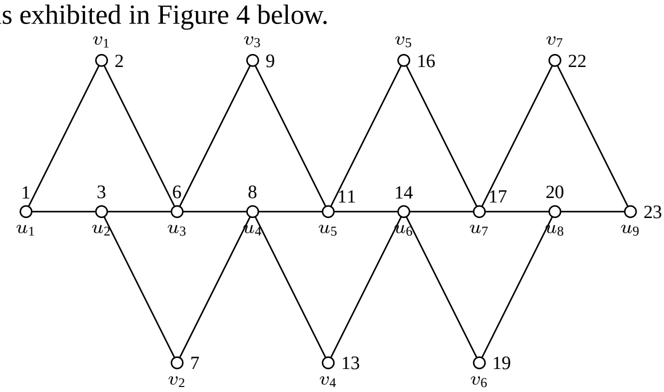

Figure 4. IT_9

Theorem 3.4. *For every integer $n \ge 2$, an alternate double triangular snake graph $AD(T_n)$ is a Heronian mean anti-magic graph.*

Proof. Consider $AD(T_n)$ as an alternate double triangular snake graph.
Case 1: The alternate double triangular snake $AD(T_n), n \ge 2$ alternatively begins from the first vertex.

Let the vertex set be $V(G) = \{u_i, x_j, y_j / 1 \le i \le n, 1 \le j \le \lceil \frac{n-1}{2} \rceil\}$ and the edge set be $E(G) = \{u_iu_{i+1}/1 \le i \le n - 1\} \cup \{u_{2i-1}x_i, u_{2i}x_i, u_{2i-1}y_i, u_{2i}y_i / 1 \le i \le \lceil \frac{n-1}{2} \rceil\}$.

Here $|V(G)| = \begin{cases} 2n & \text{if n is even;} \\ 2n-1 & \text{if n is odd.} \end{cases}$

$|E(G)| = \begin{cases} 3n-1 & \text{if n is even;} \\ 3n-3 & \text{if n is odd.} \end{cases}$

Define $f : V(G) \to \{1, 2, \ldots, q+1\}$ by,
$f(u_1) = 2;$
$f(u_i) = 3i - 2,\ 2 \le i \le n.$
$f(x_1) = 1;$
$f(x_i) = 6i - 4,\ 2 \le i \le \lceil \frac{n-1}{2} \rceil.$
$f(y_i) = 6i,\ 1 \le i \le \lceil \frac{n-1}{2} \rceil.$
Then the induced edge labels are,
$f^*(u_iu_{i+1}) = 3i,\ 1 \le i \le n-1.$
$f^*(u_{2i-1}x_i) = 6i - 5,\ 1 \le i \le \lceil \frac{n-1}{2} \rceil.$
$f^*(u_{2i}x_i) = 6i - 4,\ 1 \le i \le \lceil \frac{n-1}{2} \rceil.$
$f^*(u_{2i-1}y_i) = 6i - 2,\ 1 \le i \le \lceil \frac{n-1}{2} \rceil.$
$f^*(u_{2i}y_i) = 6i - 1,\ 1 \le i \le \lceil \frac{n-1}{2} \rceil.$
Thus, the obtained edge labels are all distinct and so the labeling f satisfies the Heronian mean condition.
Now, we have to show that $w(v) = \sum_{u \in N(v)} f^*(uv)$ are distinct for all the vertices in $AD(T_n), n \ge 2$.
For $1 \le i \le n-1$,

$w(u_i) = \begin{cases} 4(3i-1) & \text{for } i \equiv 1\ (mod\ 2); \\ 4(3i-2) & \text{for } i \equiv 0\ (mod\ 2). \end{cases}$

$w(u_n) = \begin{cases} 3(n-1) & \text{for } n \equiv 1\ (mod\ 2); \\ 9n-8 & \text{for } n \equiv 0\ (mod\ 2). \end{cases}$

$w(x_i) = 3(4i-3),\ 1 \le i \le \lceil \frac{n-1}{2} \rceil.$
$w(y_i) = 3(4i-1),\ 1 \le i \le \lceil \frac{n-1}{2} \rceil.$
Therefore, for every vertex v in $V(G)$, the sum of the resulting edge labels incident with v are distinct and the labeling f satisfies the Heronian mean anti-magic condition. Hence the graph $AD(T_n), n \ge 2$ is a Heronian mean anti-magic graph.

As described in the case 1 of Theorem 3.4, a Heronian mean anti-magic label is obtained for $AD(T_9)$ and the same is exhibited in Figure 5 below.

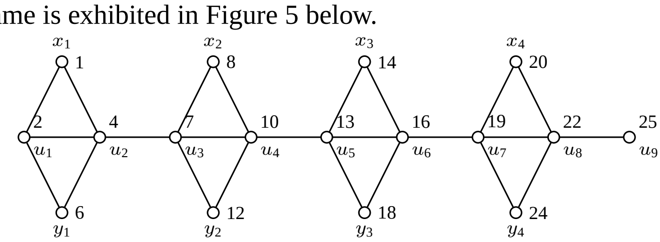

Figure 5. $AD(T_9)$

Case 2: The alternate double triangular snake $AD(T_n), n \ge 3$ alternatively begins from the second vertex.
Let the vertex set be $V(G) = \{u_i, x_j, y_j / 1 \le i \le n, 1 \le j \le \lfloor \frac{n-1}{2} \rfloor\}$ and the edge set be $E(G) = \{u_iu_{i+1} / 1 \le i \le n-1\} \cup \{u_{2i+1}x_i, u_{2i}x_i, u_{2i+1}y_i, u_{2i}y_i / 1 \le i \le \lfloor \frac{n-1}{2} \rfloor\}$.

Here $|V(G)| = \begin{cases} 2n-2 & \text{if n is even;} \\ 2n-1 & \text{if n is odd.} \end{cases}$

$|E(G)| = \begin{cases} 3n-5 & \text{if n is even;} \\ 3n-3 & \text{if n is odd.} \end{cases}$

Define $f : V(G) \to \{1, 2, \ldots, q+1\}$ by,
For $1 \le i \le n$,

$f(u_i) = \begin{cases} 3i-2 & \text{for } i \equiv 1\ (mod\ 2); \\ 3i-4 & \text{for } i \equiv 0\ (mod\ 2). \end{cases}$
$f(x_i) = 6i-3,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
$f(y_i) = 6i-2,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
Then the induced edge labels are,
$f^*(u_iu_{i+1}) = \begin{cases} 3i-2 & \text{for } i \equiv 1\ (mod\ 2); \\ 3i-1 & \text{for } i \equiv 0\ (mod\ 2). \end{cases}$
$f^*(u_{2i+1}x_i) = 6i-2,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
$f^*(u_{2i}x_i) = 6i-4,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
$f^*(u_{2i+1}y_i) = 6i,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
$f^*(u_{2i}y_i) = 6i-3,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
Thus, the obtained edge labels are all distinct and so the labeling f satisfies the Heronian mean condition.
Now, we have to show that $w(v) = \sum_{u \in N(v)} f^*(uv)$ are distinct for all the vertices in $AD(T_n), n \ge 3$.
$w(u_1) = 1;$
$w(u_2) = 11;$
For $1 \le i \le n-2$,
$w(u_i) = \begin{cases} 12i-14 & \text{for } i \equiv 1\ (mod\ 2); \\ 12i-13 & \text{for } i \equiv 0\ (mod\ 2). \end{cases}$
If n is odd,
$w(u_{n-1}) = 12n-25,\ w(u_n) = 9n-12.$
If n is even,
$w(u_{n-1}) = 12n-26,\ w(u_n) = 3n-5.$
$w(x_i) = 12i-6,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
$w(y_i) = 12i-3,\ 1 \le i \le \lfloor \frac{n-1}{2} \rfloor.$
Therefore, for every vertex v in $V(G)$, the sum of the resulting edge labels incident with v are distinct and the labeling f satisfies the Heronian mean anti-magic condition. Hence the graph $AD(T_n), n \ge 3$ is a Heronian mean anti-magic graph.

As described in the case 2 of Theorem 3.4, a Heronian mean anti-magic label is obtained for $AD(T_8)$ and the same is exhibited in Figure 6 below.

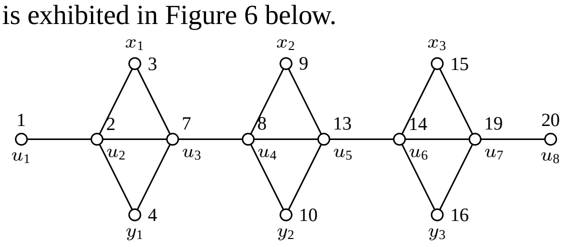

Figure 6. $AD(T_8)$

Thus, in both the cases the graph satisfies the Heronian mean anti-magic condition and so an alternate double triangular snake graph $AD(T_n)$ is a Heronian mean anti-magic graph. □

4 Conclusion

In this paper, we proved that the Triple triangular snake, Double quadrilateral snake, Irregular triangular snake and Alternate double triangular snake are all Heronian mean anti-magic. In future, we plan to investigate the labeling for several graph products.

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