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Coupon Coloring of Rooted Product Graphs

Reji T and Pavithra R

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Abstract A k-coupon coloring of a graph G without isolated vertices is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G. The rooted product, $G \circ_v H$, of two graphs G and H without isolated vertices is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H, for every $i = 1, 2, \ldots, |V(G)|$. We have studied the coupon coloring of rooted product graphs and found the coupon coloring number of some graphs. We also found some sharp bounds for the coupon coloring number of $G \circ_v H$. If G and H are two graphs without isolated vertices and $\delta(G) = 1$, then the coupon coloring number of the rooted product $G \circ_v H$ is either k or $k + 1$, where k is the coupon coloring number of H.

1 Introduction

The concept of coupon coloring number was introduced by Chen et al. in [4]. Let G be a graph without isolated vertices. A k-vertex coloring or simply a k-coloring of G is a mapping c from the vertices of G to $[k] = \{1, 2, \ldots, k\}$. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G and it is denoted by $\chi_c(G)$. The coupon coloring of a graph G without isolated vertices is well defined, since we may assign every vertex the same color.

Applications of coupon coloring is in the network science and some other related fields. If we imagine the colors as coupons of different types, then in a coupon coloring every vertex collects coupons of all different types from its neighbors. Imagine that there are n users $v_1, v_2, \ldots, v_n$ and that every user has contact with a set of other users. Suppose we have to transfer a k-bit message to all the n users $v_1, v_2, \ldots, v_n$, but each user is assigned only a bit from the k-bit message. Then every user can reconstruct the entire message from her contacts if and only if the graph of contacts has a k-coupon coloring. To maximize the length of the message that can be transmitted, we have to determine the coupon coloring number of the graph of contacts.

The results on coupon colorings have concrete applications in network science. One application is to large multi-robot networks [1]. We can imagine a network large enough so that robots must act based on local information. A graph can be constructed with robots in the network as vertices and there is an edge between two vertices if the corresponding robots are able to communicate with each other. An example is described in [1]: a group of robots is assigned to monitor an environment. A robot must monitor many different statistics like temperature, humidity, etc., of the environment. But each robot is only equipped with a single sensor (thermometer, barometer, etc.) due to power limitations. Thus, each robot must communicate with its neighbors to obtain the remaining data. A similar example arises in allocating resources to a network [1]. Suppose that there is a graph of contacts in which each vertex of the graph may only use resources available at the vertex or its neighbors. If some resource (e.g. a printer) must be available to every vertex in the network, then copies of that resource must be allocated to a dominating set of the network. If every node in the network can accommodate one resource, then finding the coupon coloring number of the network is equivalent to finding the maximum number of resources that can be made available to every node in the network.

In this paper, we have studied the coupon coloring number of rooted product of two graphs. we have found the exact coupon coloring number of the rooted product $G \circ_v H$, when G is any graph and H is either a cycle or a complete graph. We also found some sharp bounds for the coupon coloring number of rooted product graphs. If G and H are two graphs without isolated vertices and $\delta(G) = 1$, then the coupon coloring number of the rooted product $G \circ_v H$ is either k or $k + 1$, where k is the coupon coloring number of H. We found some class of graphs for which the coupon coloring number of $G \circ_v H$ is k and $k + 1$.

2 Preliminaries

All graphs considered in this paper are simple, finite and undirected. As usual K_n and C_n denote the complete graph and the cycle with n vertices. The minimum and maximum degrees of vertices in a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. Let v be a vertex of a graph G and $V(G)$ is the set of vertices of G. Then $G - v$ is the graph induced by $V(G) \setminus \{v\}$ and $|V(G)|$ denotes the number of vertices of the graph G.

Let G be a graph without isolated vertices. A k-vertex coloring, or simply a k-coloring of G is a mapping c from the vertex set of G to $[k] = \{1, 2, ..., k\}$. A vertex v is said to be a *bad vertex* in a k-coloring c, if its neighborhood does not contain vertices of all colors from $[k]$ and obviously, there is no bad vertices in a coupon coloring. Clearly, coupon coloring is an improper coloring and $\chi_c(G) \leq \delta(G)$.

Let $G = (V, E)$ be a graph. $D \subseteq V$ is a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. Let $G = (V, E)$ be a graph without isolated vertices. $D' \subseteq V$ is a total dominating set if every vertex of G is adjacent to at least one vertex in D'. The minimum cardinality among all the total dominating sets in G is called the total domination number, $\gamma_t(G)$. The coupon coloring number is also referred to as the total domatic number introduced in [2], which is the maximum number of disjoint total dominating sets. In [9] Y Shi et al. determined coupon coloring number of complete graphs, complete k-partite graphs, wheels, cycles, unicyclic graphs and bicyclic graphs. Coupon coloring is also studied in [5, 6, 8]. P Francis and Deepak Rajendraprasad studied the coupon coloring of Cartesian product of some graphs in [6].

The rooted product of two graphs was introduced by Godsil and McKay in 1978 [7]. The rooted product of two graphs G and H is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i = 1, 2, \ldots, |V(G)|$. It is denoted by $G \circ_v H$. A large number of research papers are published in different domination parameters of rooted product graphs. The total domination of rooted product graphs were studied in [3].

The following results will be useful for the upcoming sections.

Theorem 2.1. [9]

(1) Let G be a complete graph with n vertices. Then $\chi_c(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

(2) Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph where $k \geq 3$ and $n_1 \leq n_2 \leq, \ldots, \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $n = \sum_{i=1}^{k} n_i$. Then

$$\chi_c(G) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \textit{if } s \geq \frac{n}{2}, \\ s & \textit{otherwise.} \end{cases}$$

Theorem 2.2. [9] *Let C_n be the cycle with n vertices. Then*

$$\chi_c(C_n) = \begin{cases} 1 & \textit{if } n \equiv 0 \pmod 4, \\ 2 & \textit{otherwise.} \end{cases}$$

3 Coupon coloring of $G \circ_v H$

In this section we study the coupon coloring number of $G \circ_v H$. We first show that the root vertex in $G \circ_v H$ has the following property which will be useful throughout this paper.

Lemma 3.1. *Let G and H be two graphs without isolated vertices and let $\chi_c(H) = k$. If v is the root vertex of H and H^i be the i^{th} copy of H in $G \circ_v H$, then the adjacent vertices of v in H^i can have at most k colors in any coupon coloring of $G \circ_v H$.*

Proof. Suppose that c is a coupon coloring of $G \circ_v H$ with at least $k+1$ colors. If the adjacent vertices of v that are in H^i have $k+1$ colors, then the coloring c restricted to the vertices of H^i is a $(k+1)$-coupon coloring of H, because c is the coupon coloring of $G \circ_v H$ and v is the only vertex having neighbors from other copy of H. Therefore, $\chi_c(H) \geq k+1$, which is a contradiction. □

The next result gives the exact coupon coloring number of rooted product of a graph without isolated vertices with cycle C_n.

Theorem 3.2. *Let G be a graph without isolated vertices. Then*

$$\chi_c(G \circ_v C_n) = \begin{cases} \chi_c(C_n) + 1, & \textit{if } n \equiv 1, 3 \pmod 4 \\ \chi_c(C_n), & \textit{otherwise.} \end{cases}$$

Proof. Note that $\delta(G \circ_v C_n) = 2$. So, $\chi_c(G \circ_v C_n) \leq 2$. Imagine the vertices of $G \circ_v C_n$ as $v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$, where $v_{11}, v_{21}, \ldots, v_{m1}$ are the vertices of the copy of G in $G \circ_v C_n$ and $v_{i1} - v_{i2} - \cdots - v_{in} - v_{i1}$ is the i^th copy C_n^i of cycle C_n in $G \circ_v C_n$ corresponding to the root vertex v_{i1} for all $i = 1, 2, \ldots, m$.

If $n \equiv 0 \pmod 4$, then $\chi_c(C_n) = 2$ and C_n has a 2-coupon coloring. Color all the vertices of each copy of C_n in $G \circ_v C_n$ with this coloring. Clearly it is a coupon coloring of $G \circ_v C_n$. So, $\chi_c(G \circ_v C_n) = 2 = \chi_c(C_n)$.

If $n \equiv 1 \pmod 4$, then $\chi_c(C_n) = 1$. Define the coloring c_1 of $G \circ_v H$ as follows. color the vertices $v_{11}, v_{21}, \ldots, v_{m1}$ of the copy of G in such a way that a vertex with color 1 is adjacent to at least one vertex with color 2 and vice versa. This is possible since G has no isolated vertices. If $c_1(v_{i1}) = 1$, then define

$$c_1(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 1, 2 \pmod 4 \\ 2, & \text{otherwise.} \end{cases}$$

If $c_1(v_{i1}) = 2$, then define

$$c_1(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 0, 3 \pmod 4 \\ 2, & \text{otherwise.} \end{cases}$$

Clearly, the only possible bad vertex is v_{i1}. Because v_{i1}, v_{i2} and v_{in} have the same color. But v_{i1} has a neighbor in the copy of G in $G \circ_v C_n$ with a different color. So, c_1 is a coupon coloring and $\chi_c(G \circ_v C_n) = 2 = \chi_c(C_n) + 1$.

If $n \equiv 2 \pmod 4$, then $\chi_c(C_n) = 1$. It is enough to show that there does not exist a 2-coupon coloring of $G \circ_v C_n$. Suppose that c_2 is any 2-coupon coloring of $G \circ_v C_n$. Then by Lemma 3.1, v_{i2} and v_{in} must have same color. Without loss of generality assume that $c_2(v_{i2}) = 1 = c_2(v_{in})$. If $c_2(v_{i1}) = 1$ and define for all $j = 2, 3, \ldots, n$,

$$c_2(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 1, 2 \pmod 4 \\ 2, & \text{otherwise,} \end{cases}$$

then $c_2(v_{i(n-1)}) = 1$ and v_{in} is a bad vertex. If $c_2(v_{i1}) = 2$ and define for all $j = 2, 3, \ldots, n$,

$$c_2(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 2, 3 \pmod 4 \\ 2, & \text{otherwise,} \end{cases}$$

then $c_2(v_{i(n-1)}) = 2$ and v_{in} is a bad vertex. Hence, c_2 cannot be a 2-coupon coloring. Therefore, $\chi_c(G \circ_v C_n) = 1 = \chi_c(C_n)$.

If $n \equiv 3 \pmod 4$, then $\chi_c(C_n) = 1$. Define the coloring c_3 of $G \circ_v H$ by

$$c_3(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 2, 3 \pmod 4 \\ 2, & \text{otherwise.} \end{cases}$$

Here, the only possible bad vertex is v_{i1} and it is adjacent to v_{i2} and v_{in} with color 2 and v_{s1} with color 1 for some $s \in \{1, 2, \ldots, m\}$, since G has no isolated vertices. Hence, c_3 is a 2-coupon coloring of $G \circ_v C_n$ and $\chi_c(G \circ_v C_n) = 2 = \chi_c(C_n) + 1$. □

4 Bounds for $\chi_c(G \circ_v H)$

In this section, we establish some sharp bounds for the coupon coloring number of $G \circ_v H$. To show that k is an lower bound for the coupon coloring number, it is enough to show that there exist a k-coupon coloring.

Theorem 4.1. *Let G and H be two graphs without isolated vertices. Then*

$$\chi_c(G \circ_v H) \geq \chi_c(H).$$

Proof. Let $\chi_c(H) = k$ and c be a k-coupon coloring of H. Note that $G \circ_v H$ is obtained by taking $|V(G)|$ copies of H and one copy of G and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i = 1, 2, \ldots, |V(G)|$. So, if we give the coloring c to the $|V(G)|$ copies of H in $G \circ H$, then it is a k-coupon coloring of $G \circ_v H$. Thus, $\chi_c(G \circ_v H) \geq \chi_c(H)$. □

A trivial upper bound for the coupon coloring number of a graph without isolated vertices is $\delta(G)$. So, if $G \circ_v H$ is a graph without isolated vertices, then $\chi_c(G \circ_v H) \leq \delta(G \circ_v H)$.

Theorem 4.2. *Let G and H be two graphs without isolated vertices. Then*

$$\chi_c(G \circ_v H) \leq \chi_c(H) + \delta(G).$$

Proof. Suppose that $\chi_c(H) = k$. By Lemma 3.1, the root vertex can be adjacent to vertices with at most k colors in the corresponding copy of H. Note that the root vertex v identifies a vertex with degree $\delta(G)$ in G. In that case, v can be adjacent to at most $\delta(G)$ vertices in the copy of G. Thus, in $G \circ_v H$, v can be adjacent to vertices with at most $k + \delta(G)$ colors. Hence, $\chi_c(G \circ_v H) \leq k + \delta(G)$. □

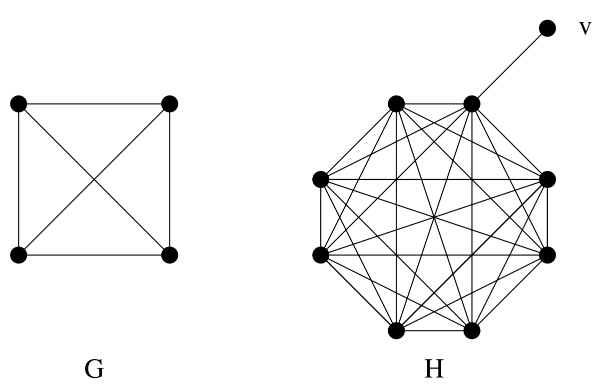

Figure 1.

Theorem 3.2 shows that the lower bound in Theorem 4.1 is sharp. The upper bound in Theorem 4.2 is also sharp. For, let $G = K_n$ and H be the graph obtained by adjoining a vertex v with an edge to a vertex of K_{2n}. The above figure 1 gives the the graphs G and H with $n = 4$. Since the degree of the vertex v is one, $\chi_c(H) = 1$. Define a coloring c of $G \circ_v H$ as follows: color all the vertices of G with n different colors. If the root vertex has color s in a copy of H, then color the vertex in H which is adjacent to v with s and color the remaining vertices with appropriate colors so that K_{2n} has an n- coupon coloring. Clearly, c is an n- coupon coloring of $G \circ_v H$. Hence, $\chi_c(G \circ_v H) \geq n = 1 + (n-1) = \chi_c(H) + \delta(G)$. So, $\chi_c(G \circ_v H) = \chi_c(H) + \delta(G)$.

Corollary 4.3. *Let G and H be two graphs without isolated vertices and let $\delta(G) = 1$. Then $\chi_c(G \circ_v H) \leq \chi_c(H) + 1$, and so $\chi_c(G \circ_v H) \in \{\chi_c(H), \chi_c(H) + 1\}$.*

Theorem 4.4. *Suppose that G and H be two graphs without isolated vertices. If v is a vertex in H such that the graph $H - v$ has no isolated vertices, then*

$$\chi_c(G \circ_v H) \leq \chi_c(H - v) + 1.$$

Proof. Let $\chi_c(H - v) = l$ and c be an $(l + 2)$- coupon coloring of $G \circ_v H$.
Claim - Every vertex in $H - v$ must be adjacent to vertices with at least $l + 1$ colors.
Otherwise, there exists a vertex u in $H - v$ such that u is not adjacent to vertices with $l + 1$ colors. That is, u is adjacent to vertices with at most l colors in H. Note that in $G \circ_v H$, u is a vertex which is adjacent only to the vertices of that copy of H. So in $G \circ_v H$, u can be adjacent to vertices with at most $l + 1$ colors (since u can be adjacent to v). Thus, $G \circ_v H$ cannot have an $(l + 2)$- coupon coloring, a contradiction. Hence the claim.

By the above claim, $H - v$ has a $(l + 1)$- coupon coloring. But it is a contradiction, since $\chi_c(H - v) = l$. Thus, $G \circ_v H$ cannot have a $(l + 2)$- coupon coloring. Therefore, $\chi_c(G \circ_v H) \leq l + 1$. □

The following corollary shows that the upper bound in Theorem 4.4 is sharp.

Corollary 4.5. *Let G be a graph without isolated vertices. Then*

$$\chi_c(G \circ_v K_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Suppose that n is odd. Then $\chi_c(K_n - v) = \chi_c(K_{n-1}) = \lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2}$. By Theorem 4.4, $\chi_c(G \circ_v K_n) \leq \lfloor \frac{n-1}{2} \rfloor + 1 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$. Define the coloring $c : V(G \circ_v K_n) \to [\frac{n+1}{2}]$ as follows: color the $n - 1$ vertices of each copy of $K_n - v$ with the colors $1, 2, \ldots, \frac{n-1}{2}$ such that each color appears twice and color the vertex v of each copy of K_n with the color $\frac{n+1}{2}$. Clearly, c is a coupon coloring of $G \circ_v K_n$ and so $\chi_c(G \circ_v K_n) \geq \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$.

If n is even, then $\chi_c(K_n) = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. By Theorem 4.1, $\chi_c(G \circ_v K_n) \geq \frac{n}{2} = \lceil \frac{n}{2} \rceil$. By Theorem 4.4, $\chi_c(G \circ_v K_n) \leq \chi_c(K_{n-1}) + 1 = \lfloor \frac{n-1}{2} \rfloor + 1 = \frac{n}{2} = \lceil \frac{n}{2} \rceil$. □

In Corollary 4.3, we have proved that $\chi_c(G \circ_v H)$ is either $\chi_c(H)$ or $\chi_c(H) + 1$ whenever $\delta(G) = 1$. The following corollary which follows from Theorem 4.1 and Theorem 4.4, gives a class of rooted product graphs for which $\chi_c(G \circ_v H) = \chi_c(H)$ holds.

Corollary 4.6. *Suppose that the graphs G, H and $H - v$ (where v is a vertex in H) have no isolated vertices and $\delta(G) = 1$. If $\chi_c(H) = k$ and $H - v$ cannot have a k-coupon coloring, then*

$$\chi_c(G \circ_v H) = k.$$

Let G and H be two graphs without isolated vertices and $\delta(G) = 1$. Assume that the root vertex $v \in V(H)$ is adjacent to all the other vertices of H. If $\chi_c(H) = k$ and $H - v$ have a k-coupon coloring, then $\chi_c(G \circ_v H) = k + 1$.

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Author information

Reji T and Pavithra R, Department of Mathematics, Government College Chittur, Palakkad, Kerala 678104, INDIA.
E-mail: rejiaran@gmail.com, pavithrajeni94@gmail.com