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The construction $(k+1;n)$-arcs and $(k+2;n)$-arcs from incomplete $(k;n)$- arc in $PG(3,q)$

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Abstract Our research is related to the projective space over the finite field. The aim of this paper is to construct new arcs of various degrees in the three-dimensional projective space over the Galois field of order 2, 3, 4, 5 . In $PG(3,q), q = 2,3,4$ and 5, an arc of degree n and order $k+1, k+2$ has found from incomplete $(k;n)-$ arc. Also, two geometrical methods are used to formed $(k+1;n)$-arc and $(k+2;n)$-arcs from incomplete $(k;n)$-arc. Many other properties of these arcs are given as T_i distributions and c_i distributions. The MATLAB programing is used to do all calculations.

1 Introduction

A projective 3-space $PG(3,q)$ over Galois field $GF(q)$, where $q = p^m$ for some prime number P and some integer m is a 3-dimensional projective space which consists of points, lines and planes with incidence relation between them. $PG(3,q)$ is satisfying the following axioms:

(a) Any three distinct non-collinear points, also any line and point not on it are contained in a unique plane.

(b) Any two distinct coplanar lines intersect in a unique point.

(c) Any line not on a given plane intersects the plane in a unique point.

(d) Any two distinct planes intersection in a unique line.

Any point in $PG(3,q)$ has the form of a quadrable (x_1,x_2,x_3,x_4), where x_1,x_2,x_3,x_4 are elements in $GF(q)$ with the exception of the quadrable consisting of four zero elements. Two quadrables (x_1,x_2,x_3,x_4) and (y_1,y_2,y_3,y_4) represent the same point if there exists λ in $GF(q)\backslash\{0\}$ such that $(x_1,x_2,x_3,x_4) = \lambda(y_1,y_2,y_3,y_4)$, Similarly, any plane in $PG(3,q)$ has the form of a quadrable $[x_1,x_2,x_3,x_4]$, where x_1,x_2,x_3,x_4 are elements in $GF(q)$ with the exception of the quadrable consisting of four zero elements.

Two quadrables $[x_1,x_2,x_3,x_4]$ and $[y_1,y_2,y_3,y_4]$ represent the same plane if there exists λ in $GF(q)\backslash\{0\}$ such that $[x_1,x_2,x_3,x_4] = \lambda[y_1,y_2,y_3,y_4]$.

A point $P(x_1,x_2,x_3,x_4)$ is incident with the plane $\pi[a_1,a_2,a_3,a_4]$ iff $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$. As a historical background, the construct new arcs of various degrees in the two-dimensional projective space over the Galois field of order 2,3,...,etc. are constructed by Hirschfeld [6]-[8].

2 Basic Concepts

Definition 2.1. [3],[7] A plane π in $PG(3,q)$ is a set of all points $P(x_1,x_2,x_3,x_4)$ satisfying a linear equation $u_1x_1+u_2x_2+u_3x_3+u_4x_4 = 0$. This plane is denoted by $\pi[u_1,u_2,u_3,u_4]$, where x_1,x_2,x_3,x_4 are elements in $GF(q)$ with the exception of the quadrable consisting of four zero elements.

Theorem 2.2. *[3],[7] The points of* $PG(3,q)$ *have unique forms which are* $(1,0,0,0)$, $(x,1,0,0)$, $(x,y,1,0)$ *and* $(x,y,z,1)$*for all* x,y,z *in* $GF(q)$*. which are* $(1,0,0,0)$ *is one point,* $(x,1,0,0)$ *are* q *points,* $(x,y,1,0)$ *are* q^2 *points, and* $(x,y,z,1)$ *are* q^3 *points, for all* x,y,z *in* $GF(3,q)$.

Theorem 2.3. *[3],[7] The planes of $PG(3,q)$ have unique forms which are $[1,0,0,0]$, $[x,1,0,0]$, $[x,y,1,0]$, $[x,y,z,1]$ for all x,y,z in $GF(q)$. which are $[1,0,0,0]$ is one plane, $[x,1,0,0]$ are q planes, $[x,y,1,0]$ are q^2 planes, and $[x,y,z,1]$ are q^3 planes, for all x,y,z in $GF(3,q)$.*

Theorem 2.4. *[3],[7] There exists q^3+q^2+q+1 of points in $PG(3,q)$.*

Theorem 2.5. *[3],[7] There exist q^3+q^2+q+1 planes in $PG(3,q)$.*

Theorem 2.6. *[3],[7] Every plane in $PG(3,q)$ contains exactly q^2+q+1 points (lines) and every point is on exactly q^2+q+1 planes.*

Theorem 2.7. *[3],[7] Every line in $PG(3,q)$ contains exactly $q+1$ points and every point is on exactly $q+1$ lines."*

Theorem 2.8. *[3],[7] Any two points in $PG(3,q)$ are on exactly $q+1$ planes.*

Theorem 2.9. *[3],[7] Any two planes in $PG(3,q)$ intersect in exactly $q+1$ points.*

Theorem 2.10. *[3],[7] there exist $(q^2+1)(q^2+q+1)$ lines in $PG(3,q)$.*

Theorem 2.11. *[3],[7] Any line in $PG(3,q)$ is on exactly $q+1$ planes.*

Definition 2.12. [3],[7] A $(k;n)$-arcA in $PG(3,q)$ is a set of k points such that at most n points of which lie in any plane, $n \geq 3$. n is called degree of the $(k;n)$-arc.

Definition 2.13. [3],[7] In $PG(3,q)$ if k is any k-set, then an n-secant of k is a line (a plane) ℓ such that $|\ell \cap k| = n$. A 0-secant is called an external line (plane) of k, a 1-secant is called a unisecant line (plane), a 2-secant is called a bisecant line and 3-secant is called a trisecant line.

Definition 2.14. [8] Let T_i be the total number of the i-secants of a $(k;n)$-arc A, then the type of A denoted by $(T_n, T_{n-1}, T_{n-2}, \ldots, T_0)$.

Definition 2.15. [8] A point R not on a $(k;n)$-arc A has index i if there exists exactly i (n-secant) of A through R, one can denoted the number of points R of index i by c_i.

It is concluded that the $(k;n)$-arc set is complete iff $c_0 = 0$. Thus the k-set is complete iff every points of $PG(3,q)$ lies on som n-secant of the $(k;n)$-set.

Definition 2.16. [8] $(k;n)$-arc A is complete if it is not contained in $(k+1;n)$-arc.

Remark 2.17. [8] A $(k;n)$-arc is complete iff $c_0 = 0$, in other words the $(k;n)$-arc is complete iff every point of $PG(3,q)$ lies on some n-secant of the $(k;n)$-arc.

Theorem 2.18. *[8] Let c_i be the number of points of index i in $PG(3,q)$ which are not on a $(k;n)$-arc A, then the constants c_i of A satisfy the following equations:*

(i) $\sum_{\alpha}^{\beta} c_i = q^3+q^2+q+1-k.$

(ii) $\sum_{\alpha}^{\beta} ic_i = \frac{k(k-1)\ldots.(k-n+1)}{n!} + (q^2+q+1-n)$

where α is the smallest i for which $c_i \neq 0$, β be the largest i for which $c_i \neq 0$.

Theorem 2.19. *[8] Let $t(p)$ represents the number of unsecants (planes) through a point P of a $(k;n)$-arc A in $PG(3,q)$, and let T_i represent the numbers of i-secants (planes)for the arc A, then:*

(i) $t = t(p) = q^2+q+2-k-\frac{(k-1)(k-2)}{2} - \cdots - \frac{(k-1)(k-2)\ldots.(k-n+1)}{(n-1)!}$

(ii) $T_1 = kt$

(iii) $T_2 = \frac{k(k-1)}{2}$

(iv) $T_3 = \frac{k(k-1)(k-2)}{3!}$

(v) $T_n = \frac{k(k-1)\ldots..(k-n+1)}{n!}$

(vi) $T_0 = q^3+q^2+q+1-kt-\frac{k(k-1)}{2}-\frac{k(k-1)(k-2)}{3!} - \cdots - \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$

Table 1. points and plaens in $PG(3,2)$.

i	P_i	π_i						
1	(1,0,0,0)	2	3	4	6	7	10	12
2	(0,1,0,0)	1	3	4	7	9	14	15
3	(0,0,1,0)	1	2	4	5	8	10	15
4	(0,0,0,1)	1	2	3	5	6	9	11
5	(1,1,0,0)	3	4	5	7	8	11	13
6	(0,1,1,0)	1	4	6	11	12	13	15
7	(0,0,1,1)	1	2	5	7	12	13	14
8	(1,1,0,1)	3	5	10	11	12	14	15
9	(1,0,1,0)	2	4	9	10	11	13	14
10	(0,1,0,1)	1	3	8	9	10	12	13
11	(1,1,1,0)	4	5	6	8	9	12	14
12	(0,1,1,1)	1	6	7	8	10	11	14
13	(1,1,1,1)	5	6	7	9	10	13	15
14	(1,0,1,1)	2	7	8	9	11	12	15
15	(1,0,0,1)	2	3	6	8	13	14	15

3 The construction $(k+1;n)$-arcs from incomplete (k,n)-arc

We can construct $(k+1;n)$-arcs from any incomplete arc with the same dgree as follows:

(i) we define the $(k;n)$-arc points in the table of points and planes for $PG(3,q)$.

(ii) we delete all points which lie in n-secant from the projective space $PG(3,q)$.

(iii) we add in each time 1 point of the remaining points to $(k;n)$-arc to obtain $(k+1;n)$- arcs.

To illustrate this method, we take the following examples:

3.1 Construction (8;5)-arcs from incomplete (7,5)-arc in PG(3,2)

A projective space $PG(3,2)$ over Galois field $GF(2)$ contains 15 points and 15 planes, each plane contain 7 points and every 2 planes intersct in three points. since a $(k;n)$-arc is a set of k points there is no $n+1$ of them are coplener, we can construct arc by choosing a set of k points such that there are no $n+1$ of them in the same plane for example let $\overline{A}=\{1,2,3,4,5,6,13\}$ is a (7,5)-arc, now we define points of $\overline{A}$ on the table 1.

We cancel out the points which lie in 5-secant from the space which are 9, 11. now we add on each time one point from the remaining poins to the (7;5)-arc $\overline{A}$, we get the following incomplete (8;5)-arc:

$$\overline{A}_i : \overline{A}_1=\{1,2,3,4,5,6,7,13\}, \quad \overline{A}_2=\{1,2,3,4,5,6,8,13\}, \overline{A}_3=\{1,2,3,4,5,6,10,13\},$$

$$\overline{A}_4=\{1,2,3,4,5,6,12,13\}, \quad \overline{A}_5=\{1,2,3,4,5,6,13,14\}, \overline{A}_6=\{1,2,3,4,5,6,13,15\}.$$

3.2 Construction of (9,5)-arcs from Incomplete (8;5)-arc in PG(3,3)

$PG(3,3)$ consists of 40 points and 40 plane, every points is on exactly 13 planes, every two plane intersect in to four points, so as to constract (k,n)-arc in $PG(3,3)$ we choose any set of space points such that lie on n-secant and there is no $n+1$ of them are coplenar for example let $\dot{B}=\{1,2,3,6,9,19,27,33\}$ where $\dot{B}$ incomplete (8;5)- arc, now to construct (9;5)-arcs from $\dot{B}$ define points of $\dot{B}$ in table 2.

Table 2. points and planes in $PG(3,3)$.

i	Points	Planes												
1	(1,0,0,0)	2	5	8	9	14	17	20	23	26	29	32	35	38
2	(0,1,0,0)	1	9	10	11	14	15	16	23	24	25	32	33	34
3	(1,1,0,0)	4	7	9	12	14	19	21	23	28	30	32	37	39
4	(2,1,0,0)	3	6	9	13	14	18	22	23	27	31	32	36	40
5	(0,1,1,0)	1	8	12	13	14	15	16	29	30	31	35	36	37
6	(1,1,1,0)	4	6	8	11	14	19	21	25	27	29	33	35	40
7	(2,1,1,0)	3	7	8	10	14	18	22	24	28	29	34	35	39
8	(0,2,1,0)	1	5	6	7	14	15	16	26	27	28	38	39	40
9	(0,0,1,0)	1	2	3	4	14	15	16	17	18	19	20	21	22
10	(1,0,1,0)	2	7	11	13	14	17	20	25	28	31	33	36	39
11	(2,0,1,0)	2	6	10	12	14	17	20	24	27	30	34	37	40
12	(1,2,1,0)	3	5	11	12	14	18	22	25	26	30	33	37	38
13	(2,2,1,0)	4	5	10	13	14	19	21	24	26	31	34	36	38
14	(0,0,0,1)	1	2	3	4	5	6	7	8	9	10	11	12	13
15	(1,0,0,1)	2	5	8	9	16	19	22	25	28	31	34	37	40
16	(2,0,0,1)	2	5	8	9	15	18	21	24	27	30	33	36	39
17	(0,1,0,1)	1	9	10	11	20	21	22	29	30	31	38	39	40
18	(1,1,0,1)	4	7	9	12	16	18	20	25	27	29	34	36	38
19	(2,1,0,1)	3	6	9	13	15	19	20	24	28	29	33	37	38
20	(0,2,0,1)	1	9	10	11	17	18	19	26	27	28	35	36	37
21	(1,2,0,1)	3	6	9	13	16	17	21	25	26	30	34	35	39
22	(2,2,0,1)	4	7	9	12	15	17	22	24	26	31	33	35	40
23	(0,0,1,1)	1	2	3	4	32	33	34	35	36	37	38	39	40
24	(1,0,1,1)	2	7	11	13	16	19	22	24	27	30	32	35	38
25	(2,0,1,1)	2	6	10	12	15	18	21	25	28	31	32	35	38
26	(0,1,1,1)	1	8	12	13	20	21	22	26	27	28	32	33	34
27	(1,1,1,1)	4	6	8	11	16	18	20	24	26	31	32	37	39
28	(2,1,1,1)	3	7	8	10	15	19	20	25	26	30	32	36	40
29	(0,2,1,1)	1	5	6	7	17	18	19	29	30	31	32	33	34
30	(1,2,1,1)	3	5	11	12	16	17	21	24	28	29	32	36	40
31	(2,2,1,1)	4	5	10	13	15	17	22	25	27	29	32	37	39
32	(0,0,2,1)	1	2	3	4	23	24	25	26	27	28	29	30	31
33	(1,0,2,1)	2	6	10	12	16	19	22	23	26	29	33	36	39
34	(2,0,2,1)	2	7	11	13	15	18	21	23	26	29	34	37	40
35	(0,1,2,1)	1	5	6	7	20	21	22	23	24	25	35	36	37
36	(1,1,2,1)	4	5	10	13	16	18	20	23	28	30	33	35	40
37	(2,1,2,1)	3	5	11	12	15	19	20	23	27	31	34	35	39
38	(0,2,2,1)	1	8	12	13	17	18	19	23	24	25	38	39	40
39	(1,2,2,1)	3	7	8	10	16	17	21	23	27	31	33	37	38
40	(2,2,2,1)	4	6	8	11	15	17	22	23	28	30	34	36	38

We delete all points that lie into 5-secant from the space and add in each time one point from the remaining points of the space to the (8;5)-arc $\dot{B}$, the resulting arcs is:

$\dot{B}_1\{1,2,3,6,9,14,19,27,33\}$, $\dot{B}_2\{1,2,3,6,9,16,19,27,33\}$,$\dot{B}_3\{1,2,3,6,9,17,19,27,33\}$,

$\dot{B}_4\{1,2,3,6,9,18,19,27,33\}$, $\dot{B}_5\{1,2,3,6,9,19,21,27,33\}$,$\dot{B}_6\{1,2,3,6,9,19,22,27,33\}$,

$\dot{B}_7\{1,2,3,6,9,19,23,27,33\}$, $\dot{B}_8\{1,2,3,6,9,19,25,27,33\}$,$\dot{B}_9\{1,2,3,6,9,19,26,27,33\}$,

$\dot{B}_{10}\{1,2,3,6,9,19,27,30,33\}$, $\dot{B}_{11}\{1,2,3,6,9,19,27,31,33\}$,$\dot{B}_{12}\{1,2,3,6,9,19,27,32,33\}$,

$\dot{B}_{13}\{1,2,3,6,9,19,27,33,34\}$, $\dot{B}_{14}=1,2,3,6,9,19,27,33,35$,$\dot{B}_{15}\{1,2,3,6,9,19,27,33,36\}$,

$\dot{B}_{16}\{1,2,3,6,9,19,27,33,39\}$, $\dot{B}_{17}\{1,2,3,6,9,19,27,33,40\}$.

3.3 Construction of (7,4)-arcs

From Incomplete (6,4)-arc in $PG(3,4)$: $PG(3,4)$ consist of (85) points and 85 plane, every points is on exactly 21 planes, every two plane intersect in five points, let $\widehat{C}=\{1,2,3,6,22,43\}$ is incomplete (6;4)-arc in $PG(3,4)$, now to construct (7;4)-arcs from $\widehat{C}$ define points of $\widehat{C}$ in the following table:

We cancel out all points that lie into 4-secant from the space and add in each time one point from the remaining points of the space to the (6;4)-arc ?, we get (7;4)- arcs which are:

$\hat{C}_1=\{1,2,3,6,22,31,43\}$. $\hat{C}_2=\{1,2,3,6,22,32,43\}$.$\hat{C}_3=\{1,2,3,6,22,33,43\}$.

$\hat{C}_4=\{1,2,3,6,22,35,43\}$. $\hat{C}_5=\{1,2,3,6,22,36,43\}$.$\hat{C}_6=\{1,2,3,6,22,37,43\}$.

$\hat{C}_7=\{1,2,3,6,22,43,46\}$. $\hat{C}_8=\{1,2,3,6,22,43,48\}$.$\hat{C}_9=\{1,2,3,6,22,43,49\}$.

$\hat{C}_{10}=\{1,2,3,6,22,43,50\}$. $\hat{C}_{11}=\{1,2,3,6,22,43,52\}$.$\hat{C}_{12}=\{1,2,3,6,22,43,53\}$.

$\hat{C}_{13}=\{1,2,3,6,22,43,62\}$. $\hat{C}_{14}=\{1,2,3,6,22,43,63\}$.$\hat{C}_{15}=\{1,2,3,6,22,43,65\}$.

$\hat{C}_{16}=\{1,2,3,6,22,43,66\}$. $\hat{C}_{17}=\{1,2,3,6,22,43,67\}$.$\hat{C}_{18}=\{1,2,3,6,22,43,69\}$.

$\hat{C}_{19}=\{1,2,3,6,22,43,78\}$. $\hat{C}_{20}=\{1,2,3,6,22,43,79\}$.$\hat{C}_{21}=\{1,2,3,6,22,43,80\}$.

$\hat{C}_{22}=\{1,2,3,6,22,43,82\}$. $\hat{C}_{23}=\{1,2,3,6,22,43,83\}$.$\hat{C}_{24}=\{1,2,3,6,22,43,84\}$.

3.4 Construction (8,4)-arcs from Incomplete (7;4)-arc in PG(3,5)

A projective space $PG(3,5)$ consists of 156 points and 156 planes every plane contains 31 points and every point is on 31 plane, any two planes from this space intersct in five points, we will construct (8;4)-arcs from incomplete (7;4)-arc.

Let $\dot{D}=\{1,2,7,32,63,100,101\}$ where $\dot{D}$ is an arc in $PG(3,5)$, the first step define points of $\dot{D}$ in table 4.

We delete all points that lie on 4- secant from the projective space , the remaining points of the space are: 12, 14, 15, 16, 22, 23, 24, 26, 37, 39, 40, 41, 42, 43, 45, 46, 58, 59, 60, 61, 67, 68, 70, 71, 114, 115, 116, 127, 128, 129, 130, 133, 134, 135, 136, 152, 153, 154, 155, add one of the remaining points in each time to incomplete (14;7)-arc $\dot{D}$,then we get (15;7)-arc $\dot{D}$, where $i=1,2,...,40$:

$\dot{D}_1=\{1,2,7,12,32,63,100,101\}$, $\dot{D}_2=\{1,2,7,14,32,63,100,101\}$,

$\dot{D}_3=\{1,2,7,15,32,63,100,101\}$, $\dot{D}_4=\{1,2,7,16,32,63,100,101\}$,

$\dot{D}_5=\{1,2,7,22,32,63,100,101\}$, $\dot{D}_6=\{1,2,7,23,32,63,100,101\}$,

$\dot{D}_7=\{1,2,7,24,32,63,100,101\}$, $\dot{D}_8=\{1,2,7,26,32,63,100,101\}$

Table 3. Points and planes in $PG(3,2)$.

i	P_i	π_i																				
1	(1,0,0,0)	2	6	10	14	18	22	26	30	34	38	42	46	50	54	58	62	66	70	74	78	82
2	(0,1,0,0)	1	6	7	8	9	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
3	(1,1,0,0)	3	6	11	16	21	22	26	30	34	39	43	47	51	56	60	64	68	73	77	81	85
4	(2,1,0,0)	5	6	13	15	20	22	26	30	34	41	45	49	53	55	59	63	67	72	76	80	84
5	(3,1,0,0)	4	6	12	17	19	22	26	30	34	40	44	48	52	57	61	65	69	71	75	79	83
6	(0,0,1,0)	1	2	3	4	5	22	23	24	25	38	39	40	41	54	55	56	57	70	71	72	73
7	(1,0,1,0)	2	7	11	15	19	22	27	32	37	38	43	48	53	54	59	64	69	70	75	80	85
8	(2,0,1,0)	2	9	13	17	21	22	29	31	36	38	45	47	52	54	61	63	68	70	77	79	84
9	(3,0,1,0)	2	8	12	16	20	22	28	33	35	38	44	49	51	54	60	65	67	70	76	81	83
10	(0,1,1,0)	1	10	11	12	13	22	23	24	25	42	43	44	45	62	63	64	65	82	83	84	85
11	(1,1,1,0)	3	7	10	17	20	22	27	32	37	39	42	49	52	56	61	62	67	73	76	79	82
12	(2,1,1,0)	5	9	10	16	19	22	29	31	36	41	42	48	51	55	60	62	69	72	75	81	82
13	(3,1,1,0)	4	8	10	15	21	22	28	33	35	40	42	47	53	57	59	62	68	71	77	80	82
14	(0,2,1,0)	1	18	19	20	21	22	23	24	25	46	47	48	49	66	67	68	69	74	75	76	77
15	(1,2,1,0)	4	7	13	16	18	22	27	32	37	40	45	46	51	57	60	63	66	71	74	81	84
16	(2,2,1,0)	3	9	12	15	18	22	29	31	36	39	44	46	53	56	59	65	66	73	74	80	83
17	(3,2,1,0)	5	8	11	17	18	22	28	33	35	41	43	46	52	55	61	64	66	72	74	79	85
18	(0,3,1,0)	1	14	15	16	17	22	23	24	25	50	51	52	53	58	59	60	61	78	79	80	81
19	(1,3,1,0)	5	7	12	14	21	22	27	32	37	41	44	47	50	55	58	65	68	72	77	78	83
20	(2,3,1,0)	4	9	11	14	20	22	29	31	36	40	43	49	50	57	58	64	67	71	76	78	85
21	(3,3,1,0)	3	8	13	14	19	22	28	33	35	39	45	48	50	56	58	63	69	73	75	78	84
22	(0,0,0,1)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
23	(1,0,0,1)	2	6	10	14	18	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83
24	(2,0,0,1)	2	6	10	14	18	25	29	33	37	41	45	49	53	57	61	65	69	73	77	81	85
25	(3,0,0,1)	2	6	10	14	18	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84
26	(0,0,1,1)	1	2	3	4	5	26	27	28	29	42	43	44	45	58	59	60	61	74	75	76	77
27	(1,0,1,1)	2	7	11	15	19	23	26	33	36	39	42	49	52	55	58	65	68	71	74	81	84
28	(2,0,1,1)	2	9	13	17	21	25	26	32	35	41	42	48	51	57	58	64	67	73	74	80	83
29	(3,0,1,1)	2	8	12	16	20	24	26	31	37	40	42	47	53	56	58	63	69	72	74	79	85
30	(0,0,2,1)	1	2	3	4	5	34	35	36	37	50	51	52	53	66	67	68	69	82	83	84	85
31	(1,0,2,1)	2	8	12	16	20	23	29	32	34	39	45	48	50	55	61	64	66	71	77	80	82
32	(2,0,2,1)	2	7	11	15	19	25	28	31	34	41	44	47	50	57	60	63	66	73	76	79	82
33	(3,0,2,1)	2	9	13	17	21	24	27	33	34	40	43	49	50	56	59	65	66	72	75	81	82
34	(0,0,3,1)	1	2	3	4	5	30	31	32	33	46	47	48	49	62	63	64	65	78	79	80	81
35	(1,0,3,1)	2	9	13	17	21	23	28	30	37	39	44	46	53	55	60	62	69	71	76	78	85
36	(2,0,3,1)	2	8	12	16	20	25	27	30	36	41	43	46	52	57	59	62	68	73	75	78	84
37	(3,0,3,1)	2	7	11	15	19	24	29	30	35	40	45	46	51	56	61	62	67	72	77	78	83
38	(0,1,0,1)	1	6	7	8	9	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
39	(1,1,0,1)	3	6	11	16	21	23	27	31	35	38	42	46	50	57	61	65	69	72	76	80	84
40	(2,1,0,1)	5	6	13	15	20	25	29	33	37	38	42	46	50	56	60	64	68	71	75	79	83
41	(3,1,0,1)	4	6	12	17	19	24	28	32	36	38	42	46	50	55	59	63	67	73	77	81	85
42	(0,1,1,1)	1	10	11	12	13	26	27	28	29	38	39	40	41	66	67	68	69	78	79	80	81
43	(1,1,1,1)	3	7	10	17	20	23	26	33	36	38	43	48	53	57	60	63	66	72	77	78	83

i	P_i	π_i																				
44	(2,1,1,1)	5	9	10	16	19	25	26	32	35	38	45	47	52	56	59	65	66	71	76	78	85
45	(3,1,1,1)	4	8	10	15	21	24	26	31	37	38	44	49	51	55	61	64	66	73	75	78	84
46	(0,1,2,1)	1	14	15	16	17	34	35	36	37	38	39	40	41	62	63	64	65	74	75	76	77
47	(1,1,2,1)	3	8	13	14	19	23	29	32	34	38	44	49	51	57	59	62	68	72	74	79	85
48	(2,1,2,1)	5	7	12	14	21	25	28	31	34	38	43	48	53	56	61	62	67	71	74	81	84
49	(3,1,2,1)	4	9	11	14	20	24	27	33	34	38	45	47	52	55	60	62	69	73	74	80	83
50	(0,1,3,1)	1	18	19	20	21	30	31	32	33	38	39	40	41	58	59	60	61	82	83	84	85
51	(1,1,3,1)	3	9	12	15	18	23	28	30	37	38	45	47	52	57	58	64	67	72	75	81	82
52	(2,1,3,1)	5	8	11	17	18	25	27	30	36	38	44	49	51	56	58	63	69	71	77	80	82
53	(3,1,3,1)	4	7	13	16	18	24	29	30	35	38	43	48	53	55	58	65	68	73	76	79	82
54	(0,2,0,1)	1	6	7	8	9	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85
55	(1,2,0,1)	4	6	12	17	19	23	27	31	35	41	45	49	53	56	60	64	68	70	74	78	82
56	(2,2,0,1)	3	6	11	16	21	25	29	33	37	40	44	48	52	55	59	63	67	70	74	78	82
57	(3,2,0,1)	5	6	13	15	20	24	28	32	36	39	43	47	51	57	61	65	69	70	74	78	82
58	(0,2,1,1)	1	18	19	20	21	26	27	28	29	50	51	52	53	62	63	64	65	70	71	72	73
59	(1,2,1,1)	4	7	13	16	18	23	26	33	36	41	44	47	50	56	61	62	67	70	75	80	85
60	(2,2,1,1)	3	9	12	15	18	25	26	32	35	40	43	49	50	55	60	62	69	70	77	79	84
61	(3,2,1,1)	5	8	11	17	18	24	26	31	37	39	45	48	50	57	59	62	68	70	76	81	83
62	(0,2,2,1)	1	10	11	12	13	34	35	36	37	46	47	48	49	58	59	60	61	70	71	72	73
63	(1,2,2,1)	4	8	10	15	21	23	29	32	34	41	43	46	52	56	58	63	69	70	76	81	83
64	(2,2,2,1)	3	7	10	17	20	25	28	31	34	40	45	46	51	55	58	65	68	70	75	80	85
65	(3,2,2,1)	5	9	10	16	19	24	27	33	34	39	44	46	53	57	58	64	67	70	77	79	84
66	(0,2,3,1)	1	14	15	16	17	30	31	32	33	42	43	44	45	66	67	68	69	70	71	72	73
67	(1,2,3,1)	4	9	11	14	20	23	28	30	37	41	42	48	51	56	59	65	66	70	77	79	84
68	(2,2,3,1)	3	8	13	14	19	25	27	30	36	40	42	47	53	55	61	64	66	70	76	81	83
69	(3,2,3,1)	5	7	12	14	21	24	29	30	35	39	42	49	52	57	60	63	66	70	75	80	85
70	(0,3,0,1)	1	6	7	8	9	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69
71	(1,3,0,1)	5	6	13	15	20	23	27	31	35	40	44	48	52	54	58	62	66	73	77	81	85
72	(2,3,0,1)	4	6	12	17	19	25	29	33	37	39	43	47	51	54	58	62	66	72	76	80	84
73	(3,3,0,1)	3	6	11	16	21	24	28	32	36	41	45	49	53	54	58	62	66	71	75	79	83
74	(0,3,1,1)	1	14	15	16	17	26	28	29	45	46	47	48	49	54	55	56	57	82	83	84	85
75	(1,3,1,1)	5	7	12	14	21	23	26	33	36	40	45	46	51	54	59	64	69	73	76	79	82
76	(2,3,1,1)	4	9	11	14	20	25	26	32	35	39	44	46	53	54	61	63	68	72	75	81	82
77	(3,3,1,1)	3	8	13	14	19	24	26	31	37	41	43	46	52	54	60	65	67	71	77	80	82
78	(0,3,2,1)	1	18	19	20	21	34	35	36	37	42	43	44	45	54	55	56	57	78	79	80	81
79	(1,3,2,1)	5	8	11	17	18	23	29	32	34	40	42	47	53	54	60	65	67	73	75	78	84
80	(2,3,2,1)	4	7	13	16	18	25	28	31	34	39	42	49	52	54	59	64	69	72	77	78	83
81	(3,3,2,1)	3	9	12	15	18	24	27	33	34	41	42	48	51	54	61	63	68	71	76	78	85
82	(0,3,3,1)	1	10	11	12	13	30	31	32	33	50	51	52	53	54	55	56	57	74	75	76	77
83	(1,3,3,1)	5	9	10	16	19	23	28	30	37	40	43	49	50	54	61	63	68	73	74	80	83
84	(2,3,3,1)	4	8	10	15	21	25	27	30	36	39	45	48	50	54	60	65	67	72	74	79	85
85	(3,3,3,1)	3	7	10	17	20	24	29	30	35	41	44	47	50	54	59	64	69	71	74	81	84

Table 4. Points and planes in $PG(3,5)$.

i	P_i	πi										
1	(1,0,0,0)	1	7	12	17	22	27	32	37	42	47	52
			57	62	67	72	77	82	87	92	97	102
			107	112	117	122	127	132	137	142	147	152
2	(0,1,0,0)	1	7	8	9	10	11	32	33	34	35	36
			57	58	59	60	61	82	83	84	85	86
			107	108	109	110	111	132	133	134	135	136
3	(1,1,0,0)	6	7	16	20	24	28	32	41	45	49	53
			57	66	70	74	78	82	91	95	99	103
			107	116	120	124	128	132	141	145	149	153
4	(2,1,0,0)	4	7	14	21	23	30	32	39	46	48	55
			57	64	71	73	80	82	89	96	98	105
			107	114	121	123	130	132	139	146	148	155
5	(3,1,0,0)	5	7	15	18	26	29	32	40	43	51	54
			57	65	68	76	79	82	90	93	101	104
			107	115	118	126	129	132	140	143	151	154
6	(4,1,0,0)	3	7	13	19	25	31	32	38	44	50	56
			57	63	69	75	81	82	88	94	100	106
			107	113	119	125	131	132	138	144	150	156
7	(0,0,1,0)	1	2	3	4	5	6	32	33	34	35	36
			37	38	39	40	41	42	43	44	45	46
			47	48	49	50	51	52	53	54	55	56
8	(1,0,1,0)	2	11	16	21	26	31	32	37	42	47	52
			61	66	71	76	81	85	90	95	100	105
			109	114	119	124	129	133	138	143	148	153
9	(2,0,1,0)	2	9	14	19	24	29	32	37	42	47	52
			59	64	69	74	79	86	91	96	101	106
			108	113	118	123	128	135	140	145	150	155
10	(3,0,1,0)	2	10	15	20	25	30	32	37	42	47	52
			60	65	70	75	80	83	88	93	98	103
			111	116	121	126	131	134	139	144	149	154
11	(4,0,1,0)	2	8	13	18	23	28	32	37	42	47	52
			58	63	68	73	78	84	89	94	99	104
			110	115	120	125	130	136	141	146	151	156
12	(0,1,1,0)	1	27	28	29	30	31	32	33	34	35	36
			77	78	79	80	81	97	98	99	100	101
			117	118	119	120	121	137	138	139	140	141
13	(1,1,1,0)	6	11	15	19	23	27	32	41	45	49	53
			61	65	69	73	77	85	89	93	97	106
			109	113	117	126	130	133	137	146	150	154
14	(2,1,1,0)	4	9	16	18	25	27	32	39	46	48	55
			59	66	68	75	77	86	88	95	97	104
			108	115	117	124	131	135	137	144	151	153
15	(3,1,1,0)	5	10	13	21	24	27	32	40	43	51	54

i	P_i	πi										
			60	63	71	74	77	83	91	94	97	105
			111	114	117	125	128	134	137	145	148	156
		3	8	14	20	26	27	32	38	44	50	56
16	(4,1,1,0)		58	64	70	76	77	84	90	96	97	103
			110	116	117	123	129	136	137	143	149	155
		1	17	18	19	20	21	32	33	34	35	36
17	(0,2,1,0)		67	68	69	70	71	102	103	104	105	106
			112	113	114	115	116	147	148	149	150	151
		5	11	14	17	25	28	32	40	43	51	54
18	(1,2,1,0)		61	64	67	75	78	85	88	96	99	102
			109	112	120	123	131	133	141	144	147	155
		6	9	13	17	26	30	32	41	45	49	53
19	(2,2,1,0)		59	63	67	76	80	86	90	94	98	102
			108	112	121	125	129	135	139	143	147	156
		3	10	16	17	23	29	32	38	44	50	56
20	(3,2,1,0)		60	66	67	73	79	83	89	95	101	102
			111	112	118	124	130	134	140	146	147	153
		4	8	15	17	24	31	32	39	46	48	55
21	(4,2,1,0)		58	65	67	74	81	84	91	93	100	102
			110	112	119	126	128	136	138	145	147	154
		1	22	23	24	25	26	32	33	34	35	36
22	(0,3,1,0)		72	73	74	75	76	87	88	89	90	91
			127	128	129	130	131	142	143	144	145	146
		4	11	13	20	22	29	32	39	46	48	55
23	(1,3,1,0)		61	63	70	72	79	85	87	94	101	103
			109	116	118	125	127	133	140	142	149	156
		3	9	15	21	22	28	32	38	44	50	56
24	(2,3,1,0)		59	65	71	72	78	86	87	93	99	105
			108	114	120	126	127	135	141	142	148	154
		6	10	14	18	22	31	32	41	45	49	53
25	(3,3,1,0)		60	64	68	72	81	83	87	96	100	104
			111	115	119	123	127	134	138	142	151	155
		5	8	16	19	22	30	32	40	43	51	54
26	(4,3,1,0)		58	66	69	72	80	84	87	95	98	106
			110	113	121	124	127	136	139	142	150	153
		1	12	13	14	15	16	32	33	34	35	36
27	(0,4,1,0)		62	63	64	65	66	92	93	94	95	96
			122	123	124	125	126	152	153	154	155	156
		3	11	12	18	24	30	32	38	44	50	56
28	(1,4,1,0)		61	62	68	74	80	85	91	92	98	104
			109	115	121	122	128	133	139	145	151	152
		5	9	12	20	23	31	32	40	43	51	54
29	(2,4,1,0)		59	62	70	73	81	86	89	92	100	103
			108	116	119	122	130	135	138	146	149	152

i	P_i	πi										
30	(3,4,1,0)	4	10	12	19	26	28	32	39	46	48	55
			60	62	69	76	78	83	90	92	99	106
			111	113	120	122	129	134	141	143	150	152
31	(4,4,1,0)	6	8	12	21	25	29	32	41	45	49	53
			58	62	71	75	79	84	88	92	101	105
			110	114	118	122	131	136	140	144	148	152
32	(0,0,0,1)	1	2	3	4	5	6	7	8	9	10	11
			12	13	14	15	16	17	18	19	20	21
			22	23	24	25	26	27	28	29	30	31
33	(1,0,0,1)	2	7	12	17	22	27	36	41	46	51	56
			61	66	71	76	81	86	91	96	101	106
			111	116	121	126	131	136	141	146	151	156
34	(2,0,0,1)	2	7	12	17	22	27	34	39	44	49	54
			59	64	69	74	79	84	89	94	99	104
			109	114	119	124	129	134	139	144	149	154
35	(3,0,0,1)	2	7	12	17	22	27	35	40	45	50	55
			60	65	70	75	80	85	90	95	100	105
			110	115	120	125	130	135	140	145	150	155
36	(4,0,0,1)	2	7	12	17	22	27	33	38	43	48	53
			58	63	68	73	78	83	88	93	98	103
			108	113	118	123	128	133	138	143	148	153
37	(0,1,0,1)	1	7	8	9	10	11	52	53	54	55	56
			77	78	79	80	81	102	103	104	105	106
			127	128	129	130	131	152	153	154	155	156
38	(1,1,0,1)	6	7	16	20	24	28	36	40	44	48	52
			61	65	69	73	77	86	90	94	98	102
			111	115	119	123	127	136	140	144	148	152
39	(2,1,0,1)	4	7	14	21	23	30	34	41	43	50	52
			59	66	68	75	77	84	91	93	100	102
			109	116	118	125	127	134	141	143	150	152
40	(3,1,0,1)	5	7	15	18	26	29	35	38	46	49	52
			60	63	71	74	77	85	88	96	99	102
			110	113	121	124	127	135	138	146	149	152
41	(4,1,0,1)	3	7	13	19	25	31	33	39	45	51	52
			58	64	70	76	77	83	89	95	101	102
			108	114	120	126	127	133	139	145	151	152
42	(0,2,0,1)	1	7	8	9	10	11	42	43	44	45	46
			67	68	69	70	71	92	93	94	95	96
			117	118	119	120	121	142	143	144	145	146
43	(1,2,0,1)	5	7	15	18	26	29	36	39	42	50	53
			61	64	67	75	78	86	89	92	100	103
			111	114	117	125	128	136	139	142	150	153
44	(2,2,0,1)	6	7	16	20	24	28	34	38	42	51	55
			59	63	67	76	80	84	88	92	101	105

i	P_i	πi										
			109	113	117	126	130	134	138	142	151	155
		3	7	13	19	25	31	35	41	42	48	54
45	(3,2,0,1)		60	66	67	73	79	85	91	92	98	104
			110	116	117	123	129	135	141	142	148	154
		4	7	14	21	23	30	33	40	42	49	56
46	(4,2,0,1)		58	65	67	74	81	83	90	92	99	106
			108	115	117	124	131	133	140	142	149	156
		1	7	8	9	10	11	47	48	49	50	51
47	(0,3,0,1)		72	73	74	75	76	97	98	99	100	101
			122	123	124	125	126	147	148	149	150	151
		4	7	14	21	23	30	36	38	45	47	54
48	(1,3,0,1)		61	63	70	72	79	86	88	95	97	104
			111	113	120	122	129	136	138	145	147	154
		3	7	13	19	25	31	34	40	46	47	53
49	(2,3,0,1)		59	65	71	72	78	84	90	96	97	103
			109	115	121	122	128	134	140	146	147	153
		6	7	16	20	24	28	35	39	43	47	56
50	(3,3,0,1)		60	64	68	72	81	85	89	93	97	106
			110	114	118	122	131	135	139	143	147	156
		5	7	15	18	26	29	33	41	44	47	55
51	(4,3,0,1)		58	66	69	72	80	83	91	94	97	105
			108	116	119	122	130	133	141	144	147	155
		1	7	8	9	10	11	37	38	39	40	41
52	(0,4,0,1)		62	63	64	65	66	87	88	89	90	91
			112	113	114	115	116	137	138	139	140	141
		3	7	13	19	25	31	36	37	43	49	55
53	(1,4,0,1)		61	62	68	74	80	86	87	93	99	105
			111	112	118	124	130	136	137	143	149	155
		5	7	15	18	26	29	34	37	45	48	56
54	(2,4,0,1)		59	62	70	73	81	84	87	95	98	106
			109	112	120	123	131	134	137	145	148	156
		4	7	14	21	23	30	35	37	44	51	53
55	(3,4,0,1)		60	62	69	76	78	85	87	94	101	103
			110	112	119	126	128	135	137	144	151	153
		6	7	16	20	24	28	33	37	46	50	54
56	(4,4,0,1)		58	62	71	75	79	83	87	96	100	104
			108	112	121	125	129	133	137	146	150	154
		1	2	3	4	5	6	132	133	134	135	136
57	(0,0,1,1)		137	138	139	140	141	142	143	144	145	146
			147	148	149	150	151	152	153	154	155	156
		2	11	16	21	26	31	36	41	46	51	56
58	(1,0,1,1)		60	65	70	75	80	84	89	94	99	104
			108	113	118	123	128	132	137	142	147	152
59	(2,0,1,1)	2	9	14	19	24	29	34	39	44	49	54

i	P_i	πi										
			61	66	71	76	81	83	88	93	98	103
			110	115	120	125	130	132	137	142	147	152
		2	10	15	20	25	30	35	40	45	50	55
60	(3,0,1,1)		58	63	68	73	78	86	91	96	101	106
			109	114	119	124	129	132	137	142	147	152
		2	8	13	18	23	28	33	38	43	48	53
61	(4,0,1,1)		59	64	69	74	79	85	90	95	100	105
			111	116	121	126	131	132	137	142	147	152
		1	27	28	29	30	31	52	53	54	55	56
62	(0,1,1,1)		72	73	74	75	76	92	93	94	95	96
			112	113	114	115	116	132	133	134	135	136
		6	11	15	19	23	27	36	40	44	48	52
63	(1,1,1,1)		60	64	68	72	81	84	88	92	101	105
			108	112	121	125	129	132	141	145	149	153
		4	9	16	18	25	27	34	41	43	50	52
64	(2,1,1,1)		61	63	70	72	79	83	90	92	99	106
			110	112	119	126	128	132	139	146	148	155
		5	10	13	21	24	27	35	38	46	49	52
65	(3,1,1,1)		58	66	69	72	80	86	89	92	100	103
			109	112	120	123	131	132	140	143	151	154
		3	8	14	20	26	27	33	39	45	51	52
66	(4,1,1,1)		59	65	71	72	78	85	91	92	98	104
			111	112	118	124	130	132	138	144	150	156
		1	17	18	19	20	21	42	43	44	45	46
67	(0,2,1,1)		77	78	79	80	81	87	88	89	90	91
			122	123	124	125	126	132	133	134	135	136
		5	11	14	17	25	28	36	39	42	50	53
68	(1,2,1,1)		60	63	71	74	77	84	87	95	98	106
			108	116	119	122	130	132	140	143	151	154
		6	9	13	17	26	30	34	38	42	51	55
69	(2,2,1,1)		61	65	69	73	77	83	87	96	100	104
			110	114	118	122	131	132	141	145	149	153
		3	10	16	17	23	29	35	41	42	48	54
70	(3,2,1,1)		58	64	70	76	77	86	87	93	99	105
			109	115	121	122	128	132	138	144	150	156
		4	8	15	17	24	31	33	40	42	49	56
71	(4,2,1,1)		59	66	68	75	77	85	87	94	101	103
			111	113	120	122	129	132	139	146	148	155
		1	22	23	24	25	26	47	48	49	50	51
72	(0,3,1,1)		62	63	64	65	66	102	103	104	105	106
			117	118	119	120	121	132	133	134	135	136
		4	11	13	20	22	29	36	38	45	47	54
73	(1,3,1,1)		60	62	69	76	78	84	91	93	100	102
			108	115	117	124	131	132	139	146	148	155

i	P_i	πi										
74	(2,3,1,1)	3	9	15	21	22	28	34	40	46	47	53
			61	62	68	74	80	83	89	95	101	102
			110	116	117	123	129	132	138	144	150	156
75	(3,3,1,1)	6	10	14	18	22	31	35	39	43	47	56
			58	62	71	75	79	86	90	94	98	102
			109	113	117	126	130	132	141	145	149	153
76	(4,3,1,1)	5	8	16	19	22	30	33	41	44	47	55
			59	62	70	73	81	85	88	96	99	102
			111	114	117	125	128	132	140	143	151	154
77	(0,4,1,1)	1	12	13	14	15	16	37	38	39	40	41
			67	68	69	70	71	97	98	99	100	101
			127	128	129	130	131	132	133	134	135	136
78	(1,4,1,1)	3	11	12	18	24	30	36	37	43	49	55
			60	66	67	73	79	84	90		96	97
			103	108	114	120	126	127	132	138	144	150
			156									
79	(2,4,1,1)	5	9	12	20	23	31	34	37	45	48	56
			61	64	67	75	78	83	91	94	97	105
			110	113	121	124	127	132	140	143	151	154
80	(3,4,1,1)	4	10	12	19	26	28	35	37	44	51	53
			58	65	67	74	81	86	88	95	97	104
			109	116	118	125	127	132	139	146	148	155
81	(4,4,1,1)	6	8	12	21	25	29	33	37	46	50	54
			59	63	67	76	80	85	89	93	97	106
			111	115	119	123	127	132	141	145	149	153
82	(0,0,2,1)	1	2	3	4	5	6	82	83	84	85	86
			87	88	89	90	91	92	93	94	95	96
			97	98	99	100	101	102	103	104	105	106
83	(1,0,2,1)	2	10	15	20	25	30	36	41	46	51	56
			59	64	69	74	79	82	87	92	97	102
			110	115	120	125	130	133	138	143	148	153
84	(2,0,2,1)	2	11	16	21	26	31	34	39	44	49	54
			58	63	68	73	78	82	87	92	97	102
			111	116	121	126	131	135	140	145	150	155
85	(3,0,2,1)	2	8	13	18	23	28	35	40	45	50	55
			61	66	71	76	81	82	87	92	97	102
			108	113	118	123	128	134	139	144	149	154
86	(4,0,2,1)	2	9	14	19	24	29	33	38	43	48	53
			60	65	70	75	80	82	87	92	97	102
			109	114	119	124	129	136	141	146	151	156
87	(0,1,2,1)	1	22	23	24	25	26	52	53	54	55	56
			67	68	69	70	71	82	83	84	85	86
			122	123	124	125	126	137	138	139	140	141
88	(1,1,2,1)	6	10	14	18	22	31	36	40	44	48	52

i	P_i	πi										
			59	63	67	76	80	82	91	95	99	103
			110	114	118	122	131	133	137	146	150	154
89	(2,1,2,1)	4	11	13	20	22	29	34	41	43	50	52
			58	65	67	74	81	82	89	96	98	105
			111	113	120	122	129	135	137	144	151	153
90	(3,1,2,1)	5	8	16	19	22	30	35	38	46	49	52
			61	64	67	75	78	82	90	93	101	104
			108	116	119	122	130	134	137	145	148	156
91	(4,1,2,1)	3	9	15	21	22	28	33	39	45	51	52
			60	66	67	73	79	82	88	94	100	106
			109	115	121	122	128	136	137	143	149	155
92	(0,2,2,1)	1	27	28	29	30	31	42	43	44	45	46
			62	63	64	65	66	82	83	84	85	86
			127	128	129	130	131	147	148	149	150	151
93	(1,2,2,1)	5	10	13	21	24	27	36	39	42	50	53
			59	62	70	73	81	82	90	93	101	104
			110	113	121	124	127	133	141	144	147	155
94	(2,2,2,1)	6	11	15	19	23	27	34	38	42	51	55
			58	62	71	75	79	82	91	95	99	103
			111	115	119	123	127	135	139	143	147	156
95	(3,2,2,1)	3	8	14	20	26	27	35	41	42	48	54
			61	62	68	74	80	82	88	94	100	106
			108	114	120	126	127	134	140	146	147	153
96	(4,2,2,1)	4	9	16	18	25	27	33	40	42	49	56
			60	62	69	76	78	82	89	96	98	105
			109	116	118	125	127	136	138	145	147	154
97	(0,3,2,1)	1	12	13	14	15	16	47	48	49	50	51
			77	78	79	80	81	82	83	84	85	86
			112	113	114	115	116	142	143	144	145	146
98	(1,3,2,1)	4	10	12	19	26	28	36	38	45	47	54
			59	66	68	75	77	82	89	96	98	105
			110	112	119	126	128	133	140	142	149	156
99	(2,3,2,1)	3	11	12	18	24	30	34	40	46	47	53
			58	64	70	76	77	82	88	94	100	106
			111	112	118	124	130	135	141	142	148	154
100	(3,3,2,1)	6	8	12	21	25	29	35	39	43	47	56
			61	65	69	73	77	82	91	95	99	103
			108	112	121	125	129	134	138	142	151	155
101	(4,3,2,1)	5	9	12	20	23	31	33	41	44	47	55
			60	63	71	74	77	82	90	93	101	104
			109	112	120	123	131	136	139	142	150	153
102	(0,4,2,1)	1	17	18	19	20	21	37	38	39	40	41
			72	73	74	75	76	82	83	84	85	86
			117	118	119	120	121	152	153	154	155	156

i	P_i	πi										
103	(1,4,2,1)	3	10	16	17	23	29	36	37	43	49	55
			59	65	71	72	78	82	88	94	100	106
			110	116	117	123	129	133	139	145	151	152
104	(2,4,2,1)	5	11	14	17	25	28	34	37	45	48	56
			58	66	69	72	80	82	90	93	101	104
			111	114	117	125	128	135	138	146	149	152
105	(3,4,2,1)	4	8	15	17	24	31	35	37	44	51	53
			61	63	70	72	79	82	89	96	98	105
			108	115	117	124	131	134	141	143	150	152
106	(4,4,2,1)	6	9	13	17	26	30	33	37	46	50	54
			60	64	68	72	81	82	91	95	99	103
			109	113	117	126	130	136	140	144	148	152
124	(2,3,3,1)	3	8	14	20	26	27	34	40	46	47	53
			60	66	67	73	79	86	87	93	99	105
			107	113	119	125	131	133	139	145	151	152
125	(3,3,3,1)	6	11	15	19	23	27	35	39	43	47	56
			59	63	67	76	80	83	87	96	100	104
			107	116	120	124	128	136	140	144	148	152
126	(4,3,3,1)	5	10	13	21	24	27	33	41	44	47	55
			61	64	67	75	78	84	87	95	98	106
			107	115	118	126	129	135	138	146	149	152
127	(0,4,3,1)	1	22	23	24	25	26	37	38	39	40	41
			77	78	79	80	81	92	93	94	95	96
			107	108	109	110	111	147	148	149	150	151
128	(1,4,3,1)	3	9	15	21	22	28	36	37	43	49	55
			58	64	70	76	77	85	91	92	98	104
			107	113	119	125	131	134	140	146	147	153
129	(2,4,3,1)	5	8	16	19	22	30	34	37	45	48	56
			60	63	71	74	77	86	89	92	100	103
			107	115	118	126	129	133	141	144	147	155
130	(3,4,3,1)	4	11	13	20	22	29	35	37	44	51	53
			59	66	68	75	77	83	90	92	99	106
			107	114	121	123	130	136	138	145	147	154
131	(4,4,3,1)	6	10	14	18	22	31	33	37	46	50	54
			61	65	69	73	77	84	88	92	101	105
			107	116	120	124	128	135	139	143	147	156
132	(0,0,4,1)	1	2	3	4	5	6	57	58	59	60	61
			62	63	64	65	66	67	68	69	70	71
			72	73	74	75	76	77	78	79	80	81
133	(1,0,4,1)	2	8	13	18	23	28	36	41	46	51	56
			57	62	67	72	77	83	88	93	98	103
			109	114	119	124	129	135	140	145	150	155
134	(2,0,4,1)	2	10	15	20	25	30	34	39	44	49	54
			57	62	67	72	77	85	90	95	100	105

i	P_i	πi										
			108	113	118	123	128	136	141	146	151	156
135	(3,0,4,1)	2	9	14	19	24	29	35	40	45	50	55
			57	62	67	72	77	84	89	94	99	104
			111	116	121	126	131	133	138	143	148	153
136	(4,0,4,1)	2	11	16	21	26	31	33	38	43	48	53
			57	62	67	72	77	86	91	96	101	106
			110	115	120	125	130	134	139	144	149	154
137	(0,1,4,1)	1	12	13	14	15	16	52	53	54	55	56
			57	58	59	60	61	87	88	89	90	91
			117	118	119	120	121	147	148	149	150	151
138	(1,1,4,1)	6	8	12	21	25	29	36	40	44	48	52
			57	66	70	74	78	83	87	96	100	104
			109	113	117	126	130	135	139	143	147	156
139	(2,1,4,1)	4	10	12	19	26	28	34	41	43	50	52
			57	64	71	73	80	85	87	94	101	103
			108	115	117	124	131	136	138	145	147	154
140	(3,1,4,1)	5	9	12	20	23	31	35	38	46	49	52
			57	65	68	76	79	84	87	95	98	106
			111	114	117	125	128	133	141	144	147	155
141	(4,1,4,1)	3	11	12	18	24	30	33	39	45	51	52
			57	63	69	75	81	86	87	93	99	105
			110	116	117	123	129	134	140	146	147	153
142	(0,2,4,1)	1	22	23	24	25	26	42	43	44	45	46
			57	58	59	60	61	97	98	99	100	101
			112	113	114	115	116	152	153			
		154	155	156								
143	(1,2,4,1)	5	8	16	19	22	30	36	39	42	50	53
			57	65	68	76	79	83	91	94	97	105
			109	112	120	123	131	135	138	146	149	152
144	(2,2,4,1)	6	10	14	18	22	31	34	38	42	51	55
			57	66	70	74	78	85	89	93	97	106
			108	112	121	125	129	136	140	144	148	152
145	(3,2,4,1)	3	9	15	21	22	28	35	41	42	48	54
			57	63	69	75	81	84	90	96	97	103
			111	112	118	124	130	133	139	145	151	152
146	(4,2,4,1)	4	11	13	20	22	29	33	40	42	49	56
			57	64	71	73	80	86	88	95	97	104
			110	112	119	126	128	134	141	143	150	152
147	(0,3,4,1)	1	17	18	19	20	21	47	48	49	50	51
			57	58	59	60	61	92	93	94	95	96
			127	128	129	130	131	137	138	139	140	141
148	(1,3,4,1)	4	8	15	17	24	31	36	38	45	47	54
			57	64	71	73	80	83	90	92	99	106
			109	116	118	125	127	135	137	144	151	153

i	P_i	πi										
		3	10	16	17	23	29	34	40	46	47	53
149	(2,3,4,1)		57	63	69	75	81	85	91	92	98	104
			108	114	120	126	127	136	137	143	149	155
		6	9	13	17	26	30	35	39	43	47	56
150	(3,3,4,1)		57	66	70	74	78	84	88	92	101	105
			111	115	119	123	127	133	137	146	150	154
		5	11	14	17	25	28	33	41	44	47	55
151	(4,3,4,1)		57	65	68	76	79	86	89	92	100	103
			110	113	121	124	127	134	137	145	148	156
		1	27	28	29	30	31	37	38	39	40	41
152	(0,4,4,1)		57	58	59	60	61	102	103	104	105	106
			122	123	124	125	126	142	143	144	145	146
		3	8	14	20	26	27	36	37	43	49	55
153	(1,4,4,1)		57	63	69	75	81	83	89	95	101	102
			109	115	121	122	128	135	141	142	148	154
		5	10	13	21	24	27	34	37	45	48	56
154	(2,4,4,1)		57	65	68	76	79	85	88	96	99	102
			108	116	119	122	130	136	139	142	150	153
		4	9	16	18	25	27	35	37	44	51	53
155	(3,4,4,1)		57	64	71	73	80	84	91	93	100	102
			111	113	120	122	129	133	140	142	149	156
		6	11	15	19	23	27	33	37	46	50	54
156	(4,4,4,1)		57	66	70	74	78	86	90	94	98	102
			110	114	118	122	131	134	138	142	151	155

,

$\dot{D}_9 = \{1,2,7,32,37,63,100,101\}$, $\dot{D}_{10} = \{1,2,7,32,39,63,100,101\}$,

$\dot{D}_{11} = \{1,2,7,32,40,63,100,101\}$, $\dot{D}_{12} = \{1,2,7,32,41,63,100,101\}$,

$\dot{D}_{13} = \{1,2,7,32,42,63,100,101\}$, $\dot{D}_{14} = \{1,2,7,32,43,63,100,101\}$,

$\dot{D}_{15} = \{1,2,7,32,45,63,100,101\}$, $\dot{D}_{16} = \{1,2,7,32,46,63,100,101\}$,

$\dot{D}_{17} = \{1,2,7,32,58,63,100,101\}$, $\dot{D}_{18} = \{1,2,7,32,59,63,100,101\}$,

$\dot{D}_{19} = \{1,2,7,32,60,63,100,101\}$, $\dot{D}_{20} = \{1,2,7,32,61,63,100,101\}$,

$\dot{D}_{21} = \{1,2,7,32,63,67,100,101\}$, $\dot{D}_{22} = \{1,2,7,32,63,68,100,101\}$,

$\dot{D}_{23} = \{1,2,7,32,63,70,100,101\}$, $\dot{D}_{24} = \{1,2,7,32,63,71,100,101\}$,

$\dot{D}_{25} = \{1,2,7,32,63,100,101,112\}$, $\dot{D}_{26} = \{1,2,7,32,63,100,101,114\}$,

$\dot{D}_{27} = \{1,2,7,32,63,100,101,115\}$, $\dot{D}_{28} = \{1,2,7,32,63,100,101,116\}$,

$\dot{D}_{29} = \{1,2,7,32,63,100,101,127\}$, $\dot{D}_{30} = \{1,2,7,32,63,100,101,128\}$,

$\dot{D}_{31} = \{1,2,7,32,63,100,101,129\}$, $\dot{D}_{32} = \{1,2,7,32,63,100,101,130\}$,

$\dot{D}_{33} = \{1,2,7,32,63,100,101,133\}$, $\dot{D}_{34} = \{1,2,7,32,63,100,101,134\}$,

$\dot{D}_{35} = \{1,2,7,32,63,100,101,135\}$, $\dot{D}_{36} = \{1,2,7,32,63,100,101,136\}$,

$\dot{D}_{37} = \{1,2,7,32,63,100,101,152\}$, $\dot{D}_{38} = \{1,2,7,32,63,100,101,153\}$,

$\dot{D}_{39} = \{1,2,7,32,63,100,101,154\}$, $\dot{D}_{40} = \{1,2,7,32,63,100,101,155\}$.

4 The Construction $(K+2,n)$-arcs from (K,n)-arc in $PG(3,q)$

There are two methods to construction $(k+2;n)$-arcs, which are explained below:

4.1 The First Method

construction of $(k+2;n)$-arcs directly from incomplete $(k;n)$-arc as following:

(i) we denote the set of arc points in the table of points and planes for $PG(3,q)$. 2- we delete all points which lie in n-secant from the projective space $PG(3,q)$.

(ii) we add two points from the remaining points to $(k;n)$-arc to obtain $(k+2;n)$- arcs provided that the two points do not lie on a plane that contain $(n-1)$-secant.

A Construction of (9;5)-arcs from (7;5)-arc in $PG(3,2)$

Let $\hat{A} = \{1,2,3,4,5,6,13\}$ is a (7;5)-arc in $PG(3,2)$,we can construct (9;5)-arcs as follow:

(i) we designate the arc points Â in table 3.

(ii) we delete the points that lie on 5-secant, the remaining points are: 7, 8, 10, 12, 14, 15.

(iii) we add two from the remaining points to the arc $\hat{A}$ (provided that the two points do not lie on a plane of type 4-secant), we obtain (9;5)-arcs

$\hat{A}_1 = \{1,2,3,4,5,6,7,13,15\}$, $\hat{A}_2 = \{1,2,3,4,5,6,8,12,13\}$, $\hat{A}_3 = \{1,2,3,4,5,6,10,13,14\}$.

A Construction of (8;4)-arcs from (6;4)-arc in $PG(3,4)$

Let $\hat{A} = \{1,2,3,6,22,43\}$ is a (6;4)-arc in $PG(3,4)$, to construction (8;4)-arc $\hat{A}_i$ we follow the following steps:

(i) determine the points of arc Â on the table 3.

(ii) eliminate the points that lie on 4-secant.

(iii) adding two of the remaining space points to the arc Â, provided that one of the points does not lie on 3-secant. we get the following (8;4)-arcs:

$1-\hat{A}=\{1,2,3,6,22,31,43,50\}$	$2-\hat{A}=\{1,2,3,6,22,31,43,52\}$
$3-\hat{A}=\{1,2,3,6,22,31,43,53\}$	$4-\hat{A}=\{1,2,3,6,22,31,43,66\}$
$5-\hat{A}=\{1,2,3,6,22,31,43,69\}$	$6-\hat{A}=\{1,2,3,6,22,31,43,82\}$
$7-\hat{A}=\{1,2,3,6,22,31,43,84\}$	$8-\hat{A}=\{1,2,3,6,22,32,43,50\}$
$9-\hat{A}=\{1,2,3,6,22,32,43,52\}$	$10-\hat{A}=\{1,2,3,6,22,32,43,66\}$
$11-\hat{A}=\{1,2,3,6,22,32,43,67\}$	$12-\hat{A}=\{1,2,3,6,22,32,43,82\}$
$13-\hat{A}=\{1,2,3,6,22,32,43,83\}$	$14-\hat{A}=\{1,2,3,6,22,32,43,84\}$
$15-\hat{A}=\{1,2,3,6,22,33,43,50\}$	$16-\hat{A}=\{1,2,3,6,22,33,43,52\}$
$17-\hat{A}=\{1,2,3,6,22,33,43,53\}$	$18-\hat{A}=\{1,2,3,6,22,33,43,66\}$
$19-\hat{A}=\{1,2,3,6,22,33,43,67\}$	$20-\hat{A}=\{1,2,3,6,22,33,43,69\}$
$21-\hat{A}=\{1,2,3,6,22,33,43,82\}$	$22-\hat{A}=\{1,2,3,6,22,33,43,83\}$
$23-\hat{A}=\{1,2,3,6,22,33,43,84\}$	$24-\hat{A}=\{1,2,3,6,22,35,43,46\}$
$25-\hat{A}=\{1,2,3,6,22,35,43,48\}$	$26-\hat{A}=\{1,2,3,6,22,35,43,49\}$
$27-\hat{A}=\{1,2,3,6,22,35,43,62\}$	$28-\hat{A}=\{1,2,3,6,22,35,43,65\}$
$29-\hat{A}=\{1,2,3,6,22,35,43,78\}$	$30-\hat{A}=\{1,2,3,6,22,35,43,80\}$
$31-\hat{A}=\{1,2,3,6,22,36,43,46\}$	$32-\hat{A}=\{1,2,3,6,22,36,43,48\}$
$33-\hat{A}=\{1,2,3,6,22,36,43,49\}$	$34-\hat{A}=\{1,2,3,6,22,36,43,62\}$
$35-\hat{A}=\{1,2,3,6,22,36,43,63\}$	$36-\hat{A}=\{1,2,3,6,22,36,43,65\}$
$37-\hat{A}=\{1,2,3,6,22,36,43,78\}$	$38-\hat{A}=\{1,2,3,6,22,36,43,79\}$
$39-\hat{A}=\{1,2,3,6,22,36,43,80\}$	$40-\hat{A}=\{1,2,3,6,22,37,43,81\}$
$41-\hat{A}=\{1,2,3,6,22,37,43,46\}$	$42-\hat{A}=\{1,2,3,6,22,37,43,49\}$
$43-\hat{A}=\{1,2,3,6,22,37,43,62\}$	$44-\hat{A}=\{1,2,3,6,22,37,43,63\}$
$45-\hat{A}=\{1,2,3,6,22,37,43,65\}$	$46-\hat{A}=\{1,2,3,6,22,37,43,78\}$
$47-\hat{A}=\{1,2,3,6,22,37,43,79\}$	$48-\hat{A}=\{1,2,3,6,22,46,43,67\}$
$49-\hat{A}=\{1,2,3,6,22,46,43,69\}$	$50-\hat{A}=\{1,2,3,6,22,46,43,83\}$
$51-\hat{A}=\{1,2,3,6,22,46,43,84\}$	$52-\hat{A}=\{1,2,3,6,22,48,43,66\}$
$53-\hat{A}=\{1,2,3,6,22,48,43,67\}$	$54-\hat{A}=\{1,2,3,6,22,48,43,82\}$
$55-\hat{A}=\{1,2,3,6,22,48,43,83\}$	$56-\hat{A}=\{1,2,3,6,22,48,43,84\}$

$57-\hat{A}=\{1,2,3,6,22,49,43,66\}$ $58-\hat{A}=\{1,2,3,6,22,49,43,67\}$

$59-\hat{A}=\{1,2,3,6,22,49,43,69\}$ $60-\hat{A}=\{1,2,3,6,22,49,43,82\}$

$61-\hat{A}=\{1,2,3,6,22,49,43,84\}$ $62-\hat{A}=\{1,2,3,6,22,50,43,63\}$

$63-\hat{A}=\{1,2,3,6,22,50,43,65\}$ $64-\hat{A}=\{1,2,3,6,22,50,43,79\}$

$65-\hat{A}=\{1,2,3,6,22,50,43,80\}$ $66-\hat{A}=\{1,2,3,6,22,52,43,62\}$

$67-\hat{A}=\{1,2,3,6,22,52,43,63\}$ $71-\hat{A}=\{1,2,3,6,22,52,43,80\}$

$68-\hat{A}=\{1,2,3,6,22,52,43,65\}$ $69-\hat{A}=\{1,2,3,6,22,52,43,78\}$

$70-\hat{A}=\{1,2,3,6,22,52,43,79\}$ $71-\hat{A}=\{1,2,3,6,22,52,43,80\}$

$72-\hat{A}=\{1,2,3,6,22,53,43,62\}$ $73-\hat{A}=\{1,2,3,6,22,53,43,63\}$

$74-\hat{A}=\{1,2,3,6,22,53,43,65\}$ $75-\hat{A}=\{1,2,3,6,22,53,43,78\}$

$76-\hat{A}=\{1,2,3,6,22,53,43,79\}$ $77-\hat{A}=\{1,2,3,6,22,62,43,67\}$

$78-\hat{A}=\{1,2,3,6,22,63,43,66\}$ $79-\hat{A}=\{1,2,3,6,22,63,43,69\}$

$80-\hat{A}=\{1,2,3,6,22,65,43,66\}$ $81-\hat{A}=\{1,2,3,6,22,65,43,67\}$

$82-\hat{A}=\{1,2,3,6,22,65,43,69\}$ $83-\hat{A}=\{1,2,3,6,22,66,43,79\}$

$84-\hat{A}=\{1,2,3,6,22,66,43,80\}$ $85-\hat{A}=\{1,2,3,6,22,67,43,78\}$

$86-\hat{A}=\{1,2,3,6,22,67,43,80\}$ $87-\hat{A}=\{1,2,3,6,22,69,43,78\}$

$88-\hat{A}=\{1,2,3,6,22,69,43,79\}$ $82-\hat{A}=\{1,2,3,6,22,65,43,69\}$

$89-\hat{A}=\{1,2,3,6,22,78,43,83\}$ $90-\hat{A}=\{1,2,3,6,22,78,43,84\}$

$91-\hat{A}=\{1,2,3,6,22,79,43,82\}$ $92-\hat{A}=\{1,2,3,6,22,79,43,84\}$

$93-\hat{A}=\{1,2,3,6,22,80,43,82\}$ $94-\hat{A}=\{1,2,3,6,22,80,43,83\}$

$95-\hat{A}=\{1,2,3,6,22,80,43,84\}$.

4.2 The Second Method

The second method of construction $(k+2;n)$-arcs involves two steps:

(i) construct $(k+1;n)$-arcs from $(k;n)$-arc as in 4.1.

(ii) construct $(k+2;n)$-arcs from incomplete $(k+1;n)$-arcs, the following example illustrates the construction method.

Construction (10;5)-arcs from (8;5)-arc in $PG(3,3)$

Let $\dot{B}=\{1,2,3,6,9,19,27,33\}$. The first step we find (9;5)-arc as in 3.2, we get 17 different arcs of type (9,5)-arc, we take (9;5)-arc $\dot{B}_1$ and try to determine the points of $\dot{B}_1$ in table 2 and delete whole points lie on 5- secant from points of $PG(3,3)$, the last step involves adding 1 point each time from the remaining points to the arc $\dot{B}_1$, we get 4 different (9;5)-arcs $\dot{B}_i$ which are:
$\dot{B}_1=\dot{B}_1\cup\{26\}=\{1,2,3,6,9,14,19,26,27,33\}$.
$\dot{B}_2=\dot{B}_2\cup\{30\}=\{1,2,3,6,9,14,19,27,30,33\}$.
$\dot{B}_3=\dot{B}_3\cup\{34\}=\{1,2,3,6,9,14,19,27,33,34\}$.
$\dot{B}_4=\dot{B}_4\cup\{39\}=\{1,2,3,6,9,14,19,27,33,39\}$.

In the same way we find the (10;5)-arc from the remainder (9;5)-arcs $\dot{B}_i$ as follows:

From the arc $\dot{B}_2=\{1,2,3,6,9,16,19,27,33\}$ we get the following arcs:

$1 - \dot{B}_1 = \{1,2,3,6,9,16,19,25,27,33\}$. $2 - \dot{B}_2 = \{1,2,3,6,9,16,19,27,30,33\}$.

$3 - \dot{B}_3 = \{1,2,3,6,9,16,19,27,31,33\}$. $4 - \dot{B}_4 = \{1,2,3,6,9,16,19,27,32,33\}$.

$5 - \dot{B}_5 = \{1,2,3,6,9,16,19,27,33,34\}$. $6 - \dot{B}_6 = \{1,2,3,6,9,16,19,27,33,35\}$.

$7 - \dot{B}_7 = \{1,2,3,6,9,16,19,27,33,40\}$.

from the arc $\dot{B}_3 = \{1,2,3,6,9,17,19,27,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,17,19,23,27,33\}$. $\dot{B}_2 = \{1,2,3,6,9,17,19,25,27,33\}$.

$\dot{B}_3 = \{1,2,3,6,9,17,19,27,33,39\}$. $\dot{B}_4 = \{1,2,3,6,9,17,19,27,33,40\}$

from the arc $\dot{B}_4 = \{1,2,3,6,9,18,19,27,33\}$

we obtain $\dot{B}_1 = \{1,2,3,6,9,18,19,25,27,33\}$.

from the arc $\dot{B}_5 = \{1,2,3,6,9,19,21,27,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,19,21,23,27,33\}$ $\dot{B}_2 = \{1,2,3,6,9,19,21,26,27,33\}$

$\dot{B}_3 = \{1,2,3,6,9,19,21,27,31,33\}$ $\dot{B}_4 = \{1,2,3,6,9,19,21,27,32,33\}$

$\dot{B}_5 = \{1,2,3,6,9,19,21,27,33,34\}$.

from the arc $\dot{B}_6 = \{1,2,3,6,9,19,22,27,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,19,22,25,27,33\}$ $\dot{B}_2 = \{1,2,3,6,9,19,22,27,30,33\}$

$\dot{B}_3 = \{1,2,3,6,9,19,22,27,33,34\}$ $\dot{B}_4 = \{1,2,3,6,9,19,22,27,33,35\}$.

from the arc $\dot{B}_7 = \{1,2,3,6,9,19,23,27,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,17,19,23,27,33\}$ $\dot{B}_2 = \{1,2,3,6,9,19,21,23,27,33\}$

$\dot{B}_3 = \{1,2,3,6,9,19,23,27,33,34\}$ $\dot{B}_4 = \{1,2,3,6,9,19,23,27,33,35\}$

from the arc $\dot{B}_8 = \{1,2,3,6,9,19,25,27,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,16,19,25,27,33\}$ $\dot{B}_2 = \{1,2,3,6,9,17,19,25,27,33\}$

$\dot{B}_3 = \{1,2,3,6,9,18,19,25,27,33\}$ $\dot{B}_4 = \{1,2,3,6,9,19,22,25,27,33\}$

$\dot{B}_5 = \{1,2,3,6,9,19,25,27,32,33\}$ $\dot{B}_6 = \{1,2,3,6,9,19,25,27,33,34\}$

$\dot{B}_7 = \{1,2,3,6,9,19,25,27,33,36\}$ $\dot{B}_8 = \{1,2,3,6,9,19,25,27,33,39\}$.

from the arc $\dot{B}_9 = \{1,2,3,6,9,19,26,27,33\}$

we obtain $\dot{B}_1 = \{1,2,3,6,9,14,19,26,27,33\}$ $\dot{B}_2 = \{1,2,3,6,9,19,21,26,27,33\}$

$\dot{B}_3 = \{1,2,3,6,9,19,26,27,32,33\}$ $\dot{B}_4 = \{1,2,3,6,9,19,26,27,33,34\}$

$\dot{B}_5 = \{1,2,3,6,9,19,26,27,33,40\}$

from the arc $\dot{B}_{10} = \{1,2,3,6,9,19,27,30,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,14,19,27,30,33\}$ $\dot{B}_2 = \{1,2,3,6,9,16,19,27,30,33\}$

$\dot{B}_3 = \{1,2,3,6,9,19,22,27,30,33\}$ $\dot{B}_4 = \{1,2,3,6,9,19,27,30,33,35\}$

$\dot{B}_5 = \{1,2,3,6,9,19,27,30,33,40\}$

$from\dot{B}_{11} = \{1,2,3,6,9,19,27,31,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,16,19,27,31,33\}$ $\dot{B}_2 = \{1,2,3,6,9,19,21,27,31,33\}$

$\dot{B}_3 = \{1,2,3,6,9,19,27,31,33,35\}$ $\dot{B}_4 = \{1,2,3,6,9,19,27,31,33,39\}$

$from\dot{B}_{12} = \{1,2,3,6,9,19,27,32,33\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,16,19,27,32,33\}$ $\dot{B}_2 = \{1,2,3,6,9,19,21,27,32,33\}$

$\dot{B}_3 = \{1,2,3,6,9,19,25,27,32,33\}$ $\dot{B}_4 = \{1,2,3,6,9,19,26,27,32,33\}$

$from\dot{B}_{13} = \{1,2,3,6,9,19,27,33,34\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,14,19,27,33,34\}$ $\dot{B}_2 = \{1,2,3,6,9,16,19,27,33,34\}$

$\dot{B}_3 = \{1,2,3,6,9,19,21,27,33,34\}$ $\dot{B}_4 = \{1,2,3,6,9,19,22,27,33,34\}$

$\dot{B}_5 = \{1,2,3,6,9,19,23,27,33,34\}$ $\dot{B}_6 = \{1,2,3,6,9,19,25,27,33,34\}$

$\dot{B}_7 = \{1,2,3,6,9,19,26,27,33,34\}$

$from\dot{B}_{14} = \{1,2,3,6,9,19,27,33,35\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,16,19,27,33,35\}$ $\dot{B}_2 = \{1,2,3,6,9,19,22,27,33,35\}$

$\dot{B}_3 = \{1,2,3,6,9,19,23,27,33,35\}$ $\dot{B}_4 = \{1,2,3,6,9,19,27,30,33,35\}$

$\dot{B}_5 = \{1,2,3,6,9,19,27,31,33,35\}$

$from\dot{B}_{15} = \{1,2,3,6,9,19,27,33,36\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,19,25,27,33,36\}$

$from\dot{B}_{16} = \{1,2,3,6,9,19,27,33,39\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,14,19,27,33,39\}$ $\dot{B}_2 = \{1,2,3,6,9,17,19,27,33,39\}$

$\dot{B}_3 = \{1,2,3,6,9,19,25,27,33,39\}$ $\dot{B}_4 = \{1,2,3,6,9,19,27,31,33,39\}$

$from\dot{B}_{17} = \{1,2,3,6,9,19,27,33,40\}$

we obtain

$\dot{B}_1 = \{1,2,3,6,9,16,19,27,33,40\}$ $\dot{B}_2 = \{1,2,3,6,9,17,19,27,33,40\}$

$\dot{B}_3 = \{1,2,3,6,9,19,26,27,33,40\}$ $\dot{B}_4 = \{1,2,3,6,9,19,27,30,33,40\}$

There are 38 repeated arcs, so the number of (10;5)-arcs equals 38 arcs.

5 Conclusions

(i) $(k+1;n)$-arcs can only be constructed from incomplete $(k;n)$-arc.

(ii) $(k+1;n)$-arcs can be complete or incomplete.

(iii) number of $(k+1;n)$-arcs that can be constructed from $(k;n)$-arc is equal $q^3+q^2+q+1-(|k|+L)$. Where $|k|$ = number of points k, L= number of points that lie on n-secant.

(iv) in the second method of construction $(k+2;n)$-arcs from $(k+1;n)$-arc, $(k;n)$-arc and $(k+1;n)$-arc must be incomplete.

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References

[1] Abdulla, A. A. A., & Yahya, N. Y. K. (2021, May). A Geometric Construction of Surface Complete (k, r)-cap in PG (3,7). In Journal of Physics: Conference Series (Vol. 1879, No. 2, p. 022112). IOP Publishing.

[2] Abdullah, F.N. & Yahya, N.Y.K. (2021), Bounds on Minimum Distance for Linear Codes Over $GF(q)$, Italian Journal of Pure and Applied Mathematics, in Italian Journal of Pure and Applied Mathematics, n.45.

[3] Al-Mukhtar, A Sh. (2008). Complete Arcs and Surfaces in Three Dimensional Projective Space Over Galois Field. Ph. D. Thesis, University of Technology, Iraq.

[4] Bartoli, D., Faina, G., Marcugini, S., & Davydov, A., (2013). New Upper Bounds on The Smallest Size of a Complete Arc in a Finite Desarguesian Projective Plane. Journal of Geometry, 104(1).

[5] Dubrovin , B.A.,& Fomenko, A.T.and Novikov,S.P. (1985), "Modren Geometry Methods and Application" Springer Verlage, New york ,Inc. Kareem,F.F.,(2013) "The Construction of Complete (k,n)-Arcs in 3-Dimensional Projective Space Over Galois Field GF(4), Journal of College of Education 183-196.

[6] Hirschfeld, J.W.P., & Storme, L. (2001). The Packing Problem in Statistics, Coding Theory and Finite Projective Spaces,Journal of Statistics,72(1).

[7] Hirschfeld, J.W.P., (1985). "Finite Projective Spaces of Three Dimensions". Oxford University Press. Hirschfeld, J.W.P, & Thas, J. A. (1991). "General Galois Geometries" (Vol. 1378). Springer.

[8] Hirshfeld, J.W.P.(1998). "Projective Geometries Over Finite Fields". Oxford Math. Monographs,Clarendon Press Oxford; 1998.

[9] Kirdar ,M.S.& AL-Mukhtar,A.Sh,(2009), "On Projective 3-Space" Engineering and Technology Journal, Vol.27(8).

[10] Kasm Yahya, N.Y.(2021)."Applications Geometry of Space in PG(3, p) "Journal of Interdisciplinary Mathematics, Doi: httpll//doi.org/10.1080/09720502.2021.1885818.

[11] Kasm Yahya, N. Y, Emad Bakr Al-Zangana (2021) The Non-existence of [1864, 3, 1828]53 Linear Code by Combinatorial Technique in International Journal of Mathematics and Computer Science, 16(2021), no. 4, 1575-1581.

[12] Li, H., Huang, L., Shao, C., & Dong, L. (2015). "Three-Dimensional Projective Geometry With Geometric Algebra". ArXiv:1507.06634.

[13] Sulaimaan A. E. M and Kasm Yahya .N .Y 2020 "The Reverse Construction of Complete (k,n)-Arcs in Three Dimensional Projective Space PG(3,4)" Journal of Physics: Conference Series 1591. doi:10.1088/1742-6596/1591/1/012078.

[14] Al-Zangana, E. B., & Yahya, N. Y. K. (2022). Subgroups and Orbits by Companion Matrix in Three Dimensional Projective Space. Baghdad Science Journal. Doi:https://doi.org/10.21123/bsj.2022.19.4.0805.

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