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D - index for a Chain of Hexagonal Cycles

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Abstract Let u and v be any distinct vertices of a connected graph Γ and $l_P(u,v) = l(P)$ be the length of $u - v$ path P, the D- distance between u and v of Γ is defined as: $d^D(u,v) = min\{l(P) + \sum_{w\in V(P)} degw\}$ where the minimum is taken over all $u - v$ path P. The D-index of Γ is defined by

$$W^D(\Gamma) = \frac{1}{2}\sum_{u,v\in V(\Gamma)} d^D(u,v) = \sum_{\{u,v\}\subseteq V(\Gamma)} d^D(u,v).$$

In this paper, we find D-index, for lollipop graph, general barbell graph, general modified barbell graph, straight hexagonal chain and a cycle hexagonal chain. Moreover, the average D-distance of these graphs will be obtained.

1 Introduction

A graph Γ is represented by $\Gamma(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is the vertex set and $E(\Gamma)$ is the edge set of Γ. The conception of standard distance $d(u,v)$ for any two distinct vertices of a graph Γ is the minimum number of edges that connects u and v, the Wiener index is the sum going over all $u - v$ paths of Γ. The D-distance and its properties were introduced and studied by Ba̱bu and Vaṟma in [4]. D- distance is different from other distances that were defined in graph theory. For u and v the D- distance is depends on the distance between u, v and the degree of vertices that lie on $u - v$ path but the other distances depend only on the distance between them. There are many works about D-distance, in [2] A̱li and Aẕiz found the relation between Wiener index and D-index for r- regular graphs of order n, they proved that

$$W^D(\Gamma) = (r+1)W(\Gamma) + r\binom{n}{2}.$$

The average D-distance between vertices and the average D-distance between edges of a graph was studied in [5] and [6]. The concept of circular D-distance was introduced by [8]. Finally, there are several studies on finding Wiener index, see [1], [3] and [7].

In this article, we found D-index for some specific graphs, such as lollipop graph, general barbell graph, and general modified barbell graph. Also, D-index of straight hexagonal chain and cycle hexagonal chain is obtained. Moreover, the average D-distance of these graphs is obtained.

Definition 1.1. For two distinct vertices u and v of a graph Γ, the D-length of a $u - v$ path s is defined as $l^D(s) = d(u,v) + deg(u) + deg(v) + \sum deg(w)$ where the sum runs over all intermediate vertices w of s.

Definition 1.2. The D-distance $d^D(u,v)$ between two vertices u, v of a graph Γ is $d^D(u,v) = min\{l^D(s)\}$, where the minimum goes over all $u - v$ paths s in Γ, in other words, $d^D(u,v) = min\{d(u,v) + deg(u) + deg(v) + \sum degw\}$, where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths in Γ.
D-index of Γ, $W^D(\Gamma)$, is the sum of all d-distances between any two distinct vertices of Γ.

Definition 1.3. For a connected graph Γ of order n, the average D-distance of Γ denoted by $\mu^D(\Gamma)$, is defined as

$$\mu^D(\Gamma) = (\binom{n}{2})^{-1} \sum_{\{u,v\}\subseteq V(\Gamma)} d^D(u,v).$$

Definition 1.4. The lollipop graph is a graph obtained by joining a complete graph K_n of order n, $n \geq 3$, and a path graph P_m of order m, $m \geq 2$ with a bridge. It is denoted by $L_{n,m}$.

Definition 1.5. A general barbell graph is a graph obtained by joining two complete graphs K_n and K_m, by a bridge, denoted by $B_{n,m}$, $n, m \geq 3$.
As a special case, if $n = m$, then the resulting graph is called barbell graph, denoted by B_n.

Definition 1.6. General modified barbell graph is a graph obtained by identifying two complete graphs K_n, K_m by identifying any two vertices of K_n and K_m, denoted by $B^*_{n,m}$, $n, m \geq 3$. As a special case, if $n = m$, then the resulting graph is called modified barbell graph, denoted by B^*_n.

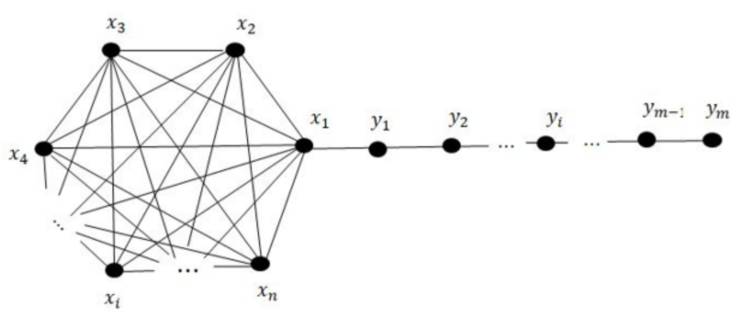

Figure 1. Lollipop Graph $L_{n,m}$

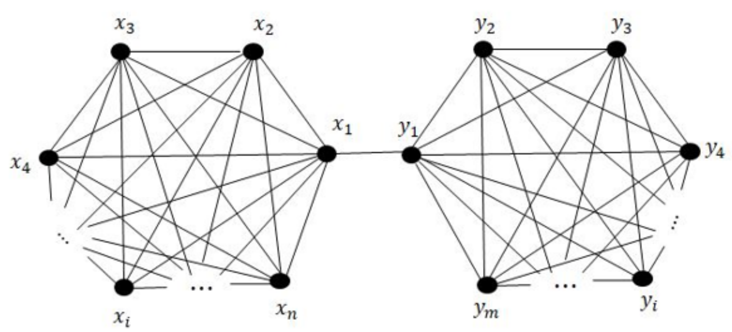

Figure 2. General Barbell Graph $B_{n,m}$

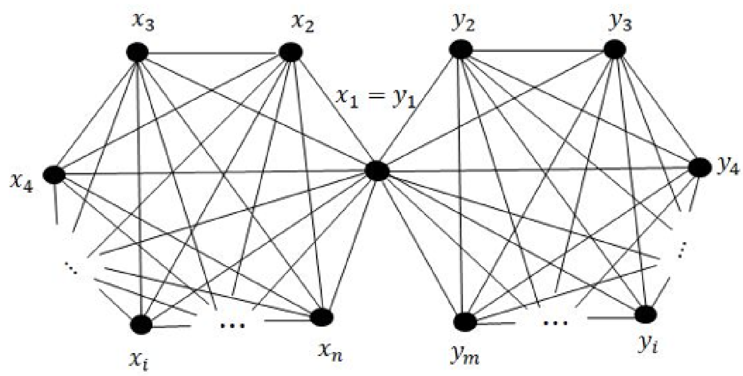

Figure 3. General Modified Barbell Graph $B^*_{n,m}$

2 D-index of Some Specific Graphs

In this section, we found D- index and average D-distance between vertices of some specific graphs, such as lollipop graph, general barbell graph and general modified barbell graph.

Theorem 2.1.

(i) The D-index of lollipop graph, for $n, m \geq 3$

$$W^D(L_{n,m}) = \frac{1}{2}[m^3 + (3n+2)m^2 + (4n^2 + n - 5)m + (2n^2 - 3n + 1)n].$$

(ii) The average D-distance for a lollipop graph, for $n, m \geq 3$

$$\mu^D(L_{n,m}) = \frac{2m(m+3)}{m+n} - \frac{m(m+1)}{m+n-1} + 2n - 1.$$

Proof.

(i) For any two vertices of lollipop graph see Figure 1, we have three main cases

a. The two vertices are in a complete graph K_n, $n \geq 3$

i. $d^D(x_1, x_i) = 2n$, for $i = 2, \ldots, n$.

ii. $d^D(x_i, x_j) = 2n - 1$, for $i = 2, \ldots, n-1, j = i+1, \ldots, n$.

b. The two vertices are in a path graph P_m, $m \geq 2$

i. $d^D(y_m, y_i) = 3(m - i) + 1$, for $i = 1, \ldots, m-1$.

ii. $d^D(y_i, y_j) = 3(j - i) + 2$, for $i = 1, \ldots, m-2$, $j = i+1, \ldots, m-1$.

c. The two vertices are in different parts

i. $d^D(x_1, y_m) = n + 3m - 1$.

ii. $d^D(x_1, y_i) = n + 3i$, for $i = 1, \ldots, m-1$.

iii. $d^D(x_i, y_m) = d(x_i, y_m) + deg(x_i) + deg(x_1) + \sum_{j=1}^{m} deg(y_j)$, $i = 2, 3, \ldots, n$.

$= 3m + 2n - 1$.

iv. $d^D(x_i, y_j) = d(x_i, y_j) + deg(x_i) + deg(x_1) + \sum_{r=1}^{j} deg(y_r)$

$= 3j + 2n$, for $i = 2, \ldots, n$, $j = 1, \ldots, m-1$.

$$W^D(L_{n,m}) = \sum_{\{u,v\}\subseteq V(L_{n,m})} d^D(u,v)$$

$$= \sum_{i=2}^{n} d^D(x_1,x_i) + \sum_{\{u,v\}\subseteq (V(K_n)-\{x_1\})} d^D(u,v) + \sum_{i=1}^{m-1} d^D(y_m,y_i)$$

$$+ \sum_{\{u,v\}\subseteq (V(P_m)-\{y_m\})} d^D(u,v) + d^D(x_1,y_m) + \sum_{i=1}^{m-1} d^D(x_1,y_i)$$

$$+ \sum_{i=1}^{n-1} d^D(x_i,y_m) + \sum_{\substack{u\in(V(K_n)-\{x_1\})\\ v\in(V(P_m)-\{y_m\})}} d^D(u,v)$$

$$= \sum_{i=2}^{n} 2n + \sum_{i=2}^{n-1}\sum_{j=i+1}^{n}(2n-1) + \sum_{i=1}^{m-1}(3(m-i)+1) + \sum_{i=1}^{m-2}\sum_{j=i+1}^{m-1}(3(j-i)+2)$$

$$+ \sum_{i=1}^{m-1}(n+3i) + \sum_{i=2}^{n}(3m+2n-1) + \sum_{i=2}^{n}\sum_{j=1}^{m-1}(3j+2n) + (n+3m-1)$$

$$= 2n(n-1) + \frac{1}{2}(2n-1)(n^2-3n+2) + \frac{1}{2}(3m^2-m-2)$$

$$+\frac{1}{2}(m^3-m^2-4m+4) + (n+3m-1) + \frac{1}{2}(m-1)(3m+2n)$$

$$+(n-1)(3m+2n-1) + \frac{1}{2}(n-1)(m-1)(3m+4n)$$

$$= \frac{1}{2}[m^3+(3n+2)m^2+(4n^2+n-5)m+n(2n^2-3n+1)].$$

(ii) We get it directly from definition.□

Corollary 2.2. *For $n \geq 3$, we have*

(i) $W^D(L_{n,n}) = n(5n^2-2)$.

(ii) $\mu^D(L_{n,n}) = \frac{5n^2-2}{2n-1}$.

Theorem 2.3. *The D-index of general barbell graph, for $n, m \geq 3$, then*

$$W^D(B_{n,m}) = \frac{1}{2}[2m^3+(4n-3)m^2+(4n^2-2n+3)m+n^2(2n-3)+3n-4].$$

Proof. We have eight cases between any two vertices of general barbell graph $B_{n,m}$, see (Figure2).

(i) $d^D(x_1,x_i) = 2n$, for $i = 2,\ldots,n$.

(ii) $d^D(x_i,x_j) = 2n-1$, for $i = 2,\ldots,n-1$, $j = i+1,\ldots,n$.

(iii) $d^D(x_1,y_1) = n+m+1$.

(iv) $d^D(x_1,y_i) = n+2m+1$, for $i = 2,\ldots,m$.

(v) $d^D(x_i,y_1) = 2n+m+1$, for $i = 2,\ldots,n$.

(vi) $d^D(x_i,y_j) = 2n+2m+1$, for $i = 2,3,\ldots,n$, $j = 2,3,\ldots,m$.

(vii) $d^D(y_1,y_i) = 2m$, for $i = 2,\ldots,m$.

(viii) $d^D(y_i,y_j) = 2m-1$, for $i = 2,\ldots,m-1$, $j = i+1,\ldots,m$.

Hence,

$$W^D(B_{n,m}) = \sum_{\{u,v\}\subseteq V(B_{n,m})} d^D(u,v)$$

$$= \sum_{i=2}^{n} d^D(x_1,x_i) + \sum_{i=2}^{n-1}\sum_{j=i+1}^{n} d^D(x_i,x_j) + d^D(x_1,y_1)$$

$$+\sum_{i=2}^{m} d^D(x_1,y_i) + \sum_{i=2}^{n} d^D(x_i,y_1) + \sum_{i=2}^{n}\sum_{j=2}^{m} d^D(x_i,y_i)$$

$$+\sum_{i=2}^{m} d^D(y_1,y_i) + \sum_{i=2}^{m-1}\sum_{j=i+1}^{m} d^D(y_i,y_j)$$

$$= 2n(n-1) + \frac{1}{2}(2n-1)(n^2-3n+2) + n + m + 1$$

$$+(n+2m+1)(m-1) + (2n+m+1)(n-1) + (2n+2m+1)(n-1)(m-1)$$

$$+2m(m-1) + \frac{1}{2}(2m-1)(m^2-3m+2)$$

.

$$= \frac{1}{2}[2m^3 + (4n-3)m^2 + (4n^2-2n+3)m + n^2(2n-3) + 3n - 4].$$

□

Corollary 2.4. *For $n = m$, $n \geq 3$, then*

$$W^D(B_n) = 6n^3 - 4n^2 + 3n - 2.$$

Corollary 2.5. *For $n, m \geq 3$*

(i) $\mu^D(B_{n,m}) = \frac{4(m^2+1)}{n+m} - \frac{2(m^2-m+1)}{n+m-1} + 2n - 1.$

(ii) $\mu^D(B_n) = \frac{2(n^2+1)}{n} - \frac{3(n^2-n+1)}{2n-1} + 2n - 1.$

Theorem 2.6. *For $n, m \geq 3$,*

(i) The D-index of general modified barbell graph is

$$W^D(B^*_{n,m}) = \frac{1}{2}[2m^3 + (4n-7)m^2 + (4n^2-8n+5)m + n(2n^2-7n+5)].$$

(ii) The average D-distance of general modified barbell graph is

$$\mu^D(B^*_{n,m}) = \frac{4m(m-1)}{m+n-1} - \frac{2(m-1)^2}{m+n-2} + 2n - 1.$$

Proof.

(i) We have five cases between any two vertices of general modified barbell graph $B^*_{n,m}$ as shown in Figure 3.
Let $u = (x_1 \equiv y_1)$ be identifying vertex.

a. $d^D(u,x_i) = 2n + m - 2$, $i = 2, 3, \ldots, n$.

b. $d^D(u,y_i) = n + 2m - 2$, $i = 2, 3, \ldots, m$.

c. $d^D(x_i,x_j) = 2n - 1$, $i = 2, 3, \ldots, n-1$, $j = i+1, \ldots, n$.

d. $d^D(y_i,y_j) = 2m - 1$, $i = 2, 3, \ldots, m-1$, $j = i+1, \ldots, m$.

e. $d^D(x_i, y_i) = 2n + 2m - 2,\ i = 2, 3, \ldots, n,\ j = 2, 3, \ldots, m.$
Hence,

$$W^D(B^*_{n,m}) = \sum_{\{w,v\}\subseteq V(B^*_{n,m})} d^D(w, v)$$

$$= \sum_{i=2}^{n} d^D(u, x_i) + \sum_{i=2}^{m} d(u, y_i) + \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} d^D(x_i, x_j)$$

$$+ \sum_{i=2}^{m-1} \sum_{j=i+1}^{m} d^D(y_i, y_j) + \sum_{i=2}^{n}\sum_{j=2}^{m} d^D(x_i, y_j)$$

$$= (2n + m - 2)(n - 1) + (n + 2m - 2)(m - 1)$$

$$+\frac{1}{2}(2n - 1)(n^2 - 3n + 2) + \frac{1}{2}(2m - 1)(m^2 - 3m + 2)$$

$$+(2n + 2m - 2)(n - 1)(m - 1)$$

$$= \frac{1}{2}[2m^3 + (4n - 7)m^2 + (4n^2 - 8n + 5)m + n(2n^2 - 7n + 5)].$$

(ii) We get it directly from definition.□

Corollary 2.7. *For $n \geq 3$, then*

(i) $W^D(B^*_n) = n(6n - 5)(n - 1)$.

(ii) $\mu^D(B^*_n) = \frac{n(6n-5)}{2n-1}$.

3 D-Index of Hexagonal Chain

A straight hexagonal chain H_p, $p \geq 3$ is a connected graph, consists of $(\frac{p-1}{2})$ hexagons, such that two hexagons are either disjoint or have exactly one edge in common, no three hexagons share a common vertex and each hexagon is adjacent to two other hexagons shown in Figure 4. A cycle hexagonal chain H_s, $p \geq 4$, as depicted in Figure 5, is a connected graph, consists of $(\frac{p}{2})$ hexagons, such that two hexagons have exactly one edge in common. In this section we derive formulas that calculate the D-index and average D-distance of these graphs.

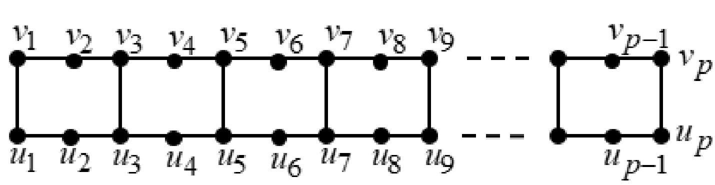

Figure 4. Straight Hexagonal chain H_s

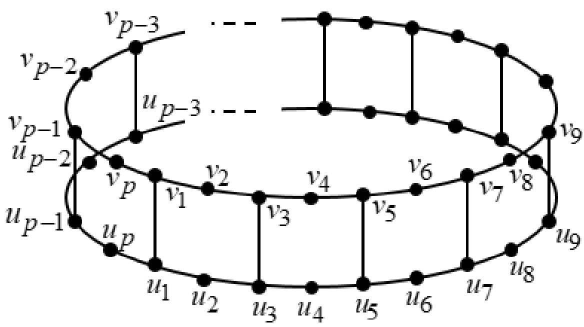

Figure 5. Cycle Hexagonal Chain H_c

Proposition 3.1.

(i) For straight hexagonal chain H_s, $p \geq 3$

$$deg(u_i) = deg(v_i) = \begin{cases} 2 + i(mod)2 & if\, 2 \leq i \leq p-1 \\ 2 & if\, i = 1, p \end{cases}.$$

(ii) For cycle hexagonal chain H_c, $p \geq 4$.

$$deg(u_i) = deg(v_i) = 2 + i(mod2), 1 \leq i \leq p.$$

Theorem 3.2. *For $p \geq 6$*

(i) The D-index of a straight Hexagonal chain

$$W^D(H_s) = 4\sum_{i=2}^{p-2}\sum_{j=i+1}^{p-1}(\lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor) + \frac{1}{3}(4p^3 + 54p^2 - 121p + 12).$$

(ii) The average D-distance of a straight Hexagonal chain

$$\mu^D(H_s) = \frac{2}{p(p-1)}[4\sum_{i=2}^{p-2}\sum_{j=i+1}^{p-1}(\lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor) + \frac{1}{3}(4p^3 + 54p^2 - 121p + 12)].$$

Proof.

(i) To find the D-index of H_s, see Figure 4, we must find $d^D(u, v)$ for each $u, v \in V(H_s)$, let $V = \{v_1, v_2, \ldots, v_p\}$, $U = \{u_1, u_2, \ldots, u_p\}$. There are two main cases

a. If $u, v \in V$ or $u, v \in U$, by using proposition 3.1 (1.)

i. For $i = 2, \ldots, p-2, j = i+1, \ldots, p-1$.

$$d^D(v_i, v_j) = 3(j-i) + \lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor + 3.$$

ii. For $2 \leq i \leq p-1$

$$d^D(v_1, v_i) = 3(i-1) + \lceil\frac{i}{2}\rceil + 1.$$

iii. For $2 \leq i \leq p-1$

$$d^D(v_i, v_p) = 3(p-1) + \lceil\frac{p}{2}\rceil - \lfloor\frac{i}{2}\rfloor + 1.$$

iv.
$$d^D(v_1, v_p) = 3(p-1) + \lceil \frac{p}{2} \rceil.$$

The same proof for u_i and u_j.

$$W_1^D(H_s) = 2\sum_{i=2}^{p-2}\sum_{j=i+1}^{p-1}((j-i) + \lceil \frac{j}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 3) + 2\sum_{i=2}^{p-1}(3(i-1) + \lceil \frac{i}{2} \rceil + 1)$$

$$+2\sum_{i=2}^{p-1}(3(p-1) + \lceil \frac{p}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 1) + 6(p-1) + 2\lceil \frac{p}{2} \rceil.$$

$$= 2\sum_{i=2}^{p-2}\sum_{j=i+1}^{p-1}(\lceil \frac{j}{2} \rceil - \lfloor \frac{i}{2} \rfloor) + \frac{1}{3}(p^3 + 33p^2 - 76p + 42).$$

b. If $v \in V$ and $u \in U$

i. If $i < j$
For $i = 2, \ldots, p-2$, $j = i+1, \ldots, p-1$.

$$d^D(v_i, u_j) = 3(j-i) + \lceil \frac{j}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 6.$$

ii. If $i > j$, the same as case (a), since the graph is symmetric.

iii. If $i = 2, \ldots, p-1$.

$$d^D(v_1, u_i) = 3(i-1) + \lceil \frac{i}{2} \rceil + 4.$$

$$d^D(v_p, u_i) = 3(p-i) + \lceil \frac{p}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 4.$$

iv. If $i = 2, \ldots, p-2$.

$$d^D(v_{p-1}, u_i) = 3(p-i-1) + \lceil \frac{p-1}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 6.$$

v. If $i = j$.

A. For $i = 3, \ldots, p-2$. If v_i and u_i are adjacent then

$$d^D(v_i, u_i) = 7.$$

If v_i and u_i are not adjacent then

$$d^D(v_i, u_i) = 13.$$

We have $\frac{p-3}{2}$ pairs of distance 7 and $\frac{p-5}{2}$ pairs of distance 13.

B. If $i = 1$ or p
$$d^D(v_1, u_1) = (v_p, u_p) = 5.$$

C. If $i = 2$ or $p-1$
$$d^D(v_2, u_2) = (v_{p-1}, u_{p-1}) = 11.$$

$$W_2^D(H_s) = 2\sum_{i=2}^{p-2}\sum_{j=i+1}^{p-1}(3(j-i) + \lceil \frac{j}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 6) + \sum_{i=2}^{p-1}(3(i-1) + \lceil \frac{i}{2} \rceil + 4)$$

$$+\sum_{i=2}^{p-1}(3(p-i) + \lceil \frac{p}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 4) + \sum_{i=2}^{p-2}(3(p-i-1) + \lceil \frac{p-1}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 6)$$

$$+7(\frac{p-3}{2}) + 13(\frac{p-5}{2}) + 32.$$

$$= 2\sum_{i=2}^{p-2}\sum_{j=i+1}^{p-1}(\lceil \frac{j}{2} \rceil - \lfloor \frac{i}{2} \rfloor) + p^3 + 7p^2 - 15p - 10.$$

Hence,

$$W^D(H_s) = 4\sum_{i=2}^{p-2}\sum_{j=i+1}^{p-1}(\lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor) + \frac{1}{3}(4p^3 + 54p^2 - 121p + 12).$$

(ii) We get it directly from definition. □

Theorem 3.3. *For* $p \geq 4$.

(i) The D-index of a cycle hexagonal chain

$$W^D(H_c) = 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}(\lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor) + 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i}^{p}(\lceil\frac{p-j+2i}{2}\rceil - \lfloor\frac{i}{2}\rfloor)$$

$$+4\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}(\lceil\frac{j}{2}\rceil + \lfloor\frac{i}{2}\rfloor) + \frac{1}{2}(3p^3 + 13p^2 + 23p - 16).$$

(ii) The average D-distance of a cycle hexagonal chain

$$\mu^D(H_c) = \frac{2}{p(p-1)}[4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}(\lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor) + 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i}^{p}(\lceil\frac{p-j+2i}{2}\rceil - \lfloor\frac{i}{2}\rfloor)$$

$$+4\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}(\lceil\frac{j}{2}\rceil + \lfloor\frac{i}{2}\rfloor) + \frac{1}{2}(3p^3 + 13p^2 + 23p - 16)].$$

Proof.

(i) To find the D-index of H_c, see Figure 5, we must find $d^D(u, v)$ for each $u, v \in V(H_c)$, by using proposition 3.1 (2.).
There are two main cases

a. For $i = 1, \ldots, p-1$, $j = i+1, \ldots, p$. If $u = v_i$ and $v = v_j$, or $u = u_i$ and $v = u_j$, we have three sub-cases

   i. For $i = 1, \ldots, \frac{p}{2} - 1$, $j = i+1, \ldots, \frac{p}{2} + i$.

$$d^D(v_i, v_j) = 3(j-i) + \lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor + 2.$$

   ii. For $i = 1, \ldots, \frac{p}{2} - 1$, $j = i+1, \ldots, p$.

$$d^D(v_i, v_j) = 3(p-j+i) + \lceil\frac{p-j+2i}{2}\rceil - \lfloor\frac{i}{2}\rfloor + 2.$$

   iii. For $i = \frac{p}{2}, \ldots, p-1$, $j = i+1, \ldots, p$.

$$d^D(v_i, v_j) = 3(j-i) + \lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor + 2.$$

   All cases are repeated for u_i and u_j.

b. If $u = u_i$ and $v = v_j$.
   For $i = 1, \ldots, p, j = 1, \ldots, p$.

   i. If $i < j$.

      A. For $i = 1, \ldots, \frac{p}{2} - 1, j = i+1, \ldots, \frac{p}{2} + i$.

$$d^D(v_i, v_j) = 3(j-i) + \lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor + 6.$$

B. For $i = 1, \ldots, \frac{p}{2} - 1, j = \frac{p}{2} + i + 1, \ldots, p$.

$$d^D(v_i, v_j) = 3(p - j + i) + \lceil \frac{p - j + 2i}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 6.$$

C. For $i = \frac{p}{2}, \ldots, p - 1, j = i + 1, \ldots, p$.

$$d^D(v_i, v_j) = 3(j - i) + \lceil \frac{j}{2} \rceil - \lfloor \frac{i}{2} \rfloor + 6.$$

ii. If $i = j$, for $i = 1, \ldots, p$.
If the two vertices are adjacent, then

$$d^D(u_i, v_i) = 7.$$

If u_i, v_i are not adjacent, then

$$d^D(u_i, v_i) = 13.$$

We have $\frac{p}{2}$ pairs of distance 7, and $\frac{p}{2}$ pairs of distance 13.

iii. If $i > j$
Since the graph is symmetric all cases of $i > j$ are the same as of $i < j$.

$$W^D(H_c) = 12\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}(j-i) + 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}\lceil\frac{j}{2}\rceil - 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}\lfloor\frac{i}{2}\rfloor + \sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}16$$

$$+12\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i+1}^{p}(p-j+i) + 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i+1}^{p}\lceil\frac{p-j+2i}{2}\rceil - 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i+1}^{p}\lfloor\frac{i}{2}\rfloor$$

$$+\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i+1}^{p}16 + 12\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}(j-i) + 4\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}\lceil\frac{j}{2}\rceil + 4\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}\lfloor\frac{i}{2}\rfloor$$

$$+\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}16 + 10p.$$

$$= \frac{3}{4}(p^3 - 4p) + 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}(\lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor) + 4p^2 - 8$$

$$+\frac{1}{2}(p^3 - 3p^2 + 2p) + 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i}^{p}(\lceil\frac{p-j+2i}{2}\rceil - \lfloor\frac{i}{2}\rfloor)$$

$$+2p^2 - 4p + \frac{1}{4}(p^3 + 6p^2 + 8p) + 4\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}(\lceil\frac{j}{2}\rceil + \lfloor\frac{i}{2}\rfloor)$$

$$+2p^2 + 4p + 10p$$

$$= 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=i+1}^{\frac{p}{2}+i}(\lceil\frac{j}{2}\rceil - \lfloor\frac{i}{2}\rfloor) + 4\sum_{i=1}^{\frac{p}{2}-1}\sum_{j=\frac{p}{2}+i}^{p}(\lceil\frac{p-j+2i}{2}\rceil - \lfloor\frac{i}{2}\rfloor)$$

$$+4\sum_{i=\frac{p}{2}}^{p-1}\sum_{j=i+1}^{p}(\lceil\frac{j}{2}\rceil + \lfloor\frac{i}{2}\rfloor) + \frac{1}{2}(3p^3 + 13p^2 + 23p - 16).$$

(ii) We get it directly from definition. □

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