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Modified Spline Polynomial for Solving Nonlinear Thin Film Flow and Three Dimensional Hybrid Non-Fluid Flow

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Abstract In this paper, we consider the nonlinear thin film flow and three dimensional hybrid fluid flows over a non-linear stretching sheet are solved numerically using the quartic spline methods. The obtained results are compared through tables and illustrative graphs with the exact solution and numerical solutions are shown. Further, the effect of the numerical techniques for solving the second order thin film flow and non-linear system of hybrid parameter on the obtained by the spline methods, velocity and shear stress profiles is presented through tables and illustrative graphs.

1 Introduction

The phrase "thin film," which refers to a layer of material with a thickness that can range from a few nanometers to several micrometers, is frequently used to describe flow. One dimension has a substantially narrower flow spectrum than the other (one or two dimensions). It is a property used to condense the Navier-Stokes equations into a straightforward set of modeling equations. Thin film flow concerns are mostly structurally significant. Researchers are very interested in working in this topic because of its many applications in areas including surface coating, chemical engineering, industrial processes, cooling, and the lubrication of heat exchange fins into the motion of contact lenses [1]. Thin film flow issues can have extremely simple or sophisticated effects in daily life, such as transmitting a raindrop onto an airplane panel. It is acknowledged that thin film flow is a factor in many industrial processes. As a result, the research community is eager to use thin film flow in a variety of natural science and engineering domains. The following fields paint coatings, wave modeling for dam breaching, and nuclear reactor fluid dynamics show the most no-table industrial uses. The representation of TFF dynamics in the realm of biophysics is similar to membranes and ocular tear films. The most typical flow and thin film flow characteristics. Additionally, thin films enable everyone to plan the architecture of nuclear reactors, verify the output of paints in surface flow coverings, and reduce the composition of equipment such as bearing fluid. Similar to red blood cell movement through eyes, veins, and lung friction is one of TFF's biological uses [2]. Its impact may also be observed in several engineering, social, and natural scientific domains. Many scientists and engineers that study nonlinear dynamic systems believe that the fin-est analysis at this time may be found in these fluids in many of these domains [3]. Examples include the exceptional study on mixed convection flow through a porous middling and the investigation of the movement of a viscous fluid across a moving solid surface [4]. Although many physical issues are nonlinear, insufficient linear analysis exists to fully describe how physical systems behave. An accurate solution for nonlinear dynamic systems is rarely found, at least in the state of science today. It is important to comprehend novel and inventive approaches capable of resolving nonlinear system dynamics in this context. Recently, some useful solutions to various nonlinear equations have been found. The weighted linearization process [5], the Lindstedt-Poincare method [6], the Adomian decomposition technique [7], the boundary element technique [8] and the optimal asymptotic homotopy method [9] are some of the analytical techniques that are used to address nonlinear problems. All of the strategies described above perform admirably for weakly non-linear situations, and several of them even perform admirably for higher-order nonlinear systems. In the current study, a boundary value issue is solved using the spline approach for the nonlinear differential equation of thin film flow [10]. Chemical engineers use thin film layers extensively in the development of distillation columns, thin film reactions, evaporators, and condensers. Due to its low density, which facilitates movement through microchannels, thin layers have great benefits. Thin film flows are related with several problems in geophysical engineering, including catastrophic flooding, lava, and flash floods [11]. These are also very important since shear thickening and pseudo-plastic or shear thinning liquids are considered to be non-Newtonian fluids. Non-Newtonian liquids are often utilized in commerce and industry and are now the focus of much study, especially when combined with nanomaterials [12, 13]. For example, whilst rheological activity is utilized by biological equipment like the homodialyser, chemical companies handle polymers and plastics extensively. These methods are often used by researchers to resolve linear and nonlinear differential equations [14]. A few recent applications include the crosswise stream fluid model involving nanomaterial over porous stretching medium [15], the hybrid rotational nanofluidic model with thermal characteristic consideration [16], The Dynamic of thin Liquid film [24], Courtain cauting flow in thin Liquid films [25], Modeling on MHD Boundary Layer Flow Unsteady Stretching Sheet [26], the mathematical models of hydrogen possessions [17] and the COVID-19 epidemical models with future generation disease control [18]. Additionally, the spline approach is appealing to the research community since it may be used to solve many different kinds of differential equations. The spline approach has recently been regarded by scholars as the best numerical method for resolving a variety of issues [19].

2 Mathematical Modeling

Since mathematics is a very exact language, it aids in the formulation of concepts and the discovery of underlying presumptions. It is possible to transform real-world issues into mathematical models and then solve them using mathematical modeling, which is a sub-field of mathematical logic. A model can be useful for studying the impacts of various components, explaining how a system works, and forecasting behavior. Relationships and variables are often the building blocks of mathematical models. Assumptions, variables, and parameters are clarified during the model formulation process. Mathematical techniques and computer simulations can be used to examine the behavior of accurate mathematical models. Using mathematical techniques and computer simulations, it is possible to examine the behavior of exact mathematical models. Mathematical modeling is the practice of utilizing different mathematical structures, such as graphs, equations, diagrams, scatterplots, tree diagrams, and more, to reflect actual world events. In actuality, mathematical modeling is not a brand-new field. It has existed there for a very long time. Through the use of mathematical models, scientists, engineers, statisticians, and astronomers have been able to examine a wide range of variety issues.

Case 1: Following are the governing equations for incompressible fluid in the absence of thermal effects. The continuity equation for incompressible fluid has the following form:

$$\nabla . V = 0. \tag{2.1}$$

The momentum equation is

$$\rho \frac{DV}{Dt} = -\nabla P + \rho f + div\tau, \tag{2.2}$$

where $\rho, V, P, \frac{DV}{Dt}$ and τ represent density, velocity, pressure, material derivative and the stress tensor respectively.

Consider a general form of TFF is given as two-point boundary value problem of second order [21].

$$\frac{d^2y}{dx^2} + 6\beta \left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} - n = 0,$$

with boundary conditions:

$$y(0) = 1 \text{ and } y^{'}(1) = 0.$$

Case 2: Three dimensional steady incompressible hybrid nanofluid flows is considered over a biaxial stretching-shrinking sheet along porous surface, Mathematical equations under assumptions are seen on [20].

$$f^{'''} + 2ff^{'} - f^{'2} = 0,$$
$$h^{''} + 2fh^{'} - hf^{'} = 0,$$
$$\frac{1}{0.5}(1+0.5)\theta^{''} + 2f\theta^{'} = 0,$$

and the boundary conditions are:

$$f = 1, \;\; f^{'} = -1, \;\; h = 0, \theta = 1 \;\text{ for }\; \eta = 0,$$
$$f^{'} = 0, \;\; h^{'} = 1, \;\; \theta = 0 \;\text{ as }\; \eta = 3.$$

3 Solution Methodologies

This section is divided into three parts: Modified Polynomial Spline in the first, Spline Collocation and Quasilinearization in the second and finally, formulation of the problems. It's a fantastic meth-od for resolving nonlinear equations. It's a great way to solve non-linear equations. This method is depending on decomposing a non-linear equation solution into Quasilinearization and series of functions. In theory, these computations are simple, but in fact, computing the polynomials and showing the convergence of the related series can be challenging.

3.1 Modified Polynomial Spline

Let $x_{i}{}_{i=0}^{n}$ be a strictly increasing sequence of points such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ consider that $f(x)$ is a continuous function over an interval $[a, b]$. An interpolate to $f(x)$ is a polynomial spline $s(x)$ of degree m such that [22].

(i) $s(x)$ is a polynomial of degree m over each subinterval $[x_i, x_{i+1}], i = 0, 1, \ldots, n-1$.

(ii) $s(x_i) = f(x_i), i = 0, 1, 2, \ldots, n$.

(iii) $s(x) \in C^{m-1}[a, b]$.

i.e. $s(x)$ and its successive derivatives of order $(m-1)$ are continuous over the entire interval $[a, b]$. The cubic and quartic spline is defined $s(x)$ to solve two and third order boundary value problem respectively as follows:

$$s(x) = a_0 + a_1(x - x_0) + \frac{1}{2}a_2(x - x_0)^2 + \frac{1}{6}\sum_{k=0}^{n-1} b_k(x - x_k)_+^3 \tag{3.1}$$

$$s(x) = a_0 + a_1(x - x_0) + \frac{1}{2}a_2(x - x_0)^2 + \frac{1}{6}a_3(x - x_0)^3 + \frac{1}{24}\sum_{k=0}^{n-1} b_k(x - x_k)_+^4 \tag{3.2}$$

where n is the number of subintervals of $[a, b]$ and $a_0, a_1, a_2, a_3, b_0, \ldots, b_{n-1}$ are $(n+3)$ unknowns. The power function $(x - x_k)_+$ has a property that

$$(x - x_k)_+ = \begin{cases} x - x_k & \text{if } x > x_k \\ 0 & \text{if } x \leq x_k \end{cases} \tag{3.3}$$

different cubics in the subinterval $[x_i, x_{i+1}]$ of $[a, b]$ are represented by $s(x)$. Due to their basic mathematical qualities, such as continuity and differentiability, as well as their ease of expression, polynomials have been used to approximate other functions for a very long time. The definition of spline functions given above makes it abundantly clear that they are also piecewise polynomials. Spline functions obtain all the basic mathematical properties as polynomials. This indicates that the spline functions and their successive derivatives are endless, differentiable, and most often, analytical functions. The finite dimensional linear spaces known as polynomial splines have very fitting bases [22].

3.2 Spline Collocation and Quasilinearization

In mathematical modeling, it is a common experience of coming across nonlinear equations. Except a very few, usually, non-linear differential equations are difficult to solve analytically. In such situations, one has to go for approximate or numerical solutions to the equations. Our aim is to solve some non-linear boundary value problems arising in the study of applied science like problems in fluid mechanics.
In order to solve the non-linear boundary value problem using spline collocation method, the boundary value problem is linearized before solving it. The linear expression of the non-linear equation takes place due to a well-known technique known as Quasilinearization [23].
The quasilinearization is shortly described as follows:
Let the nonlinear boundary value problem is given by

$$y^{''} = f(x, y, y^{'}) \tag{3.4}$$

subject to the boundary conditions

$$g_1\left[y(a), y^{'}(a)\right] = L \text{ and } g_2\left[y(b), y^{'}(b)\right] = M. \tag{3.5}$$

Rewrite the equation (3.4) as:

$$\phi(x, y, y^{'}, y^{''}) = y^{''} - f(x, y, y^{'}) = 0. \tag{3.6}$$

Now, we derive the recurrence equations. For that, let us denote the n^{th} and $(n+1)^{th}$ iterations by y_n and y_{n+1} respectively and require that for both iterations $\phi = 0$. Taking n^{th} iterations, this gives

$$y_n^{''} - f(x, y_n, y_n^{'}) = 0. \tag{3.7}$$

The $(n+1)^{th}$ iteration, is obtained as:

$$\begin{aligned}\phi\left(x, y_{n+1}, y_{n+1}^{'} y_{n+1}^{''}\right) = \phi(x, y_n, y_n^{'}, y_n^{''}\Big) + \left(\frac{\partial \phi}{\partial y}\right)_n (y_{n-1} - y_n) + \left(\frac{\partial \phi}{\partial y^{'}}\right)(y_{n-1}^{'} - y_n^{'}) \\ + \left(\frac{\partial \phi}{\partial y^{''}}\right)_n (y_{n-1}^{''} - y_n^{''}) + \cdots = 0\end{aligned} \tag{3.8}$$

Where the higher order derivatives are ignored accordingly. Expressing the equation (3.8) in terms of $f(x, y, y^{'})$ and using the equation (3.7) the ultimate equation becomes

$$-\left(\frac{\partial f}{\partial y}\right)_n (y_{n+1} - y_n) - \left(\frac{\partial f}{\partial y^{'}}\right)_n \left(y_{n+1}^{'} - y_n^{'}) + (y_{n+1}^{''} - y_n^{''}\right) = 0.$$

Finally,

$$y_{n+1}^{''} - \left(\frac{\partial f}{\partial y^{'}}\right)_n y_{n+1}^{'} - \left(\frac{\partial f}{\partial y}\right)_n y_{n+1} = f(x, y_n, y_n^{'}) - \left(\frac{\partial f}{\partial y}\right)_n y_n - \left(\frac{\partial f}{\partial y^{'}}\right)_n y_n^{'}. \tag{3.9}$$

The boundary conditions in the linearized form are expressed as

$$\left(\frac{\partial g_1}{\partial y}\right)_n \left[y(a), y^{'}(a)\right] [y_{n+1}(a) - y_n(a)] + \left(\frac{\partial g_1}{\partial y^{'}}\right)_n \left[y(a), y^{'}(a)\right] \left[y_{n+1}^{'}(a) - y_n^{'}(a)\right] = L, \tag{3.10}$$

and

$$\left(\frac{\partial g_2}{\partial y}\right)_n \left[y(b), y^{'}(b)\right] [y_{n+1}(b) - y_n(b)] + \left(\frac{\partial g_2}{\partial y^{'}}\right)_n \left[y(b), y^{'}(b)\right] \left[y_{n+1}^{'}(b) - y_n^{'}(b)\right] = M. \tag{3.11}$$

Thus, it is seen that the equation (3.9) is a linear differential equation subject to the boundary conditions (3.10) and (3.11). In this way, the quasilinearization technique is used to convert a non-linear equation into a linear equation. Now the equation (3.9) can be solved numerically by any of the known methods. An important point to be noticed in such combination of two

processes is that the solution of a linear differential equation which ultimately is a solution to a nonlinear equation.
Returning to the equation (3.9) with associated boundary conditions, the interpolant $s(x)$ of the function $y_n(x)$ is a spline function. In light of the interpolating conditions have

$$s(x_i) = y_n(x_i), \qquad i = 0, 1, 2, \ldots, n, \tag{3.12}$$

an application of these methods is developed with the combination of spline collocation and quasilinearization technique to solve a nonlinear boundary value problem. In this section, apply above method to solve linear boundary value problems and non-linear boundary value problems.

3.3 Formulation of the problems

To apply the quartic spline and quasilinearization of the equation (case 1 problem), we construct the following:

$$y_i^{''} + 6\beta y_i^{''} y_i^{'2} - n = 0, \tag{3.13}$$

where, $x_i \in [0, 1]$, satisfying boundary conditions $y_0 = 1$ and $y_n^{'} = 0$. First we set an initial guess by having $[s_i]_0 = ax_i + b$,$[s_0]_0 = 1$ and $[s_n^{'}]_0 = 0$.
We get $a = 0$ and $= 1$, $[s_i]_0 = 1$ as our first spline we find next spline approximation

$$[s_i]_1 = \sum_{m=0}^{2} a_m \frac{h^m}{m!} + \sum_{j=0}^{n-1} b_j \frac{h_+^3}{3!}$$

Now, rewrite the Equation (3.13) as:

$$y_i^{''} - \frac{n}{1 + 6\beta y_i^{'2}} = 0, \tag{3.14}$$

Now, we derive the recurrence equations. For that, let us denote the k^{th} and $(k+1)^{th}$ iterations by y_k and y_{k+1} respectively and require that for both iterations $\phi = 0$. Taking k^{th} iterations, this gives:

$$\phi(y_k, y_k^{'}, y_k^{''}) = y_i^{''} - \frac{n}{1 + 6\beta y_i^{'2}} \tag{3.15}$$

The $(k+1)^{th}$ iteration is obtained as:
Applying quasilinearization technique for the equation (3.12) produces:

$$\phi\left(y_{k+1}, y_{k+1}^{'}, y_{k+1}^{''}\right) = \phi\left(y_k, y_k^{'}, y_k^{''}\right) + \left[\frac{\partial \phi}{\partial y_k}\right]_k [y_{k+1} - y_k] + \left[\frac{\partial \phi}{y_k^{'}}\right]_k \left[y_{k+1}^{'} - y_k^{'}\right]$$
$$+ \left[\frac{\partial \phi}{y_k^{''}}\right]_k \left[y_{k+1}^{''} - y_k^{''}\right]. \tag{3.16}$$

Where the higher order derivatives are ignored accordingly. Expressing the equation (3.16) in terms of $y_{k+1}^{''}$ and using the equation (3.15) the ultimate equation becomes:

$$0 + \frac{12n\beta y_k^{'}}{\left[1 + 6\beta y_k^{'2}\right]^2} \left[y_{k+1}^{'} - y_k^{'}\right] + \left[y_{k+1}^{''} - y_k^{''}\right] = 0,$$

$$y_{k+1}^{''} + y_{k+1}^{'} \frac{12n\beta y_k^{'}}{\left[1 + 6\beta y_k^{'2}\right]^2} = y_k^{''} + \frac{12n\beta y_k^{'}}{\left[1 + 6\beta y_k^{'2}\right]^2}, \tag{3.17}$$

Substituting $y_k^{''} = \frac{n}{1+6\beta y_k^{'2}}$ into the Equation (3.17), to obtain as follows:

$$y_{k+1}^{''} + y_{k+1}^{'} \frac{12n\beta y_k^{'2}}{\left[1 + 6\beta y_k^{'2}\right]^2} = \frac{n\left[1 + 18\beta y_k^{'2}\right]}{\left[1 + 6\beta y_k^{'2}\right]^2}, \tag{3.18}$$

$$0 \times a_0 + a_1 \frac{12n\beta y_k^{'2}}{\left[1 + 6\beta y_k^{'2}\right]^2} + a_2 \frac{1 + 12n\beta y_k^{'}}{\left[1 + 6\beta y_k^{'2}\right]^2} + \sum_{j=0}^{n-1} b_j \left(h_+ + \frac{12n\beta y_k^{'} h_+^2}{2\left[1 + 6\beta y_k^{'2}\right]^2} \right)$$

$$= n \frac{\left[1 + 18\beta y_k^{'2}\right]}{\left[1 + 6\beta y_k^{'2}\right]^2}. \tag{3.19}$$

Now, From the equation (3.19), putx_i, $i = 0, 1, 2, \ldots, 5$, we have six equations with two boundary equations we get eight equations for the unknowns $a_0, a_1, a_2, b_0, b_1, b_2, b_3$ and b_4.

$$AX = B,$$

$$[s]_1 = 1 - nh + \frac{nh^2}{2},$$

Using initials polynomials $[F]_0 = x^2 - x + 1$, $[H]_0 = x$ and $[\theta]_0 = 1, x \in [0, 3]$.

$$f_k^{(3)} + 2f_k f_k^{'} - \left(f_k^{'}\right)^2 = 0, \tag{3.20}$$

$$h_k^{''} + 2f_k h_k^{'} - hf_k^{'} = 0, \tag{3.21}$$

$$3\theta_k^{''} + 2\theta_k^{'} f_k = 0, \tag{3.22}$$

and the boundary conditions are:

$$f(0) = 1, \;\; f^{'}(0) = -1 \;\text{ and }\; f^{'}(3) = 0, \tag{3.23}$$

$$h(0) = 0 \;\text{ and }\; h^{'}(3) = 1, \tag{3.24}$$

$$\theta(0) = 1 \;\text{ and }\; \theta(3) = 0, \tag{3.25}$$

To solve the equation (3.20), it is required to convert nonlinear problem into linear form. Convert the Equation (3.20) in linear form using quasilinearization method.
Applying quasilinearization technique and rewrite the equation (3.20), we obtain:

$$\phi_1 \left(f_k, f_k^{'}, f_k^{''}, f_k^{(3)}\right) = f_k^{(3)} + 2f_k f_k^{'} \left(f_k^{'}\right)^2 = 0, \tag{3.26}$$

The $(k+1)^{th}$ iteration is obtained as:

$$\phi_1 \left(f_{k+1}, f_{k+1}^{'}, f_{k+1}^{''}, f_{k+1}^{(3)}\right) = \phi_1 \left(f_k, f_k^{'}, f_k^{''}, f_k^{(3)}\right) + \left[\frac{\partial \phi_1}{\partial f_k}\right]_k [f_{k+1} - f_k] + \left[\frac{\partial \phi_1}{\partial f_k^{'}}\right]_k \left[f_{k+1}^{'} - f_k^{'}\right]$$

$$+ \left[\frac{\partial \phi_1}{\partial f_k^{''}}\right]_k \left[f_{k+1}^{''} - f_k^{''}\right] + \left[\frac{\partial \phi_1}{\partial f_k^{(3)}}\right]_k \left[f_{k+1}^{(3)} - f_k^{(3)}\right], \tag{3.27}$$

where the higher order derivatives are ignored accordingly. Expressing the Equation (3.27) and using the equation (3.26) the ultimate equation becomes:

$$f_{k+1}^{(3)} + 2f_{k+1}^{'} \left[f_k - f_k^{'}\right] + 2f_{k+1} f_k^{'} = 2f_k f_k^{'} - \left(f_k^{'}\right)^2, \tag{3.28}$$

$$[F]_1 = \sum_{m=0}^{3} a_m \frac{h^m}{m!} + \sum_{j=0}^{n-1} b_j \frac{h_+^4}{4!}.$$

Using boundary conditions of the equation (3.23), substituting the equation (3.1) into the equation (3.28), following collocations are obtained:

$$2a_0 f_k^{'} + a_1[2f_k^{'}h + 2(f_k - f_k^{'})] + a_2[f_k^{'}h^2 + 2h(f_k - f_k^{'})] + a_3\left[\frac{f_k^{'}h^3}{3} + (f_k - f_k^{'})h^2 + 1\right]$$
$$+ \sum_{j=0}^{n-1} b_j\left[f_k^{'}\frac{h_+^4}{12} + \frac{\left(f_k - f_k^{'}h_+^3\right)}{3} + h_+\right] = 2f_k f_k^{'} - \left(f_k^{'}\right)^2.$$

Applying quasilinearization technique and rewrite the Equation (3.21), we obtain:

$$\phi_2(h_k, h_k^{'}, h_k^{''}) = h_k^{''} + 2f_k h_k^{'} - h_k f_k^{'} = 0. \tag{3.29}$$

The $(k+1)^{th}$ iteration is obtained as:

$$\phi_2\left(h_{k+1}, h_{k+1}^{'}, h_{k+1}^{''}\right) = \phi_2\left(h, h_k^{'}, h_k^{''}\right) + \left[\frac{\partial \phi_2}{\partial h_k}\right]_k [h_{k+1} - h] + \left[\frac{\partial \phi_2}{\partial h_k^{'}}\right]_k \left[h_{k+1}^{'} - h_k^{'}\right]$$
$$+ \left[\frac{\partial \phi_2}{\partial h_k^{''}}\right]_k \left[h_{k+1}^{''} - h_k^{''}\right], \tag{3.30}$$

where the higher order derivatives are ignored accordingly. Expressing the equation (3.30) and using the equation (3.29) the ultimate equation becomes:

$$h_{k+1}^{''} + 2f_k h_{k+1}^{'} - f_k^{'} h_{k+1} = 0, \tag{3.31}$$

$$[H]_1 = \sum_{m=0}^{2} c_m \frac{h^m}{m!} + \sum_{j=0}^{n-1} d_j \frac{h_+^3}{3!}.$$

Using boundary conditions of the equations (3.23)-(3.24), substituting the equation (3.1) into the equation (3.31), following collocations are obtained:

$$-c_0 f_k^{'} + c_1\left[2f_k - f_k^{'}h\right] + c_2\left[1 + 2f_k h_k - f_k^{'}\frac{h^2}{2}\right] + \sum_{j=0}^{n-1} d_j\left[h_+ + 2f_k h_+^2 - f_k^{'}\frac{h_+^3}{6}\right] = 0.$$

Applying quasilinearization technique and rewrite the equation (3.22), we obtain:

$$\phi_3\left(\theta_k, \theta_k^{'}, \theta_k^{''}\right) = 3\theta^{''} + 2\theta_k^{'} f_k = 0, \tag{3.32}$$

The $(k+1)^{th}$ iteration is obtained as:

$$\phi_3\left(\theta_k, \theta_k^{'}, \theta_k^{''}\right) = \phi_3\left(\theta_k, \theta_k^{'}, \theta_k^{''}\right) + \left[\frac{\partial \phi_3}{\partial \theta_k}\right]_k [\theta_{k+1} - \theta_k] + \left[\frac{\partial_2}{\partial \theta_k^{'}}\right]_k \left[\theta_{k+1}^{'} - \theta_k^{'}\right]$$
$$+ \left[\frac{\partial \phi_2}{\partial \theta_k^{''}}\right]_k \left[\theta_{k+1}^{''} - \theta_k^{''}\right], \tag{3.33}$$

where the higher order derivatives are ignored accordingly. Expressing the equation (3.33) and using the equation (3.32) the ultimate equation becomes:

$$3\theta_{k+1}^{''} + 2\theta_{k+1}^{'} f_{k+1} = 0, \tag{3.34}$$

$$[\theta]_1 = \sum_{m=0}^{2} v_m \frac{h^m}{m!} + \sum_{j=0}^{n-1} u_j \frac{h_+^3}{3!}.$$

Using boundary conditions of the equations (3.23)-(3.25), substituting the equation (3.1) into the Equation (3.34), following collocations are obtained:

$$0 \times v_0 + v_1 \frac{2}{3} f_k + v_2\left[1 + \frac{2}{3} f_k h\right] + \sum_{j=0}^{n-1} u_j\left[\frac{(f_k h_+^2)}{3 + h_+}\right] = 0.$$

Finally is always a linear differential equation and can be solved recursively, where iteration k known and one can use it to get iteration $k+1$.

4 Numerical Results

In this section, we will apply the scheme described in section 3 to test the problems to demonstrate the efficiency, accuracy, and applicability of the present scheme.
In this porous, the obtained approximate solutions using collocation method based on cubic and quartic spline function equations (3.1 & 3.4) are discussed. Also solving the second order differential equation, the condition $B = 0.5, n = 0.3$ is used instead for some sufficiently large B in [21]. The solution obtained by spline polynomial methods of the proposed problem is investigated to show the characteristic of hybrid nanofluid flow for various magnetic field and velocity parameter values. The non-dimensional parameters are considered as μ, ρ, Rd and Pr used the same values in [20], in numerical results except the variations. In illustrative graphs, the figure 1 profiles and figure 2 profiles are signalizing the flow of Aluminum nanofluid and the flow of fluid respectively. The following figures and tables, Describes the influence of the shrinking parameter B and n on the velocity profiles for both of nanofluid to regular fluid, and the tables described the errors estimation of the mean, maximum and standard deviations.

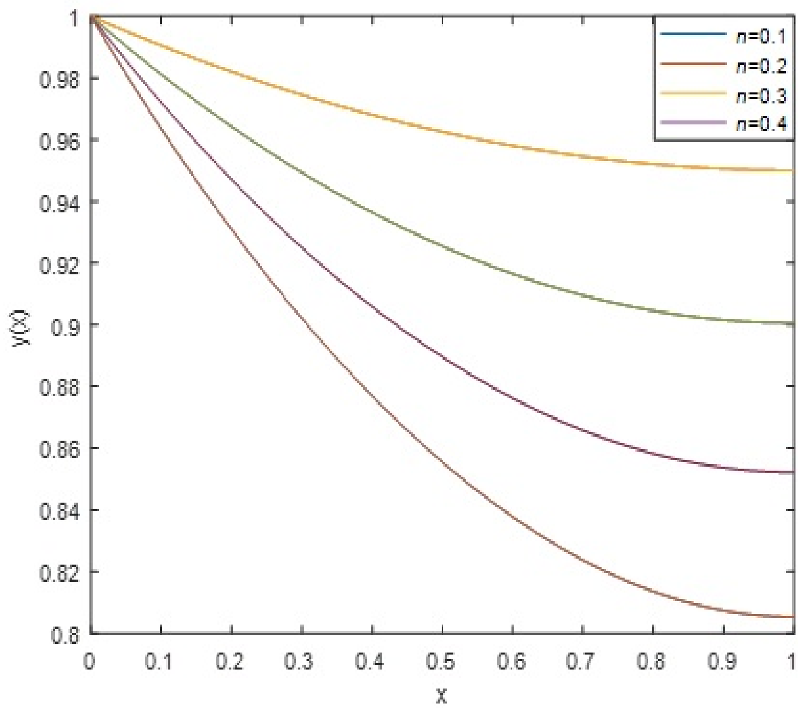

Figure 1. Comparison of the solution $y(x)$ with respect values of n obtained by Scheme-spline of equations (3.19).

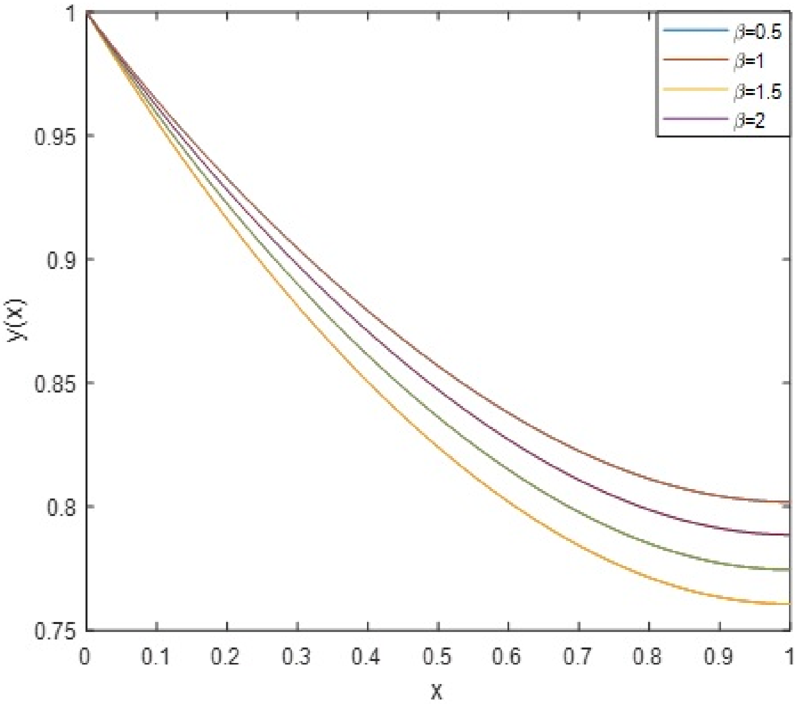

Figure 2. Comparison of the solution $y(x)$ with respect values of B obtained by Scheme-spline of equations (3.19).

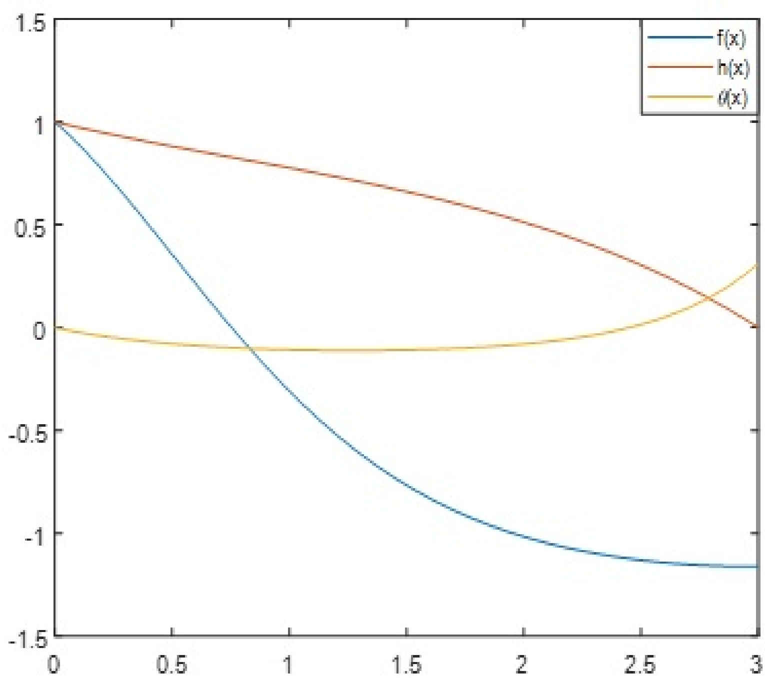

Figure 3. Comparison of the solution ϕ_1, ϕ_2, ϕ_3 obtained by Scheme-spline of equations (3.30), (3.31) and (3.32).

Table 1. Relative analysis using the absolute of the system from case 1 in section 2 errors using Cubic spline for ϕ_1, order 6 with $h = 0.001$ and $n = 0.5$.

Itaration ϕ_1	**Max**	**Min**	**Mean**	**Standard deviation**
$\beta = 0$				
1	7.894736842105265*10-02	0	2.815697650470592*10-02	2.412269437773161*10-02
2	1.416573824443238*10-01	5.381931765310261*10-02	1.707640324452051*10-03	5.670473240876488*10-02
3	1.447965850482579*10-01	5.835529044252974*10-02	1.148357094943844*10-03	5.948715285515747*10-02
4	1.448752879309030*10-01	5.855888113393221*10-02	1.130486014505150*10-03	5.959340873067134*10-02
5	1.448768450314582*10-01	5.855478354227367*10-02	1.132674447403552*10-03	5.959360592367963*10-02
6	1.448765411788454*10-01	5.855341337140774*10-02	1.132952483827189*10-03	5.959316796495750*10-02
$\beta = 0.5$	**Max**	**Min**	**Mean**	**Standard deviation**
1	1.363636363636364*10-01	0	5.137869297774935*10-02	4.239561378446506*10-02
2	2.228483514225428*10-01	9.344795533323136*10-02	6.298911282231505*10-03	9.587573357122692*10-02
3	2.196258882147988*10-01	1.044781549438103*10-01	3.667478642816133*10-03	9.919730766847099*10-02
4	2.192305986622645*10-01	1.048872475794099*10-01	3.610562532496218*10-03	9.918662594680179*10-02
5	2.192104597125460*10-01	1.047828124908745*10-01	3.635941908331039*10-03	9.914741475679041*10-02
6	2.192125685594191*10-01	1.047600310835255*10-01	3.640431728797615*10-03	9.914167358886802*10-02
$\beta = 1.5$	**Max**	**Min**	**Mean**	**Standard deviation**
1	1.800000000000000*10-01	0	7.101826174273480*10-02	5.671873959193382*10-02
2	2.692901988286111*10-01	1.234555190898874*10-01	1.164300451520183*10-02	1.231696544393663*10-01
3	2.489579228571077*10-01	1.390629129985898*10-01	5.885461567829465*10-03	1.227621811756120*10-01
4	2.467503496098790*10-01	1.392354702940378*10-01	5.826713460360161*10-03	1.222572926632027*10-01
5	2.467327834531945*10-01	1.388893167227034*10-01	5.910442028499966*10-03	1.221552569504196*10-01
6	2.467672936784404*10-01	1.388133086397143*10-01	5.926661386141495*10-03	1.221451260825144*10-01
$\beta = 2$	**Max**	**Min**	**Mean**	**Standard deviation**
1	2.142857142857143*10-01	0	8.795025070364379*10-02	6.824101244815493*10-02
2	2.945469974718948*10-01	1.469566763050957*10-01	1.651175430515143*10-02	1.422375475496879*10-01
3	2.531016370279627*10-01	1.651569879668941*10-01	7.192637792418951*10-03	1.358020131328365*10-01
4	2.485304313223083*10-01	1.648424626848376*10-01	7.274491417051896*10-03	1.348935695117291*10-01
5	2.486514047327587*10-01	1.642031673670021*10-01	7.446144449571263*10-03	1.347737736599676*10-01
6	2.487573309213280*10-01	1.640553080777843*10-01	7.478146222844833*10-03	1.347652664250729*10-01

Table 2. Relative analysis using the absolute of the system from case 1 in section 2 errors using Cubic spline for ϕ_2, order 6 with $h = 0.001$ and $\beta = 0.5$.

Itaration ϕ_1 $\beta = 0$	**Max**	**Min**	**Mean**	**Standard deviation**
1	2.999100269918120*10-05	0	1.000319768628892*10-05	8.955941383834370*10-06
2	6.045122901779387*10-05	1.361123624500776*10-04	7.962673327189603*10-05	5.231239348251850*10-05
3	6.094110200643987*10-05	1.372529182606386*10-04	8.022195569905751*10-05	5.275362987557723*10-05
4	6.094436476480891*10-05	1.372590938816115*10-04	8.022512987293158*10-05	5.275606394232343*10-05
5	6.094438213642694*10-05	1.372591149998992*10-04	6.094438220151377*10-05	1.372591150207297*10-04
6	6.094438220151377*10-05	1.372591150207297*10-04	8.022514178569575*10-05	5.275607299978148*10-05
$\beta = 0.5$	**Max**	**Min**	**Mean**	**Standard deviation**
1	9.554140127388255*10-04	0	3.192401733291184*10-04	2.854884247604817*10-04
2	1.950654574967292*10-03	1.973370203929709*10-03	9.305487675347942*10-04	1.027200210532221*10-03
3	1.999545822922638*10-03	2.028237732421390*10-03	9.530691985628691*10-04	1.054970798410739*10-03
4	2.000437819588585*10-03	2.029156368895363*10-03	9.534226677168675*10-04	1.055457490287414*10-03
5	2.000451702498407*10-03	2.029163433289682*10-03	9.534253994344448*10-04	1.055462876442645*10-03
6	2.000451807812387*10-03	2.029163339793610*10-03	9.534253693742450*10-04	1.055462876925303*10-03
$\beta = 1.5$	**Max**	**Min**	**Mean**	**Standard deviation**
1	7.117055550131812*10-03	0	2.396337505273268*10-03	2.132417634185868*10-03
2	1.451107207941954*10-02	8.975515353563501*10-03	2.890936818238309*10-03	6.279122401047496*10-03
3	1.508025523466627*10-02	9.434048055603106*10-03	3.017100560860625*10-03	6.573613814813055*10-03
4	1.509446459723457*10-02	9.447856118620146*10-03	3.020493556278070*10-03	6.582477690466286*10-03
5	1.509483665148442*10-02	9.447991083463669*10-03	3.020504718589314*10-03	6.582647734080373*10-03
6	1.509483768226449*10-02	9.447981772608582*10-03	3.020501653017330*10-03	6.582646436228047*10-03
$\beta = 2$	**Max**	**Min**	**Mean**	**Standard deviation**
1	2.852897473997029*10-02	0	9.797042556570549*10-03	8.606779182191720*10-03
2	5.606093597365652*10-02	2.518674344389077*10-02	3.660594846593367*10-03	2.251650518513378*10-02
3	5.828721918616819*10-02	2.700698175201588*10-02	3.949738851259722*10-03	2.380851821943954*10-02
4	5.834313446146561*10-02	2.708513719228928*10-02	3.960763058901483*10-03	2.385891976918472*10-02
5	5.834508037678704*10-02	2.708579602513328*10-02	3.960536903662511*10-03	2.386010818354099*10-02
6	5.834504147365005*10-02	2.708563837927341*10-02	3.960494530828423*10-03	2.386007086010117*10-02

Table 3. Relative analysis using the absolute of the system from case 2 in section 2 errors using Cubic spline for $\phi_2,\ \phi_3$ and quartic Spline for ϕ_1, order 10 with $h = 0.1$.

Iteration ϕ_1	**Max**	**Min**	**Mean**	**Standard deviation**
1	4.140323153797758*10-01	1.969164508960211*10-01	8.471832753662527*10-02	1.866709925466592*10-01
2	1.646437321023164*10-02	2.761713958274292*10-03	4.934252067374655*10-03	6.580309467679718*10-03
3	3.569040098025389*10-05	3.108147466956979*10-06	1.330945397769343*10-05	1.424814241558397*10-05
4	1.944629596239977*10-10	1.402566951469453*10-11	7.411576884125660*10-11	7.745988039655105*10-11
5	1.877912235476169*10-14	1.233995544636102*10-14	1.503403108782384*10-15	4.909248855427772*10-15
6	1.003710621615408*10-14	1.474992003536180*10-14	1.876752729653788*10-16	4.369492126399187*10-15
7	1.355263557628605*10-14	2.886579864025407*10-15	1.981709157221836*10-15	3.519001474086504*10-15
8	8.333611578592581*10-15	3.941291737419306*10-15	9.551905022021586*10-16	2.551970829616746*10-15
9	7.105427357601002*10-15	9.200105954842996*10-15	2.896311663728248*10-17	3.318008225719942*10-15
10	1.776356839400250*10-15	1.301389551677801*10-14	2.726478177412927*10-15	4.278176984703715*10-15
Iteration ϕ_2	**Max**	**Min**	**Mean**	**Standard deviation**
1	0	1.699100862490146*10+00	5.194991321520223*10-01	4.629584791517645*10-01
2	3.534287669754381*10-01	8.243832402559681*10-17	4.919716527263826*10-02	8.635072932353401*10-02
3	1.695048968385970*10-02	1.479933840734306*10-16	2.626803451771443*10-03	4.230822472978780*10-03
4	3.918366575755697*10-05	2.728199168348121*10-16	5.969135102780457*10-06	9.759062263719038*10-06
5	2.026073268584920*10-10	7.106167702764026*10-16	3.049880453727852*10-11	5.034685902215766*10-11
6	1.524821935383613*10-15	3.820401903446577*10-15	4.358520210225613*10-16	1.147287369910599*10-15
7	2.892218453459804*10-15	1.334002353026165*10-15	4.407787893580501*10-17	7.829204805641124*10-16
8	1.254205073131232*10-15	1.141448047192739*10-14	9.090213395369740*10-16	2.560382034494635*10-15
9	1.269991056762620*10-14	3.330669073875470*10-16	1.481606432772191*10-15	3.179277739169777*10-15
10	3.108624468950438*10-15	1.010476424756490*10-15	3.571196444610888*10-16	9.099979748442500*10-16
Iteration ϕ_3	**Max**	**Min**	**Mean**	**Standard deviation**
1	7.219921264765374*10-01	1.110223024625157*10-16	4.013603531874011*10-01	2.229086180922683*10-01
2	6.106226635438361*10-16	2.910002286722586*10-01	8.942508422566953*10-02	8.772370745294902*10-02
3	1.110223024625157*10-16	1.248000580210129*10-02	3.805868996084922*10-03	3.836755738549251*10-03
4	5.551115123125783*10-16	2.870431437673737*10-05	8.410235213287157*10-06	8.776138835071381*10-06
5	2.220446049250313*10-16	1.484175005117550*10-10	4.262364867419829*10-11	4.521598968552813*10-11
6	2.442490654175344*10-15	8.881784197001252*10-16	8.828063728067777*10-16	7.051180931204019*10-16
7	2.442490654175344*10-15	1.110223024625157*10-15	8.774343259134301*10-17	6.437123486414294*10-16
8	8.215650382226158*10-15	8.881784197001252*10-16	9.554409235105807*10-16	1.872317229695979*10-15
9	2.220446049250313*10-16	1.199040866595169*10-14	2.393023055699103*10-15	3.305756100012035*10-15
10	1.332267629550188*10-15	3.108624468950438*10-15	4.513638566848041*10-16	8.215400107624890*10-16

Table 4. Relative analysis using the absolute of the system from case 2 in section 2 errors using Cubic spline for $\phi_2,\ \phi_3$ and quatic Spline for ϕ_1, order 10 with $h = 0.01$.

Iteration ϕ_1	**Max**	**Min**	**Mean**	**Standard deviation**
1	4.126943918321302*10-01	1.971827793222096*10-01	8.631761678669202*10-02	1.856507288083475*10-01
2	1.636775638715060*10-02	2.787352535466470*10-03	5.015175493261235*10-03	6.468898941104339*10-03
3	3.498587524859709*10-05	3.087509751420470*10-06	1.342770899690993*10-05	1.375178714712246*10-05
4	1.872784080592105*10-10	1.362554513661962*10-11	7.352673825277870*10-11	7.329322474594301*10-11
5	2.415320547732858*10-14	5.501805608321142*10-14	5.328052887034594*10-15	1.049301971370877*10-14
6	3.596819023177211*10-14	2.731322112925483*10-14	3.399581098265174*10-15	7.620945938106923*10-15
7	3.758842195833445*10-14	1.779826286352204*10-14	3.546296356062325*10-15	7.947321318792020*10-15
8	3.757356838857140*10-14	3.337803124170424*10-14	2.018081307130694*10-15	7.380446709788958*10-15
9	2.615095640035037*10-14	3.695307948525794*10-14	2.168399665799499*10-15	1.125958003348733*10-14
10	3.272902782125442*10-14	2.598295927372374*10-14	1.409180577963838*10-15	7.754048816277836*10-15
Iteration ϕ_2	**Max**	**Min**	**Mean**	**Standard deviation**
1	2.575717417130363*10-14	1.694595310938617*10+00	5.065549177290124*10-01	4.326974355119805*10-01
2	3.516311583644137*10-01	8.312056521678554*10-16	4.528865125076022*10-02	7.630746931759723*10-02
3	1.677458990388520*10-02	4.250110964704060*10-16	2.433717187919441*10-03	3.756389209057476*10-03
4	3.842561440208905*10-05	3.430003627529208*10-16	5.468567147990468*10-06	8.576003345418822*10-06
5	1.951602274948434*10-10	6.227455267005897*10-16	2.740618659538620*10-11	4.341436412756515*10-11
6	2.466776782839020*10-15	4.700146521985360*10-14	3.039077082884590*10-15	6.634945498403793*10-15
7	3.763048900262689*10-14	6.150462084075770*10-15	1.487579768937693*10-15	4.485704910535061*10-15
8	4.185887747532036*10-14	9.462049199715494*10-15	1.465920423792281*10-15	4.720152674578898*10-15
9	2.407796184655808*10-15	3.883026712669935*10-14	1.919782723558007*10-15	6.046159140867161*10-15
10	3.592395478313470*10-14	2.431825838029339*10-15	2.405517435413479*10-15	5.186746863380768*10-15
Iteration ϕ_3	**Max**	**Min**	**Mean**	**Standard deviation**
1	7.202502690516377*10-01	4.440892098500626*10-15	4.019177575952539*10-01	2.131523086046362*10-01
2	2.886579864025407*10-15	2.890500627895294*10-01	8.727478099333183*10-02	8.275102505737479*10-02
3	2.220446049250313*10-16	1.233840452045243*10-02	3.693194654505104*10-03	3.609967467191330*10-03
4	4.440892098500626*10-16	2.812304761823370*10-05	8.068521630334169*10-06	8.169797993253620*10-06
5	2.220446049250313*10-16	1.428350770993347*10-10	4.011633496668522*10-11	4.131005118016108*10-11
6	4.218847493575595*10-14	8.881784197001252*10-16	3.912765333681046*10-15	6.678087569955814*10-15
7	5.773159728050814*10-15	2.953193245502916*10-14	3.268018757508444*10-15	3.855140197084554*10-15
8	1.998401444325282*10-15	2.997602166487923*10-14	2.886886754474230*10-15	5.079575615716583*10-15
9	3.441691376337985*10-14	8.881784197001252*10-16	6.951843816367427*10-15	4.964556177873176*10-15
10	4.440892098500626*10-16	2.842170943040401*10-14	6.201606169812272*10-15	3.641742804559507*10-15

Table 5. Relative analysis using the absolute of the system from case 2 in section 2 errors using Cubic spline for $\phi_2,\ \phi_3$ and quartic Spline for ϕ_1, order 10 with $h = 0.001$.

Iteration ϕ_1	**Max**	**Min**	**Mean**	**Standard deviation**
1	4.126924770553270*10-01	1.972230796081391*10-01	8.656711756616264*10-02	1.855860455090289*10-01
2	1.636654844931845*10-02	2.787328268244682*10-03	5.029543920588495*10-03	6.462504866515269*10-03
3	3.497870126867308*10-05	3.086915799421774*10-06	1.346515844984876*10-05	1.372901422728667*10-05
4	1.872133836844370*10-10	1.362243651215067*10-11	7.371958323578844*10-11	7.315468551819396*10-11
5	1.597812458514411*10-13	1.077005882209586*10-13	6.443878828729815*10-15	2.084217592264253*10-14
6	8.278734685632227*10-14	1.688669684770869*10-13	9.738923115633085*10-15	2.669865606272009*10-14
7	1.587740355857292*10-13	1.612958898389305*10-13	3.704784837942029*10-15	2.635115657050993*10-14
8	9.294821856631330*10-14	1.828255276526856*10-13	5.477020064468861*10-15	2.344983705097809*10-14
9	1.786840803357642*10-13	8.454955485737159*10-14	1.301588854166074*10-14	3.192999640504145*10-14
10	1.846971468995318*10-13	1.365043874376054*10-13	1.246987251558913*10-15	2.370631755261380*10-14
Iteration ϕ_2	**Max**	**Min**	**Mean**	**Standard deviation**
1	7.016609515630989*10-14	1.694549981879830*10+00	5.055021144104167*10-01	4.297710840920775*10-01
2	3.516130326723234*10-01	3.486407348591355*10-15	4.489625820478437*10-02	7.534531603266352*10-02
3	1.677281440250943*10-02	3.001299380603386*10-15	2.415680362036380*10-03	3.713099762871405*10-03
4	3.841798686720296*10-05	6.181396818929503*10-15	5.426417569427605*10-06	8.475199458148121*10-06
5	1.951752960787599*10-10	6.593683932187844*10-15	2.718548545067617*10-11	4.289161216120518*10-11
6	1.483647189479131*10-13	8.207583918062866*10-14	5.519120218485943*10-15	1.143011774360030*10-14
7	1.060798938554787*10-13	1.088911946722781*10-13	1.569010829955317*10-15	9.743030653722699*10-15
8	1.266048897663463*10-13	7.897458728645113*10-14	2.532504707191075*10-15	1.174429774286837*10-14
9	7.698485945950573*10-14	1.362327134782348*10-13	3.656762826612299*10-15	1.302651850268060*10-14
10	9.581484911036497*10-14	1.003693655965421*10-13	1.927640160752340*10-15	1.035946346127103*10-14
Iteration ϕ_3	**Max**	**Min**	**Mean**	**Standard deviation**
1	7.202327260601936*10-01	1.432187701766452*10-14	4.020365359967633*10-01	2.122183948278585*10-01
2	5.384581669432009*10-15	2.890304480798656*10-01	8.709833026255363*10-02	8.230471026593310*10-02
3	9.992007221626409*10-16	1.233697735400097*10-02	3.685367529172710*10-03	3.591086122217240*10-03
4	6.217248937900877*10-15	2.811720510620397*10-05	8.048903148930080*10-06	8.124951389187111*10-06
5	5.218048215738236*10-15	1.428266394043476*10-10	4.001321419907555*10-11	4.106918163738754*10-11
6	4.973799150320701*10-14	1.125766146969909*10-13	1.308017601820230*10-15	1.082489938815868*10-14
7	8.371081605673680*10-14	6.394884621840902*10-14	6.539468937247546*10-15	9.380304435707813*10-15
8	4.440892098500626*10-14	9.947598300641403*10-14	7.512595997123063*10-16	1.104774997688174*10-14
9	1.105782132526656*10-13	4.951594689828198*10-14	2.931216135088762*10-15	1.495311959626038*10-14
10	7.460698725481052*10-14	7.949196856316121*10-14	7.263239522979649*10-16	9.596421015375272*10-15

5 Conclusions

This paper considers the scheme spline polynomial method for various degrees for solving second order differential equations and three-dimensional with nonlinear thin film flow, and Aluminum nanofluiod model for three dimensional hybrids was constructed over biaxial porous stretching/shrinking sheet with heat transfer. The quasi-linearization technique is used to convert the non-linear equations of the model to a system of linear equations and then solved by the collocation method based on the cubic and quantic spline function. The comparison of the present results with the numerical results previously reported shows the efficiency of the suggested technique. It is found that the new results have an excellent agreement with the exact solution for the second order and best approximation for the system and the early published works.

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