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The Restricted Detour Radial Graphs

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Abstract In this paper, the restricted detour radial graph was introduced. We obtained some results connecting restricted detour radial graph and graph of certain particular graphs. Also, the restricted detour radial graph of the sum of two graphs and one-vertex union of graphs are determined.

1 Introduction

Let $G = (V, E)$, be a finite undirected simple connected graph. For basic graph terminology and details, we refer to [[5], 7]. Distance is one of the basic concepts of graph theory. For two vertices u and v in a graph G, the **distance** (ordinary distance) from u to v is denoted by $d_G(u, v)$ or simply $d(u, v)$ and defined as the length of a shortest $u - v$ path in graph G. **Detour distance** $D(u, v)$ between two distinct vertices u and v in a connected graph G is the maximum lengths of $u - v$ paths in G [3, 4]. In 2012, Ali and MohammedSaleh[1] defined the **restricted detour distance** $D^*{}_G\,(u, v)$ between two vertices $u,\ v$ in a graph G as the length of the longest $u - v$ path P such that $< V\,(P) >= P$. A **chord** of a path P is an edge of G joining two non-adjacent vertices of P. Thus**,** $D^*{}_G\,(u, v)$ **is the length of the longest** $u-v$ path P having no chords. Such a $u-v$ path is called **u−v restricted detour**. In [13], the authors called $D^*{}_G(u, v)$ **detour monophonic distance** between vertices u and v. It is clear that $D^*{}_G(u, v) = 0$ if and only if $u = v$, and $D^*{}_G(u, v) = 1$ if and only if uv is an edge in G. A graph G is called **restricted detour** if $D^*{}_G(u, v) = d(u, v)$ for every pair u and v of vertices in G. It is clear that all trees, complete graphs, and complete bipartite graphs are restricted detour graphs. Some researchers studied the radial graph and detour radial graphs such as [2, 6, 8, 9, 10, 11, 12, 14]. In [9], Kathiresan and Marimuthu introduced the concept of radial graph $R(G)$ using $d(u, v)$ and proved characterization for $R(G)$. In [2], Avadayappan and Ganeshwari introduced a detour radial graph using $D(u, v)$, length of the longest $u - v$ path of G. The **restricted detour radial graph** $R_{D^*}(G)$ has vertex set as in G and vertices u and v in $R_{D^*}\,(G)$ are adjacent in $R_{D^*}(G)$ if and only if the restricted detour distance $D^*{}_G(u, v)$ is equal to the **restricted detour radius** $r_{D^*}(G)$ of G. In this paper, we derive formulas for restricted detour radial graphs of some particular graphs.

2 Some Basic Results

Theorem 2.1 $R_{D^*}\,(G) = G$ if and only if $\triangle\,(G) = p - 1$, where $p \geq 2$.
Proof. First, assume that $\triangle\,(G) = p - 1$. Let v be a vertex of G of degree $p - 1$. Then $D^*{}_G\,(u, v) = 1$ for every vertex $u\ (\neq v)$ of G. Thus $e_{D^*}\,(v) = 1$ and so $r_{D^*}\,(G) = 1$. For any pair of adjacent vertex $(x, y)\,, \quad x \neq y$, of G, $D^*{}_G\,(u, v) = 1 = r_{D^*}\,(G)$. Thus, $xy \in E(R_{D^*}\,(G))$. If $x',\ y'$ are nonadjacent in G then $D^*{}_G\,(x', y') \geq 2$ and so $D^*{}_G\,(x', y') > r_{D^*}\,(G)$ which implies that $x'y' \notin E(R_{D^*}\,(G))$. Therefore $E\,(R_{D^*}\,(G)) = E(G)$ and so $R_{D^*}\,(G) = G$.

For the proof of converse, let $R_{D^*}\,(G) = G$, we shall prove that $\triangle\,(G) = p - 1$. Let $uv \in E\,(G)$, then $uv \in E\,(R_{D^*}\,(G))$. Thus, $1 = D^*{}_G\,(u, v) = r_{D^*}\,(G)$. Therefore, G contains a vertex, say w such that $e_{D^*}\,(w) = r_{D^*}\,(G) = 1$. Thus, w is adjacent to every other vertex of G, that is $deg_G\ w = p - 1$. Hence the proof is completed.■

Definition 2.2 A connected graph G is called **self-restricted detour radial graph** if and only if $R_{D^*}\,(G) = G$.

Corollary 2.3 Every complete graph K_p , every wheel graph W_p, $p \geq 5$, every star graph $K_{1,p-1}$ and every fan-graph $F_n,\ n \geq 2$ is self-restricted detour radial graphs.

Remarks 2.4 We deduce from the proof of Theorem 2.1 that:

(i) If $r_{D^*}(G) \geq 2$ then G is not self-restricted detour radial graph, that is $R_{D^*}(G) \neq G$.

(ii) If $r_{D^*}(G) \geq 2$ then $E(G) \cap E(R_{D^*}(G)) = \emptyset$, that is $R_{D^*}(G) \subseteq \overline{G}$.

We may have $R_{D^*}(G)$ which is isomorphic to G, as for cycle graphs of odd orders.

Proposition 2.5 Let C_{2n+1} be a cycle graph of order $p = 2n+1,\ \ n \geq 2$. Then $R_{D^*}(C_{2n+1}) \cong C_{2n+1}$.

Proof. Let $C_{2n+1} = (u_1,\ u_2,\ \ldots,\ u_{2n+1}, u_1)$, the $D^*(u_i, u_{i+2}) = 2n-1$, for $i = 1,\ 2,\ \ldots,\ 2n+1$, with $u_{2n+2} \equiv u_1$ and $u_{2n+3} \equiv u_2$. Therefore $e_{D^*}(u) = 2n-1$ for every $u \in V(C_{2n+1})$, and so $r_{D^*}(C_{2n+1}) = 2n-1$. Hence, in $R_{D^*}(C_{2n+1})$, the pair of vertices u_i, u_{i+2}, are adjacent, for $i = 1,\ 2,\ \ldots,\ 2n+1$. For any other pairs of vertices in C_{2n+1}, that is $u_i,\ u_j$ for $|i-j| > 2$, $D^*(u_i,\ u_j) < 2n-1$ and so $u_i u'_j \notin E(R_{D^*}(C_{2n+1}))$. Thus $R_{D^*}(C_{2n+1})$ is the cycle graph $(u_1,\ u_3,\ u_5 \ldots,\ u_{2n+1},\ u_2,\ u_4,\ u_6, \ldots,\ u_{2n},\ u_1) = C'_{2n+1}$. It is clear that $C'_{2n+1} \cong C_{2n+1}$.■

The following examples illustrate Proposition 2.5.

Example 2.6.

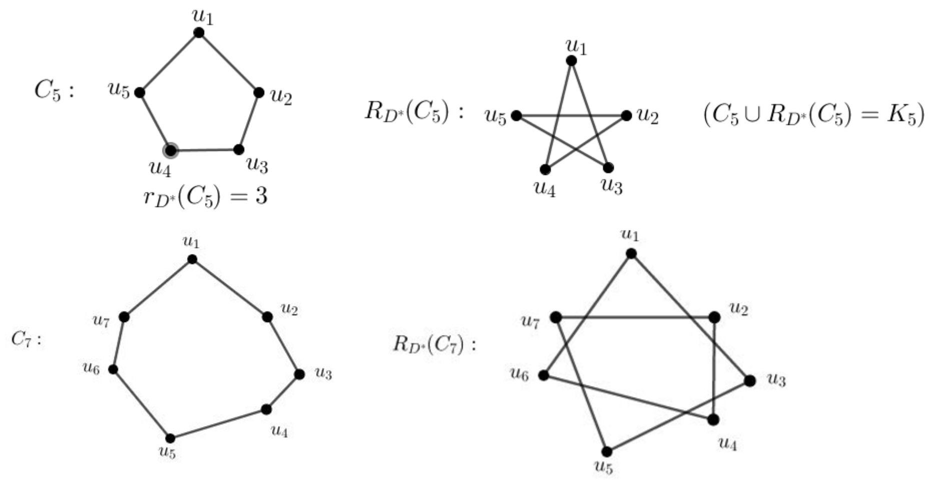

Figure 2.1

Now, we consider cycle graphs of even orders. **Proposition 2.7** Let C_{2n} be a cycle graph of order $2n,\ \ n \geq 3$, then $R_{D^*}(C_{2n}) = C_n \cup C'_n$, in which $V(C_n) \cap V(C'_n) = \emptyset$.

Proof. As in the proof of Proposition 2.5, $r_{D^*}(C_{2n}) = 2n-2$, and $D^*(u_i, u_{i+2}) = 2n-2,\ \ i = 1, 2,\ \ldots,\ 2n$ with $u_{2n+1} = u_1$ and $u_{2n+2} = u_2$.

It is clear that $D^*(u_i, u_j) < 2n-2$, for $|j-i| > 2$; thus $u_i u_j \notin E(R_{D^*}(C_{2n}))$. Hence $E(R_{D^*}(C_{2n})) = \{u_i u_{i+2}\ ;; i = 1, 2,\ \ldots,\ 2n\}$.

Therefore $R_{D^*}(C_{2n}) = (u_1, u_3, u_5,\ \ldots,\ u_{2n-1},\ u_1) \cup (u_2, u_4,\ u_6,\ \ldots,\ u_{2n}, u_2) = C_n \cup C'_n$.■

Example 2.8 Consider $C_8 = (u_1, u_2,\ \ldots,\ u_8,\ u_1)$. Then

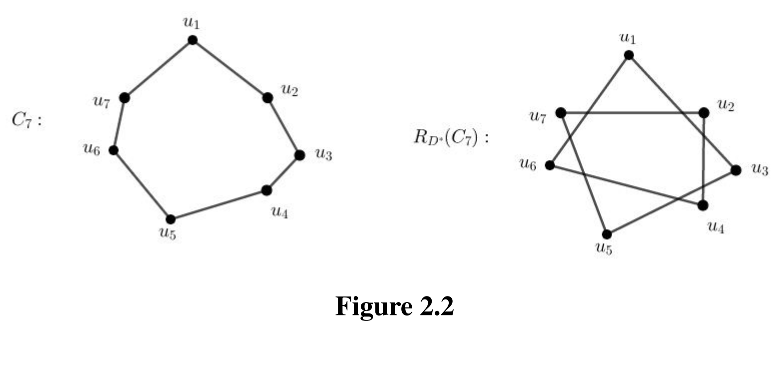

Figure 2.2

Proposition 2.9 Let $K_{m,n}$ be a complete bipartite graph with $m, n \geq 2$, then $R_{D^*}(K_{m,n}) = K_m \cup K_n = \overline{K_{m,n}}$, $V(K_m) \cap V(K_n) = \emptyset$.
Proof. Let $V(K_{m,n}) = V \cup U$, $V = \{v_1, v_2, \ldots, v_m\}$, $U = \{u_1, u_2, \ldots, u_n\}$, $D^*(x, y) = 2$ for $x, y \in V$ and $x, y \in U$.
Thus $e_{D^*}(v) = e_{D^*}(u) = 2$, for $v \in V$ and $u \in U$. Therefore $r_{D^*}(K_{m,n}) = 2$. It is clear that $D^*(u, v) = 1$ and so $uv \notin E(R_{D^*}(K_{m,n}))$. Hence, in $R_{D^*}(K_{m,n})$ every two vertices of V, and also every two vertices of U are adjacent. Therefore, $R_{D^*}(K_{m,n})$ consists of two vertex disjoint complete graphs K_m and K_n. $R_{D^*}(K_{m,n}) = \overline{K_{m,n}}$. Hence, the proof is completed.■
Corollary 2.10 For a complete $t-$partite graph $K_{n_1,n_2,\ldots,n_t}$, $t \geq 2$, we have $R_{D^*}(K_{n_1,n_2,\ldots,n_t}) = \overline{K_{n_1,n_2,\ldots,n_t}} = K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_t}$. ■
Proposition 2.11 If $R_{D^*}(G) = K_p$, where G is a connected graph of order $p \geq 2$, then $G = K_p$.
Proof. Let u, v be any two distinct vertices of G, then u, v are in K_p, and $uv \in E(K_p)$. But, by definition of restricted detour radial graph, $D_G^*(u, v) = r_{D^*}(G) = 1$ if $uv \in E(G)$. Since G consists of at least one edge, say xy, then $D_G^*(x, y) = 1$ and so $r_{D^*}(G) = 1$. Therefore, every two vertices, u, v of G are adjacent, implies that $G = K_p$.■
Corollary 2.12 $R_{D^*}(G) = K_p$ if and only if $G = K_p$.
Proof. The proof follows from Theorem 2.1 and Proposition 2.11. ■
Proposition 2.13 Let P_p be a path-graph of order $p \geq 4$, then

$$R_{D^*}(P_p) = \begin{cases} P_3 \cup (n-1) K_2, & \text{if } p = 2n+1, \quad n \geq 2, \\ nK_2, & \text{if } p = 2n, \quad n \geq 2. \end{cases}$$

Proof.
(1) $P_{2n+1} = (u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{2n+1})$. It is clear that $r_{D^*}(P_{2n+1}) = n$, and u_{n+1} is the center of P_{2n+1}. One may check that $D^*(x, y) = n$ for every pair (x, y) of vertices in $S = \{(u_i, u_{i+n}) : i = 1, 2, \ldots, n+1\}$; and $D^*(x, y) \neq n$ for $(x, y) \notin S$. Thus $E(R_{D^*}(P_{2n+1})) = \{u_i u_{i+n}; i = 1, 2, \ldots, n+1\}$. Since $V(R_{D^*}(P_{2n+1})) = V(P_{2n+1})$, then $R_{D^*}(P_{2n+1}) = P_3 \cup (n-1)K_2$.
(2) $P_{2n} = (v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n})$. It is clear that $r_{D^*}(P_{2n}) = n$, and u_{n+1} and u_n are the only centers of P_{2n}. One may see $D^*(x', y') = n$ for every pair (x', y') of vertices in $S' = \{(v_i, v_{i+n}) : i = 1, 2, \ldots, n\}$; and $D^*(x', y') \neq n$ for $(x', y') \notin S'$. Thus, $E(R_{D^*}(P_{2n})) = \{v_i v_{i+n};; i = 1, 2, \ldots, n\}$, and so $R_{D^*}(P_{2n}) = nK_2$.■
Definition 2.14 A graph H is called **restricted detour radial** (RDR) graph if there exists a connected graph G such that $R_{D^*}(G) = H$.
Let F^* be the set of all RDR graphs. From the results above, every graph G of order p and $\triangle(G) = p-1$ belongs to F^* and K_p, $K_{p_1} \cup K_{p_2}$, $\overline{K_{t_1,t_2,\ldots,t_m}}$, C_{2n+1}, $C_n \cap C'_n$, mK_2, $P_3 \cup nK_2$, S_p, W_p, $F_n \in F^*$.
We shall find other graphs of F^*.
Problem 2.15. Let T be a tree of order $p \geq 4$ and $T \neq K_{1,p-1}$, then $R_{D^*}(T)$ is not a tree.
Observation 2.16. Let T be any tree of order $p \geq 2$, then $R_{D^*}(T)$ contains no isolated vertex, that is $\delta(R_{D^*}(T)) \geq 1$.
Proof. Let v be any vertex of T and u be a center of T, and $rad(T) = n$. Then, there is one $u - v$ of length $m \geq 1$. Therefore, $e(v) = n + m$ which implies that there is a vertex v' such that $d(v, v') = n$. Thus, $vv' \in E(R_{D^*}(T))$. ■

Let $S_{m,n}$ be a double-star, $m, n \geq 2$, of order $p = 2 + m + n$ (see the Fig. 2.3)

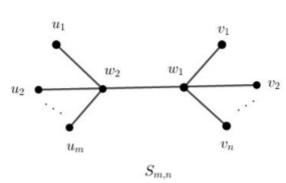

Figure 2.3

Proposition 2.17. Let $S_{m,n}$ be a double-star, then $R_{D^*}(S_{m,n}) = K_{m+1} \cup K_{n+1}$, in which $V(K_{m+1}) = \{w_1, u_1, u_2, \ldots, u_m\}$ and $V(K_{n+1}) = \{w_2, v_1, v_2, \ldots, v_n\}$.
Proof. It is clear that $e(w_1) = e(w_2) = 2$, and $rad(S_{m,n}) = 2$.

$\therefore$ $d(w_1, u_i) = d(w_2, v_j) = d(u_k, u_h) = d(v_r, v_s) = 2$ for $1 \le i \le m$, $1 \le j \le n$, $k \ne h \in \{1, 2, \ldots, m\}$, $r \ne s \in \{1, 2, \ldots, n\}$. Moreover, $d(u_i, v_j) = 3$. Thus, in $R_{D^*}(S_{m,n})$ every two vertices in $\{w_1, u_1, u_2, \ldots, u_m\}$ are adjacent and every two vertices in $\{w_2, v_1, v_2, \ldots, v_n\}$ are adjacent. But (u_i, v_j) are nonadjacent, for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Hence $R_{D^*}(S_{m,n}) = K_{m+1} \cup K_{n+1}$. ■

Proposition 2.18. Let $S_m^{(2)}$ be a star of m rays, each ray of two edges, shown in Fig. 2.4:

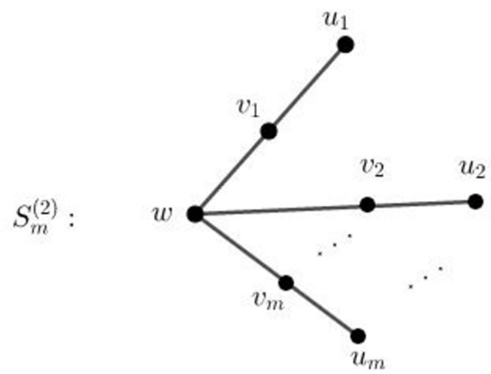

Figure 2.4

Then $R_{D^*}\left(S_m^{(2)}\right) = K_m \cup K_{1,m}$ in which $(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(K_{1,m}) = \{w, u_1, u_2, \ldots, u_m\}$.

Proof. The proof follows from the fact: $e(w) = 2 = rad\left(S_m^{(2)}\right)$, $d(v_i, u_j) = 2$, $d(w, u_j) = 2$, $d(u_i, u_j) = 4$, $d(u_i, v_j) = 3$, $i \ne j$, $i, j \in \{1, 2, \ldots, m\}$.■

Proposition 2.19. Let $S_m^{(3)}$ be a star of m rays, each ray of 3 edges, then $R_{D^*}\left(S_m^{(3)}\right) = (K_{m,m} - mK_2) \cup K_{1,m}$.

Proof. The proof is like that of Proposition 2.18.■

3 On the RDR of the Sum of Two Graphs

Let G_1 and G_2 be the connected graphs of order p_1 and p_2 respectively, and denote $G = G_1 + G_2$. If $\triangle(G_1) = p_1 - 1$ or $\triangle(G_2) = p_2 - 1$, then $\triangle(G) = p_1 + p_2 - 1$; and so $R_{D^*}(G_1 + G_2) = G_1 + G_2$. From now on, we assume $\triangle(G_i) < p_i - 1$, for $i = 1$ and 2. Thus, it is assumed that $r_{D^*}(G_i) \ge 2$, $i = 1, 2$. One can easily check that:

Theorem 3.1 $r_{D^*}(G_1 + G_2) = \min\{r_{D^*}(G_1), r_{D^*}(G_2)\}$.

Proof. Let P be a restricted detour between v_i and v_j in $G_1 + G_2$. If P contains a vertex u of G_2, then uv_i and uv_j are chords of P, a contradiction. Thus there is a $u_i - v_j$ restricted detour in G not containing vertices of G_2. Therefore, $D_G^*(u_i, v_j) = D_{G_1}^*(u_i, v_j)$ and $D_G^*(u_i, u_j) = D_{G_2}^*(u_i, u_j)$. This implies that:

$e_{D^*}(v_i : G) = e_{D^*}(v_i : G_1)$ for $i = 1, 2, \ldots, p_1$ and $e_{D^*}(u_j : G) = e_{D^*}(u_j : G_2)$ for $j = 1, 2, \ldots, p_2$.

Thus

$$\begin{aligned} r_{D^*}(G) &= \min\{\{e_{D^*}(v_i) : i = 1, 2, \ldots, p_1\} \cup \{e_{D^*}(u_j) : j = 1, 2, \ldots, p_2\}\} \\ &= \min\{r_{D^*}(G_1), r_{D^*}(G_2)\} .■ \end{aligned}$$

From our assumption on G_1 and G_2, and from this theorem we have:

Corollary 3.2 Let $\triangle(G_i) \le p_i - 1$, $i = 1, 2$, and $r_{D^*}(G_1) = r_{D^*}(G_2)$ then $R_{D^*}(G_1 + G_2) = R_{D^*}(G_1) \cup R_{D^*}(G_2)$.■

Example 3.3. Let C_n and C'_n be a two-cycle graph of order $n \ge 4$, then $R_{D^*}(C_n + C'_n) = R_{D^*}(C_n) \cup R_{D^*}(C'_n) = C_n^{''} \cup C_n^{'''}$.

Example 3.4. Consider C_4 and C_5. Then

$$r_{D^*}(C_4 + C_5) = \min\{2, 3\} = 2.$$

$$R_{D^*}(C_4+C_5) = R_{D^*}(C_4) \cup 5K_1 = 2K_2 \cup 5K_1.$$

Example 3.5. Consider C_5 and C_6. Then

$$r_{D^*}(C_5+C_6) = \min\{3,4\} = 3.$$

$$R_{D^*}(C_5+C_6) = C_4' \cup 3K_2.$$

Proposition 3.6. Let C_n and C_m be two distinct cycle graphs, such that $n < m$, then

$$R_{D^*}(C_n+C_m) = \begin{cases} C_n' \cup (n-2)K_2 \text{ if } & m = 2(n-2), \\ C_n' \cup mK_1, & \text{otherwise.} \end{cases}$$

Proof. Obvious.■

Let P_n and P_m be two distinct path graphs with order $n,\ m \geq 4$. Then, by Theorem 3.1,

$$r_{D^*}(P_n+P_m) = \min\{r_{D^*}(P_n),\ r_{D^*}(P_m)\} = n \text{ if } n \leq m.$$

Proposition 3.7. If $n = m \geq 4$ or $m - n + 1 \geq 5$, and m is odd, then

$$R_{D^*}(P_n+P_m) = R_{D^*}(P_n) \cup R_{D^*}(P_m).$$

Proof. It is clear that $rad(P_n) = rad(P_m)$. Thus, the results follow from Corollary 3.2.■

Now, we consider $P_n + P_m$ when $rad(P_n) < rad(P_m)$.

Theorem 3.8. Let $m > n+1$ or $m = n+1$ is even, then

$$R_{D^*}(P_n+P_m) = R_{D^*}(P_n) \cup rP_{k+1} \cup (d-r)P_k,$$

where $k = \left\lfloor \frac{m}{d} \right\rfloor$, $r = m - kd$, and d is the radius of P_n.

Proof. It is clear that $rad(P_n) < rad(P_m)$. Thus, by Theorem 3.1,

$$rad_{D^*}(P_n+P_m) = rad_{D^*}(P_n) = d.$$

Therefore, two vertices x, y of $V(P_n + P_m)$ are adjacent if and only if $x, y \in V(P_n)$ and $d(x,y) = d$, or $x, y \in V(P_n)$ and $d(x,y) = d$. Thus, $R_D(P_n)$ is a subgraph of $R_{D^*}(P_n+P_m)$. Since $d \geq 2$, then no vertex of P_n is adjacent to a vertex of P_m in $R_{D^*}(P_n+P_m)$. Now, let $P_n = v_1, v_2, \ldots, v_{m-1}, v_m$. In $R_{D^*}(P_n+P_m)$, denote H, each of the vertices $v_i,\ i = 1, 2, \ldots, d$ is of degree one and adjacent to v_{i+d}. Also, each of the vertices $v_j, j = m-d+1,\ \ m-d+2, \ldots, m$ is of degree one and adjacent to v_{j-d}, in H. All other vertices v_k of P_m are of degree 2 in H; v_k is adjacent to v_{k+d} and v_{k-d}. There are the only edges between vertices of P_m in H. It is clear that no vertex of P_n is adjacent to a vertex of P_m in H. Thus, the proof is completed.■

The following examples illustrate the proof of the theorem:

Example 3.9. Consider the path graph P_8 and P_{23}. It is clear that $rad_{D^*}(P_8+P_{23}) = rad_{D^*}(P_8) = 4 = d$. Then $k = \left\lfloor \frac{23}{4} \right\rfloor = 5,\ r = 23-4(5) = 3$. Thus, the edges of (P_8+P_{23}) other than those in $R_{D^*}(P_8)$ are v_1v_5, v_2v_6, v_3v_7, v_4v_8; $v_{16}v_{20}$, $v_{17}v_{21}$, $v_{18}v_{22}$, $v_{19}v_{23}$; v_5v_9, v_6v_{10}, v_7v_{11}, v_8v_{12}, v_9v_{13}, $v_{10}v_{14}$, $v_{11}v_{15}$, $v_{12}v_{16}$, $v_{13}v_{17}$, $v_{14}v_{18}$, $v_{15}v_{19}$. Moreover by Proposition 2.13, $R_{D^*}(P_8) = 4K_2$. Thus $R_{D^*}(P_8+P_{23})$ is given in Fig. 3.1.

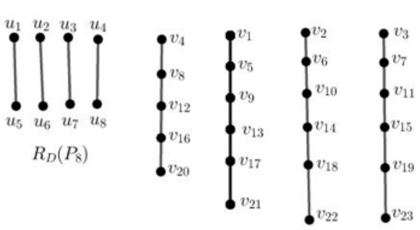

Figure 3.1

$R_{D^*}(P_8+P_{23}) = 4K_2 \cup P_5 \cup 3P_6.$

4 One-Vertex Union of Graphs

Definition 4.1. Let $H_1, H_2, \ldots, H_m$ be distinctly connected graphs, and let u_i be a vertex of H_i, $i = 1,\ 2,\ \ldots,\ m$. The **one-vertex union** of $H_1, H_2, \ldots, H_m$ is the graph, denoted $G(H_1, H_2, \ldots, H_m)$
Or simply G, obtained by identifying $u_1, u_2, \ldots, u_m$ to one vertex w. If $H_1, H_2, \ldots, H_m$ are m copies of a graph H of order p, we denote $G(H_1, H_2, \ldots, H_m) = G_p^{(m)}(H)$.
It is clear that: for each $i = 1,\ 2,\ \ldots,\ m$, $D_G^*(v, v') = D_{H_i}^*(v, v')$, for each pair v, v' of H_i; and for $i \neq j$,
$D_G^*(v, u) = D_{H_i}^*(v, w) + D_{H_i}^*(w, u)$, for each $v \in V(H_i)$, and $u \in V(H_j), u, v \neq 0$. Moreover:

$$e_{D^*}(w : G) = \max\{e_{D^*}(w : H_i) : i = 1,\ 2,\ \ldots,\ m\},$$

in which $e_{D^*}(w : G)$ is the **restricted detour eccentricity** of vertex w in the graph G; Similarly for $e_{D^*}(w : H_i)$.
Now, we give the following simple results:
Proposition 4.2. Let u_i be a restricted detour center of H_i for $i = 1,\ 2,\ \ldots,\ m$. Then

$$rad_{D^*}(G) = \max\{rad_{D^*}(H_i) : i = 1,\ 2,\ \ldots,\ m\}.$$

Proof. By definition of restricted detour center:

$$rad_{D^*}(H_i) = e_{D^*}(u_i : H_i), \quad i = 1,\ 2,\ \ldots,\ m.$$

Thus,

$$\begin{aligned} e_{D^*}(w : G) &= \max\{e_{D^*}(u_i : H_i) : i = 1,\ 2,\ \ldots,\ m\} \\ &= \max\{rad_{D^*}(H_i) : i = 1,\ 2,\ \ldots,\ m\}. \end{aligned}$$

It is clear that the restricted detour eccentricity of every vertex of G is not more than that of w in G. Thus, w is a restricted detour center of G, and so $rad_{D^*}(G) = e_{D^*}(w : G)$. Hence, the proof is completed.■
Corollary 4.3. Let u be a restricted detour center of H, and $H_1, H_2, \ldots, H_m$ be m copies of H. Then

$$rad_{D^*}\left(G_p^{(m)}(H)\right) = rad_{D^*}(H).■$$

We notice that it is difficult to find a simple formula for $R_{D^*}(G(H_1, H_2, \ldots, H_m))$ and $R_{D^*}\left(G_p^{(m)}(H)\right)$ if H_i is any connected graph. Therefore, we try to find restricted detour radial graphs for such one-vertex union graphs in which $H_1, H_2, \ldots, H_m$ are special graphs.
If $H_i = K_{p_i}$, $i = 1,\ 2,\ \ldots,\ m$, then for $m \geq 2$:

$$R_{D^*}(G(K_{p_1}, K_{p_2}, \ldots, K_{p_m})) = G(K_{p_1}, K_{p_2}, \ldots, K_{p_m}),$$

because $\triangle(G) = p_1 + p_2 + \cdots + p_m - m + 1$

$$= deg_G(w) = p(G) - 1.$$

Now, we determine $R_{D^*}(G(H_1, H_2, \ldots, H_m))$ in which $m = 2$ and H_1, H_2 are cycle graphs. Usually, $G(H_1, H_2)$ is denoted by $H_1 \circ H_2$ [2].
Let $C_{2n+1} = (v_1,\ v_2, \ldots,\ v_{2n+1},\ v_1)$ and $C'_{2n+1} = (u_1,\ u_2, \ldots,\ u_{2n+1},\ u_1)$ be two distinct cycle graphs of orders, $n \geq 1$.
Theorem 4.4. For $n \geq 3$

$$R_{D^*}(C_{2n+1} \circ C'_{2n+1}) = (C''_{2n+1} \circ C'''_{2n+1}) \bigcup C_4 \bigcup C'_4,$$

where,

$$C''_{2n+1} = (w,\ v_3,\ v_5,\ v_7, \ldots,\ v_{2n+1},\ v_2,\ v_4, \ldots,\ v_{2n-2},\ v_{2n}, w)$$
$$C'''_{2n+1} = (w,\ u_3,\ u_5,\ u_7, \ldots,\ u_{2n+1},\ u_2,\ u_4, \ldots,\ u_{2n-2},\ u_{2n}, w)$$

$C_4 = (v_2,\ u_4,\ v_{2n+1},\ u_{2n},\ v_2)$ and $C'_4 = (u_2,\ v_4,\ u_{2n+1},\ v_{2n},\ u_2)$.

Proof. In $C_{2n+1} \circ C'_{2n+1}$ we assume that vertex w is obtained by identifying vertices v_1 and u_1. By Proposition 4.2,

$$rad_{D^*}\left(C_{2n+1} \circ C'_{2n+1}\right) = rad_{D^*}\left(C_{2n+1}\right) = 2n - 1.$$

Let $x, y \in V(C_{2n+1} \circ C'_{2n+1})$, then $D^*_G(x, y) = 2n - 1$ if and only if :

(i) $x, y \in V(C_{2n+1})$ and $x, y \in E(R_{D^*}(C_{2n+1}))$,

(ii) $x, y \in V(C'_{2n+1})$ and $x, y \in E(R_{D^*}(C'_{2n+1}))$, (see Proposition 2.5)

(iii) $x = v_2$, and $y = u_4$ or u_{2n-1},

(iv) $x = v_{2n+1}$, and $y = u_4$ or u_{2n-1},

(v) $x = u_2$, and $y = v_4$ or v_{2n-1}, or

(vi) $x = u_{2n+1}$, and $y = v_4$ or v_{2n-1}.

Therefore, $R_{D^*}(C_{2n+1}) \cup R_{D^*}(C'_{2n+1}) = \left(C''_{2n+1} \circ C'''_{2n+1}\right)$ is a spanning subgraph of $R_{D^*}(C_{2n+1} \circ C'_{2n+1})$. Moreover, from (iii)-(vi), $C_4 \cup C'_4$ is a subgraph of $R_{D^*}(C_{2n+1} \circ C'_{2n+1})$. Hence, the proof.■
The following example illustrates Theorem 4.4.
Example 4.5. For $n = 2$, we have $rad_{D^*}(C_5 \circ C'_5) = 3$, and $R_{D^*}(C_5 \circ C'_5) = (C''_5 \circ C'''_5)$, as shown in Fig. 4.1.

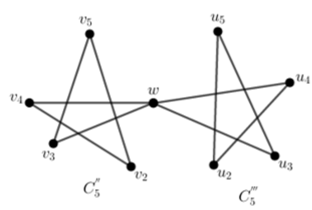

Figure 4.1 $C''_5 \circ C'''_5$.

For $n = 3$, we have $rad_{D^*}(C_7 \circ C'_7) = 5$, and $R_{D^*}(C_7 \circ C'_7) = (C''_7 \circ C'''_7) \cup C_4 \cup C'_4 = (C''_7 \circ C'''_7) \cup (v_2, u_4, v_7, u_6, v_2) \cup (u_2, v_4, u_7, v_6, u_2)$, as illustrated in Fig. 4.2.

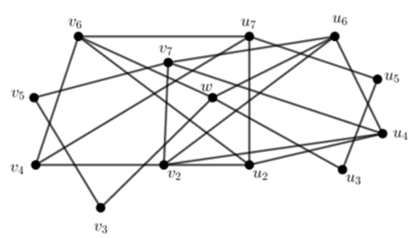

Figure 4.2 $R_{D^*}(C_7 \circ C'_7)$.

Now, we find $R_{D^*}(C_p \circ C'_p)$ for even p, say $p = 2n$, $n \geq 3$. For $n = 3$, we have $R_{D^*}(C_6 \circ C'_6)$ given in Fig. 4.3.

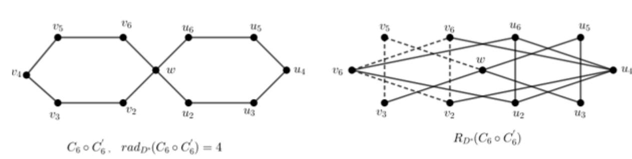

Figure 4.3 $R_{D^*}(C_7 \circ C'_7)$.

It is clear that $R_{D^*}(C_6 \circ C'_6) = R_{D^*}(C_6) \circ R_{D^*}(C'_6) \cup P_3 \cup P'_3$, in which $P_3 = (v_2, u_4, v_6)$ and $P'_3 = (u_2, v_4, u_6)$.

For $n \geq 6$, we have the following proposition:

Proposition 4.6. Let $n \geq 4$, and $C_{2n} = (v_1, v_2, \ldots, v_{2n}, v_1)$, $C'_{2n} = (u_1, u_2, \ldots, u_{2n}, u_1)$. Then

$$R_{D^*}(C_{2n} \circ C'_{2n}) = (C''_n \circ C^*_n) \cup C'''_n \cup C^{**}_n \cup C_4 \cup C'_4,$$

where

$$C''_n = (w, v_3, v_5, \ldots, v_{2n-1}, w), C'''_n = (v_2, v_4, v_6, \ldots, v_{2n}, v_2),$$

$$C^*_n = (w, u_3, u_5, \ldots, u_{2n-1}, w), C^{**}_n = (u_2, u_4, u_6, \ldots, u_{2n}, u_2),$$

$$C_4 = (v_2, u_4, v_{2n}, u_{2n-2}, v_2), C^*_4 = (u_2, v_4, u_{2n}, v_{2n-2}, u_2).$$

Proof. The proof is similar to that of Theorem 4.4.■

Proposition 4.7. Let $C_p = (v_1, v_2, \ldots, v_p)$ and $C'_p = (u_1, u_2, \ldots, u_{p-1})$ be distinct cycle graphs, $p \geq 7$ and $w = u_1 = v_1$. Then

$$R_{D^*}\left(C_p \circ C'_{p-1}\right) = R_{D^*}(C_p) \cup C_4 \cup C'_4 \cup S, \tag{4.1}$$

where
$C_4 = (v_2, u_4, v_{p-1}, u_{p-2}, v_2)$, $C'_4 = (v_3, u_2, v_{p-2}, u_p, v_3)$, and $S = (v_4, v_5, \ldots, v_{p-3})$ is a set of $(p-6)$ isolated vertices.

Proof. Using the procedure used in proving Theorem 4.4, we obtain (4.1).■

For $p = 4, 5, 6$ we have

$$R_{D^*}(C_4 \circ C'_3) = (v_2, u_2, v_3, u_4, v_2) \cup (wu_3, u_2u_4) = C''_4 \cup 2K_2.$$

$R_{D^*}(C_5 \circ C'_4) = (w, u_3, u_5, u_2, w) \cup (u_2, v_2, u_5) = C''_4 \cup 2K_2 = R_{D^*}(C_5) \cup P_3$, where $P_3 = (u_2, v_2, u_5)$.

$$R_{D^*}(C_6 \circ C'_5) = R_{D^*}(C_6) \cup (u_2, v_3, u_6, v_4, u_2) \cup (v_2, u_4, v_5).$$

Moreover, we illustrate Proposition 4.7 in the next example.

Example 4.8. For $p = 7$, we give $R_{D^*}(C_7 \circ C'_6)$ in Fig.4.4.

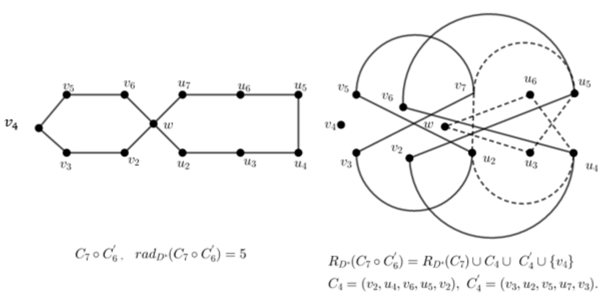

Figure 4.4

Example 4.9. For $p = 8$, we give $R_{D^*}(C_8 \circ C'_7)$ in Fig.4.5.

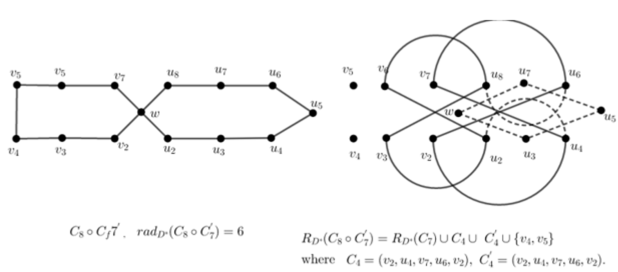

Figure 4.5

Proposition 4.10.

(1) For $p \geq 7$, $R_{D^*}(C_p \circ C_3') = R_{D^*}(C_p) \cup (v_2, u_4,\ v_3,\ u_{p-2},\ v_2)$,
(2) For $p \geq 9$, $R_{D^*}(C_p \circ C_4') = R_{D^*}(C_p) \cup (v_2, u_4,\ v_4,\ u_{p-2},\ v_2) \cup (u_5, v_3,\ u_{p-3})$,
in which $C_p = (u_1,\ u_2, \ldots, u_p,\ u_1)$, $C_3' = (v_1,\ v_2, v_3,\ v_1)$, $C_4' = (v_1,\ v_2, v_3,\ v_4,\ v_1)$, and $w = u_1 = v_1$.■
In Case (1), for $p = 5,\ 6$, we have:

$$R_{D^*}(C_5 \circ C_3') = R_{D^*}(C_5) \cup (v_2,\ v_3),$$

$$R_{D^*}(C_6 \circ C_3') = R_{D^*}(C_6) \cup (v_2, u_4,\ v_3).$$

In Case (2), for $p = 6,\ 7, 8$, we have:

$$R_{D^*}(C_6 \circ C_4') = R_{D^*}(C_6) \cup (v_2, u_4,\ \ u_4),$$

$$R_{D^*}(C_7 \circ C_4') = R_{D^*}(C_7) \cup (v_2, u_4,\ v_4,\ u_6,\ v_2) \cup \{v_3\},$$

$$R_{D^*}(C_8 \circ C_4') = R_{D^*}(C_8) \cup (v_2, u_4,\ v_4,\ u_6,\ v_2) \cup \{u_3 u_5\}.$$

Problem. We think it is possible to find $R_{D^*}\left(C_p \circ C_{p'}'\right)$, $p \geq p'$ for $p' = 5$ and 6, or $p - 2$ and $p - 6$.

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