Palestine Journal of Mathematics

Vol. 12(Special Issue I)(2023) , 1–4

On the sigma-conjugate maps

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Communicated by Rostam Kareem Saeed

MSC 2010 Classifications: 47H05, 39B22.
Keywords and phrases: Generalized convexity, Generalized Conjugat,· sigma-subdifferential.

Abstract In this note, we investigate the properties of the sigma-conjugate map. Also, we drive various mutual results regarding the sigma-subdifferential and sigma-conjugate. Indeed, we prove that some well-known properties regarding the Fenchel conjugate and ϵ-conjugate remain valid for the sigma-conjugate.

1 Introduction and preliminaries

Throughout this note, E is a Banach space and we will denote its topological dual by E^*. The evaluation of a functional $x^* \in E^*$ at a point $x \in E^*$ is written as $\langle \cdot, \cdot \rangle$.

In what follows, $f : E \to \mathbb{R} \cup \{+\infty\}$ is a map. The domain of f will be defined by dom $f = \{x \in E : f(x) < +\infty\}$. We say that f is proper if dom $f \neq \emptyset$. Moreover, we call a map f is convex if for each $x, y \in E$ and for any $\lambda \in [0, 1]$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a map. Its Fenchel subdifferential at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) = \{x^* \in E^* : \langle y - x, x^* \rangle \leq f(y) - f(x) \qquad \forall y \in E\} \tag{1.1}$$

and $\partial f(x) = \emptyset$ if $x \notin \operatorname{dom} f$.

We recall that from [10], for a mapping $f : E \to \mathbb{R} \cup \{+\infty\}$ its generalized directional derivative (in the sense of Clarke-Rockafellar) at x in a direction $z \in E$ is defined as follows:

$$f^{\uparrow}(x, z) = \sup_{\delta > 0} \limsup_{(y,\alpha) \xrightarrow{f} x, \lambda \searrow 0} \inf_{u \in B(z,\delta)} \frac{f(y + \lambda u) - \alpha}{\lambda}$$

where $(y, \alpha) \xrightarrow{f} x$ means that $y \to x, \alpha \to f(x)$ and $\alpha \geq f(y)$. We recall that the subdifferential of f at $x \in \operatorname{dom} f$ (in the sense of Clarke-Rockafellar) is defined in the following way:

$$\partial^{CR} f(x) = \left\{x^* \in E^* : \langle x^*, z \rangle \leq f^{\uparrow}(x, z) \quad \forall z \in E\right\}.$$

As in [7] we say that a map $f : E \to \mathbb{R} \cup \{+\infty\}$ is ε-convex if for all $a, b \in E$, and $\alpha \in]0, 1[$

$$f(\alpha a + (1-\alpha) b) \leq \alpha f(a) + (1-\alpha) f(b) + \alpha(1-\alpha)\varepsilon ||a - b||.$$

We also recall that:

Definition 1.1. [3].Given a map $f : E \to \mathbb{R} \cup \{+\infty\}$ and a function σ from E to $\mathbb{R}_+ \cup \{+\infty\}$, such that dom $f \subseteq \operatorname{dom} \sigma$. Then f is called *σ-convex* if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) + \lambda(1-\lambda) \min\{\sigma(x), \sigma(y)\} ||x - y|| \tag{1.2}$$

for every $x, y \in E$, and $\lambda \in [0, 1]$.

It should be noticed that from now on, to simplify the writing, we define $\bar{\sigma}(x, y) := \min\{\sigma(x), \sigma(y)\}$. Note that σ and $\bar{\sigma}$ are different maps because their domains are different, i.e., dom $\bar{\sigma} \subset X \times X$ and dom $\sigma \subset X$.

In a word,

$$\sigma\text{-convexity} \implies \epsilon\text{-convexity} \implies \text{convexity}.$$

In [8], various properties of ε-convex maps are presented. Some links between ϵ-subdifferential and ϵ-monotonicity were found in [7]. These concepts are generalized to σ-convexity and σ-monotonicity in [1, 2, 3, 4, 6]. In this note, we provide additional results regarding the concepts σ-convexity and σ-subdifferential.

2 Main results

Here, we study the concept of $\bar{\sigma}$-conjugate which is reduced to the notion of "ϵ-conjugate" [9] and conjugate if we choose $\bar{\sigma}(x,y) = \epsilon$ and $\bar{\sigma}(x,y) = 0$, respectively.

Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a σ-convex map and $y \in E$ be fixed. Then the function $f_y^* : E^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f_y^*(\bar{\sigma}, x^*) = \sup_{x \in E} \{\langle x^*, x\rangle - f(x) - \bar{\sigma}(x,y)\,||x-y||\}, \qquad \forall x^* \in E^* \tag{2.1}$$

is named the $\bar{\sigma}$-conjugate of f, where $\bar{\sigma}(x,y) = \min\{\sigma(x), \sigma(y)\}$. In the whole of this section, we will use this notation regarding $\bar{\sigma}$.

Note that when $\bar{\sigma}(x,y) \equiv 0$ then $f_y^*(\bar{\sigma}, x^*)$ reduces to $f^*(x^*)$ and if $\bar{\sigma}(x,y) = \epsilon$,($\epsilon \in]0,+\infty[$) then $\bar{\sigma}$-conjugacy coincides with ϵ-conjugacy [9].

As for the convex case, the map $f_y^{**} : E \to \mathbb{R} \cup \{\pm\infty\}$ is defined as follows:

$$f_y^{**}(\bar{\sigma}, x) = \sup_{x^* \in E^*} \{\langle x^*, x\rangle - f_y^*(\bar{\sigma}, x^*)\}, \qquad \forall x \in E$$

and it is called the $\bar{\sigma}$-biconjugate of f.

From (2.1), we obtain a generalized Fenchel inequality as follows:

$$f_y^*(\bar{\sigma}, x^*) + f(x) + \bar{\sigma}(x)\,||x-y|| \geq \langle x^*, x\rangle, \qquad \forall x \in E, \forall x^* \in E^*. \tag{2.2}$$

The next proposition represents some properties regarding $\bar{\sigma}$-conjugate maps.

Proposition 2.1. *Suppose that $f, g : E \to \mathbb{R} \cup \{+\infty\}$ are σ-convex maps. Then we have*
(i) $f_y^*(\bar{\sigma}, x^*) \leq f^*(x^*)$, and therefore if $f^*(\cdot)$ is proper, then $f_y^*(\bar{\sigma}, \cdot)$ is too;
(ii) $g_y^*(\bar{\sigma}, x^*) \leq f_y^*(\bar{\sigma}, x^*)$ whenever $f \leq g$;
(iii) $f_y^*(\bar{\sigma}', x^*) \leq f_y^*(\bar{\sigma}, x^*)$ when $\bar{\sigma} \leq \bar{\sigma}'$;
(iv) If $k(x) := f(x) + M$ for some real M, then $k_y^*(\bar{\sigma}, x^*) = f_y^*(\bar{\sigma}, x^*) - M$;
(v) When M is a positive real number and $k(x) := Mf(x)$, then $h_y^*(\bar{\sigma}, x^*) = Mf_y^*\left(\frac{\bar{\sigma}}{M}, \frac{x^*}{M}\right)$;
(vi) Assume that M is a positive real number and σ is positively homogeneous of degree l. If $k(x) := f(Mx)$, then $k_y^*(\bar{\sigma}, x^*) = f_{M^l y}^*\left(\frac{\bar{\sigma}}{M^{2l}}, \frac{x^*}{M^l}\right)$;
(vii) Assume that $\operatorname{dom}\sigma = E$ and σ is bounded from above. Then f_y^* is Lipschitz with respect to y with a Lipschitz rank $l = \sup_{x \in E} \sigma(x)$;
(viii)The σ-conjugation map is convex, that is for each $\alpha \in]0,1[$ we have

$$(\alpha f + (1-\alpha)g)_y^*(\bar{\sigma}, x^*) \leq \alpha f_y^*(\bar{\sigma}, x^*) + (1-\alpha)\,g_y^*(\bar{\sigma}, x^*);$$

Proof. We note that the proofs of parts (i), (ii), (iii), and (iv) are easy consequences of the definition.

For part (v), we have

$$\begin{aligned} k_y^*(\bar{\sigma}, x^*) &= \sup_{x \in E} \{\langle x^*, x\rangle - Mf(x) - \bar{\sigma}(x,y)\,||x-y||\} \\ &= M \sup_{x \in E} \left\{\left\langle \frac{x^*}{M}, x\right\rangle - f(x) - \frac{1}{M}\bar{\sigma}(x,y)\,||x-y||\right\} = Mf_y^*\left(\frac{\bar{\sigma}}{M}, \frac{x^*}{M}\right). \end{aligned}$$

To prove the part (vi), by assumptions $M > 0$, and σ is positively homogeneous of degree l, then $\bar{\sigma}$ is positively homogeneous of rank l with respect to (x,y). Now by setting $z = Mx, u = y$, we infer the desired formula. For the proof of part (vii), note that σ is bounded above. Thus $\bar{\sigma}$ has the same bound. So for $y_1, y_2 \in E$ and $x^* \in E^*$, we get

$$\begin{aligned} f^*_{y_1}(\bar{\sigma}, x^*) &= \sup_{x\in E}\{\langle x^*, x\rangle - f(x) - \bar{\sigma}(x, y_1)\|x - y_1\|\} \\ &\le \sup_{x\in E}\{\langle x^*, x\rangle - f(x) - \bar{\sigma}(x_1)\|x - y_2\| + \bar{\sigma}(x, y_2)\|y_2 - y_1\|\} \\ &\le \sup_{x\in E}\{\langle x^*, x\rangle - f(x) - \bar{\sigma}(x, y_2)\|x - y_2\|\} + \sup_{x\in E}\sigma(x)\|y_2 - y_1\| \\ &= f^*_{y_2}(\sigma, x^*) + l\|y_2 - y_1\|. \end{aligned}$$

If in the above inequality we change x to y, the desired statement will be obtained.

To show part (viii), let $\alpha \in [0, 1]$, then

$$\begin{aligned} (\alpha f + (1-\alpha) g)^*_y(\bar{\sigma}, x^*) &= \sup_{x\in E}\{\langle x^*, x\rangle - (\alpha f + (1-\alpha) g)(x) - \bar{\sigma}(x, y)\|x - y\|\} \\ &\le \alpha \sup_{x\in E}\{\langle x^*, x\rangle - f(x) - \bar{\sigma}(x, y)\|x - y\|\} \\ &+ (1-\alpha)\sup_{x\in E}\{\langle x^*, x\rangle - g(x) - \bar{\sigma}(x, y)\|x - y\|\} \\ &= \alpha f^*_y(\bar{\sigma}, x^*) + (1-\alpha) g^*_y(\bar{\sigma}, x^*). \end{aligned}$$

The proof is completed. □

In the next results, we generalize some results from [9].

Proposition 2.2. *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be σ-convex. Then for each $y^* \in \partial f(y)$ we have $\partial^\sigma f(x) \subset \Omega$, where*

$$\Omega := \{x^* \in E^* : \langle x^* - y^*, y - x\rangle \le \bar{\sigma}(x, y)\|x - y\|\}.$$

Proof. By assumptions $x^* \in \partial^\sigma f(x)$ and $y^* \in \partial f(y)$. Therefore by using the related definitions, we obtain

$$\begin{aligned} \langle x^*, x - y\rangle &\ge f(x) - f(y) + \bar{\sigma}(y, x)\|x - y\|, \\ \langle y^*, y - x\rangle &\ge f(y) - f(x). \end{aligned}$$

Now if we add the above inequalities, then we infer $x^* \in \Omega$ i.e., $\partial^\sigma f(x) \subset \Omega$. □

Proposition 2.3. *Let $l = \sup_{z\in E}\bar{\sigma}(z, y)$ and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lsc and σ-convex map. Then for each $y \in E$, the maps f^*_y and f^{**}_y are proper.*

Proof. Since f is proper and lsc map, therefore, according to [5] there is $x \in \operatorname{dom} f$ such that $\partial f^{CR}(x) \neq \emptyset$. By [2, Proposition 9], $\partial^{CR} f(x) \subset \partial^\sigma f(x)$, thus $\partial^\sigma f(x) \neq \emptyset$. So, we assume that $x^* \in \partial^\sigma f(x)$. Thus, for every $u \in E$ we have:

$$\begin{aligned} \langle x^*, u - x\rangle &\le f(u) - f(x) + \bar{\sigma}(u, x)\|x - u\| \\ &\le f(u) - f(x) + \bar{\sigma}(u, y)\|x - y\| + \bar{\sigma}(u, y)\|u - y\|. \end{aligned}$$

From the above inequalities, we conclude that $\langle x^*, u\rangle - f(u) - \bar{\sigma}(u, y)\|u - y\| \le \bar{\sigma}(u, y)\|x - y\| + \langle x^*, x\rangle - f(x)$. Now get the supremum over $u \in E$. This implies that

$$f^*_y(\sigma, x^*) \le l\|x - y\| + \langle x^*, x\rangle - f(x).$$

Since $l = \sup_{z\in E}\bar{\sigma}(z, y)$, consequently f^*_y is proper. Thus at least for one $x^* \in E$, we have $f^*_y(\bar{\sigma}, x^*) \in \mathbb{R}$. Using the definition of f^{**}_y we obtain $f^{**}_y(\bar{\sigma}, x) > -\infty$ for every $x \in E$. Applying Proposition 2.1(ix) implies that f^{**}_y is proper too. □

We call $f : E \to \mathbb{R} \cup \{+\infty\}$ strongly coercive, if $\lim_{\|x\|\to+\infty}\left(\frac{f(x)}{\|x\|}\right) = +\infty$.

Proposition 2.4. *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be minorized by an affine map and be a σ-convex and strongly coercive map. Then $\operatorname{dom} f^*_y(\bar{\sigma}, \cdot) = E^*$ and also f is bounded on every bounded set.*

Proof. By using [10, Lemma 3.6.1] we get that $f^*(x^*) < +\infty$ and also, it is bounded on any bounded set. Thus by Proposition 2.1(i) we obtain $f^*_y(\bar{\sigma}, x^*) < +\infty$. The proof is completed. □

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