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THE NEW INTEGRAL TRANSFORM "SUM TRANSFORM" AND ITS PROPERTIES

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Abstract In this study, a new integral transform called the SUM transform is introduced and some of its properties are studied. In addition, the application of the SUM integral transform in solving ordinary and partial differential equations is demonstrated with several examples. If the parameters of the SUM integral transform are appropriately and accurately substituted, it will reduce to some of the integral transforms that exist in the literature as special cases.

1 Introduction

Integral transform due to their applications in science and technology in areas such as automatic missile control, mechanical engineering, population growth, electric circuits, nuclear physics, pharmacokinetics, precision deflection of beams, cryptography, etc. are studied, extended, modified and generalized by many mathematicians such as N.H. Abel, J. Fourier, C.G.J. Jacobi and many others, see for example Kuffi et al., [19], Kuffi and Abbas [20], Liang et al., [24] and Monsour et al.,[28, 29, 30]. Recently, the following new type of the exponential kernel Laplace integral transform is studied by Gaur and Agarwal [9]:

$$L_{\beta}\{f(t)\}_{(s)} = \int_{t=0}^{\infty} f(t)\beta^{-st}dt, \tag{1.1}$$

where $\partial\ (>0)$ is the exponential order of the function $f(t)$, $\beta > 0$ and $Re(s) > \frac{\partial}{log(a)}$. Gaur and Agarwal in [10] studied application of transform in (1.1) and similar transform was also investigated by Saif et al., [33].
Fractional order integral transform was also provided by Gaur and Aarwal [11] as an extension of transform in (1.1) as follows:

$$L^{\alpha}_{\beta}\{f(t)\}_{(s)} = \int_{t=0}^{\infty} f(t)E_{\alpha}\left(-\left[stIn(\beta)\right]^{\alpha}\right)(dt)^{\alpha}, \tag{1.2}$$

where $\beta > 0$ and $s \in \mathbb{C}$ when it converges.
Follow similar methodology in (1.1) and (1.2) Duran [6] presented the following modified Sumudu transform:

$$S_a\{f(t)\}_{(s)} = \frac{1}{s}\int_{t=0}^{\infty} f(t)a^{-\frac{t}{s}}dt, \tag{1.3}$$

where $s \in (-n_1, n_2)\,, a \in (0, \infty) \setminus \{1\}$.

Definition 1.1. A function $f(t)$ is called exponential order $\partial\ (>0)$ as $t \longrightarrow \infty$ if there exists $L > 0$ such that (see, Kadhem and Hassan [15] and Kharrat and Toma [17])

$$|f(t)| < Le^{\partial t}, \quad L, \partial > 0,\ \forall t \geqslant 0. \tag{1.4}$$

Definition 1.2. A function $f(t)$ is said to be piecewise continuous on closed interval $[a, b]$, if it is defined and continuous on $[a, b]$ except for a finite number of points $p_1, p_2, p_3, \ldots, p_n$ at each of which left and right limits of $f(t)$ exist (see Samah et al., [34]).

2 A new integral transform

Following the order of the integral transforms above, the following integral transform is introduced

$$S_a\{f(t)\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{\infty} f(t)a^{-st}dt = G_a(s), \tag{2.1}$$

where $t \geq 0$, $r \in \mathbb{Z}$, $a \in (0,\infty)\setminus\{1\}$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$ and $f(t)$ are sectionally continuous and exponential order. The sufficient condition for the existence of SUM transform is given in the following theorem:

Theorem 2.1. *Suppose $f(t)$ is piecewise continuous in all finite interval $0 \leq t \leq t_0$ and and is of exponential order $\partial(>0)$ as $t \to \infty$, the $S_a\{f(t)\}_{(s)}$ exist for all $Re(s) > \frac{\partial}{log(a)}$.*

Proof. Suppose $t_0 > 0$, then (2.1) can rewritten as

$$S_a\{f(t)\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{t_0} f(t)e^{-stlog(a)}dt + \frac{1}{s^r}\int_{t=t_0}^{\infty} f(t)e^{-stlog(a)}dt. \tag{2.2}$$

Since $f(t)$ is piecewise continuous in the interval $0 \leq t \leq t_0$, the first integral of (2.2) exists. Similarly, since $f(t)$ is of exponential order ∂ as $t \to \infty$, using (1.4) $\lim_{x\to\infty} f(t)e^{-\partial t}$ is finite. Now

$$\begin{aligned}\left|\frac{1}{s^r}\int_{t=t_0}^{\infty} f(t)e^{-stlog(a)}dt\right| &\leq \frac{1}{s^r}\int_{t=t_0}^{\infty} \mid f(t)\mid e^{-stlog(a)}dt \\ &< \frac{1}{s^r}\int_{t=t_0}^{\infty} e^{-\{[slog(a)]-\partial\}t}dt \\ &= \frac{Le^{-\{[slog(a)]-\partial\}t_0}}{s^r\,\{[slog(a)]-\partial\}}, \quad if\ Re(s) > \frac{\partial}{log(a)}.\end{aligned}$$

Thus, $\frac{Le^{-\{[slog(a)]-\partial\}t_0}}{s^r\{[slog(a)]-\partial\}}$ can be made as small as we want by choosing t_0 as large enough for $Re(s) > \frac{\partial}{log(a)}$ and so the second integral in (2.2) exist and subsequently $S_a\{f(t)\}_{(s)}$ exist for all $Re(s) > \frac{\partial}{log(a)}$. The proof is complete. □

Remark 2.2. Many known integral transforms are obtained as special cases of the SUM transform, for example (1.3) are obtained if $r = 1$ and $s = 1/p$, other integral transforms are listed below:

(i) If $a = e$ then (2.1) becomes:

$$S_a\{f(t)\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{\infty} f(t)e^{-st}dt = G(s), \tag{2.3}$$

where $t \geq 0$, $r \in \mathbb{Z}$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is called Sadiq-Emad-Eman transform which is also known as SEE transform [31].

(ii) Setting $a = e$ and $r = 1$ then (2.1) reduces to

$$A\{f(t)\}_{(s)} = \frac{1}{s}\int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.4}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is known as Aboodh transform [2, 3].

(iii) Letting $a = e$ and $r = -1$ then (2.1) gives

$$M\{f(t)\}_{(s)} = s\int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.5}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is named Mohand transform [27].

(iv) Substitute $a = e$ and $r = 2$ then (2.1) reduce immediately to

$$ES\{f(t)\}_{(s)} = \frac{1}{s^2}\int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.6}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform refers to Emad-Sara transform [26].

(v) Putting $a = e$ and $r = -2$ then (2.1) leads to

$$M\{f(t)\}_{(s)} = s^2\int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.7}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is known to be Mahgoub transform [25].

(vi) On considering $a = e$ and $r = 3$ then (2.1) coincides precisely with

$$RU\{f(t)\}_{(s)} = \frac{1}{s^3}\int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.8}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is known as Gupta transform [13].

(vii) Swapping $a = e$ and $r = -3$ then (2.1) gives

$$R\{f(t)\}_{(s)} = s^3\int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.9}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is known to be Rohit transform [14].

(viii) Changing $a = e$ and $r = -5$ then (2.1) correspond to

$$D\{f(t)\}_{(s)} = s^5\int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.10}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is called Dinesh Verma transform (DVT) [35].

(ix) Replacing $a = e$, $r = 2$ and transformation $s = \frac{1}{p}$ then (2.1) yields

$$A\{f(t)\}_{(s)} = s^2\int_{t=0}^{\infty} f(t)e^{-\frac{t}{s}}dt, \tag{2.11}$$

where $t \geq 0$, and $s > 0$. This integral transform is called Anju transform [23].

(x) Interchanging $a = e$ and $r = 0$ then (2.1) becomes:

$$L\{f(t)\}_{(s)} = \int_{t=0}^{\infty} f(t)e^{-st}dt, \tag{2.12}$$

where $Re(s) > 0$. This integral transform is the well-known Laplace transform (cf. [12, item 1.1.3, page 2].

(xi) Allowing $a = e$, $r = 0$ and transformation $s = \frac{1}{p}$ then (2.1) leads to

$$K\{f(t)\}_{(s)} = \int_{t=0}^{\infty} f(t)e^{-\frac{t}{s}}dt, \tag{2.13}$$

where $Re(s) > 0$. This is called Kamal transform [16].

(xii) On setting $a = e$, $r = 0$ and $f(t) \to t^n f(t)$ then (2.1) leads us to

$$M_n\{f(t)\}_{(s)} = \int_{t=0}^{\infty} t^n f(t)e^{-st}dt, \tag{2.14}$$

where $s > 0$. This transform is called Atangana-Kilicman integral transform [5].

(xiii) Replacing $a = e$, $r = -1$ and $f(t) \to t^{n-1} f(t)$ then (2.1) reduces to

$$G_n\{f(t)\}_{(s)} = s \int_{t=0}^{\infty} t^{n-1} f(t) e^{-st} dt, \tag{2.15}$$

where $s > 0$. This type of integral transform is named ARA integral transform can be obtained [32].

(xiv) Taking $a = e$, $r = -1$ and transformation $s = \frac{1}{p}$ then (2.1) leads to

$$E\{f(t)\}_{(s)} = s \int_{t=0}^{\infty} f(t) e^{-\frac{t}{s}} dt, \tag{2.16}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$.This integral transform is known as Elzaki transform [8].

(xv) Adopting $a = e$, $r = 2$ and the transformation $s = \frac{1}{p}$ then (2.1) gives

$$S\{f(t)\}_{(s)} = \frac{1}{s^2} \int_{t=0}^{\infty} f(t) e^{-\frac{t}{s}} dt, \tag{2.17}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is Sawi transform [18].

(xvi) Considering $a = e$, $r = -i$ and transformation $s = \frac{1}{p}$ then (2.1) one may obtain

$$E\{f(t)\}_{(s)} = s \int_{t=0}^{\infty} f(t) e^{-\frac{t}{s}} dt, \tag{2.18}$$

where $t \geq 0$, $i \in \mathbb{Z}$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$.This integral transform is G-transform [1].

(xvii) Inserting $a = e$, $r = 2$ and the transformation $s = \frac{1}{p}$ then (2.1) yields

$$S\{f(t)\}_{(s)} = \frac{1}{s} \int_{t=0}^{\infty} f(t) e^{-\frac{t}{s}} dt, \tag{2.19}$$

where $t \geq 0$, $n_1 \leq s \leq n_2$, $n_1, n_2 > 0$. This integral transform is Sumudu transform [36].

3 Some properties of the SUM integral transform

Some properties like linearity, change of scale, first shifting and second shifting are set up in this section.

Theorem 3.1. *(Linearity property) If $f(t)$ and $g(t)$ are piecewise continuous and exponential order functions, δ and μ are constants, then*

$$S_a\{\delta f(t) + \mu g(t)\}_{(s)} = \delta S_a\{f(t)\}_{(s)} + \mu S_a\{g(t)\}_{(s)}. \tag{3.1}$$

Proof. We can express the right-hand side of (3.1) by

$$\begin{aligned} S_a\{\delta f(t) + \mu g(t)\}_{(s)} &= \frac{1}{s^r} \int_{t=0}^{\infty} \{\delta f(t) + \mu g(t)\}\, a^{-st} dt \\ &= \delta \left\{ \frac{1}{s^r} \int_{t=0}^{\infty} f(t) a^{-st} dt \right\} + \mu \left\{ \frac{1}{s^r} \int_{t=0}^{\infty} g(t) a^{-st} dt \right\} \\ &= \delta S_a\{f(t)\}_{(s)} + \mu S_a\{g(t)\}_{(s)}. \end{aligned}$$

The proof is complete. □

Theorem 3.2. *(First shifting property) If $S_a\{f(t)\}_{(s)} = G_a(s)$, then*

$$S_a\{e^{\rho t} f(t)\}_{(s)} = G(slog(a) - \rho). \tag{3.2}$$

Proof. By definition SUM transform in (2.1) we have

$$S_a\{e^{\rho t}f(t)\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{\infty} f(t)e^{-(slog(a)-\rho)}dt = G(slog(a)-\rho), \tag{3.3}$$

which is obtained by using the SEE integral transform in equation (2.3). The proof is complete. □

Theorem 3.3. *(Second shifting property) If* $S_a\{f(t)\}_{(s)} = G_a(s)$ *and* $g(t) = \begin{cases} f(t-\beta), & if t > \beta, \\ 0, & if t < \beta, \end{cases}$ *then*

$$S_a\{g(t)\}_{(s)} = a^{-s\beta}S_a\{f(t)\}_{(s)}. \tag{3.4}$$

Proof. In view of (2.1) we have

$$\begin{aligned} S_a\{g(t)\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\beta} g(t)a^{-st}dt + \frac{1}{s^r}\int_{t=\beta}^{\infty} g(t)a^{-st}dt \\ &= \frac{1}{s^r}\int_{t=0}^{\beta} 0\cdot a^{-st}dt + \frac{1}{s^r}\int_{t=\beta}^{\infty} f(t-\beta)a^{-st}dt = \frac{1}{s^r}\int_{t=\beta}^{\infty} f(t-\beta)a^{-st}dt. \end{aligned}$$

Setting $t-\beta=\gamma$ and change of variables, leads to

$$S_a\{g(t)\}_{(s)} = a^{s\beta}\left\{\frac{1}{s^r}\int_{t=0}^{\infty} f(t)a^{-st}dt\right\} = a^{-s\beta}S_a\{f(t)\}_{(s)}.$$

The proof is complete. □

Theorem 3.4. *(Change of scale property) If* $S_a\{f(t)\}_{(s)} = G_a(s)$, *then*

$$S_a\{f(\rho t)\}_{(s)} = \frac{1}{\rho}G\left(\frac{s}{\rho}\right). \tag{3.5}$$

Proof. Utilizing (2.1) we obtain

$$S_a\{f(\rho t)\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{\infty} f(\rho t)a^{-st}dt.$$

Putting $\rho t = \gamma$ and change of variable, gives

$$S_a\{f(\rho t)\}_{(s)} = \frac{1}{\rho}\left\{\frac{1}{s^r}\int_{t=0}^{\infty} f(t)a^{-st}dt\right\} = \frac{1}{\rho}G\left(\frac{s}{\rho}\right).$$

The proof is complete. □

4 The SUM integral transform of some elementary functions

The SUM integral transform of some elementary functions is evaluated in this section

(i) If $f(t)=k$, where k is constant, then

$$S_a\{k\}_{(s)} = \frac{k}{s^r[slog(a)]}. \tag{4.1}$$

Proof. By considering left-hand side of (4.1) leads to

$$S_a\{k\}_{(s)} = k\left\{\frac{1}{s^r}\int_{t=0}^{\infty} a^{-st}dt\right\} = \frac{k}{s^r[slog(a)]}.$$

The proof is complete. □

(ii) If $f(t) = t^n$, $n \in \mathbb{N}_0$, then

$$S_a\{t^n\}_{(s)} = \frac{\Gamma(n+1)}{s^r[slog(a)]^{n+1}}, \quad Re(n+1) > 0. \tag{4.2}$$

Proof. We can represent the left-hand side of (4.2) by

$$S_a\{t^n\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{\infty} t^n e^{-stlog(a)}dt.$$

Applying integration by part by setting $v = stlog(a)$ and $dt = \frac{dv}{slog(a)}$, so that

$$S_a\{t^n\}_{(s)} = \frac{1}{s^r[slog(a)]^{n+1}}\frac{1}{s^r}\int_{t=0}^{\infty} v^n e^{-v}dv.$$

Using the definition of the gamma function in Abubakar [34], the required result in equation (4.2) will be obtained. The proof is complete. □

(iii) If $f(t) = e^{\rho t}$ where ρ is a constant, then

$$S_a\{e^{\rho t}\}_{(s)} = \frac{1}{s^r\{[slog(a)] - \rho\}}. \tag{4.3}$$

Proof. By definition of the SUM integral transform in (2.1) gives

$$S_a\{e^{\rho t}\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{\infty} e^{-\{[slog(a)]-\rho\}t}dt = \frac{1}{s^r\{[slog(a)] - \rho\}}.$$

The proof is complete. □

(iv) Suppose $f(t) = sin(\rho t)$ where ρ is a constant, then

$$S_a\{sin(\rho t)\}_{(s)} = \frac{\rho}{s^r\left\{[slog(a)]^2 + \rho^2\right\}}. \tag{4.4}$$

Proof. Using (2.1) leads to

$$\begin{aligned} S_a\{sin(\rho t)\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\infty} a^{-st}\left(\frac{e^{\rho it} - e^{\rho it}}{2i}\right) \\ &= \frac{1}{2is^r}\int_{t=0}^{\infty}\left\{e^{-[slog(a)-i\rho]t} - e^{-[slog(a)+i\rho]t}\right\} \\ &= \frac{\rho}{s^r\left\{[slog(a)]^2 + \rho^2\right\}}. \end{aligned}$$

The proof is complete. □

(v) Suppose $f(t) = cos(\rho t)$ where ρ is a constant, then

$$S_a\{sin(\rho t)\}_{(s)} = \frac{[slog(a)]}{s^r\left\{[slog(a)]^2 + \rho^2\right\}}. \tag{4.5}$$

Proof. Using (2.1) one may obtain

$$\begin{aligned} S_a\{cos(\rho t)\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\infty} a^{-st}\left(\frac{e^{\rho it} + e^{\rho it}}{2}\right) \\ &= \frac{1}{2is^r}\int_{t=0}^{\infty}\left\{e^{-[slog(a)-i\rho]t} + e^{-[slog(a)+i\rho]t}\right\} \\ &= \frac{[slog(a)]}{s^r\left\{[slog(a)]^2 + \rho^2\right\}}. \end{aligned}$$

The proof is complete. □

(vi) Suppose $f(t) = sinh(\rho t)$ where ρ is a constant, then

$$S_a\{sinh(\rho t)\}_{(s)} = \frac{\rho}{s^r\left\{[slog(a)]^2 - \rho^2\right\}}. \tag{4.6}$$

Proof. Using (2.1) leads us to

$$\begin{aligned} S_a\{sinh(\rho t)\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\infty} a^{-st}\left(\frac{e^{\rho t} - e^{\rho t}}{2i}\right) \\ &= \frac{1}{2is^r}\int_{t=0}^{\infty}\left\{e^{-[slog(a)-\rho]t} - e^{-[slog(a)+\rho]t}\right\} \\ &= \frac{\rho}{s^r\left\{[slog(a)]^2 - \rho^2\right\}}. \end{aligned}$$

The proof is complete. □

(vii) Suppose $f(t) = cosh(\rho t)$ where ρ is a constant, then

$$S_a\{cosh(\rho t)\}_{(s)} = \frac{[slog(a)]}{s^r\left\{[slog(a)]^2 - \rho^2\right\}}. \tag{4.7}$$

Proof. Using (2.1) gives

$$\begin{aligned} S_a\{cosh(\rho t)\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\infty} a^{-st}\left(\frac{e^{\rho t} + e^{\rho t}}{2}\right) \\ &= \frac{1}{2is^r}\int_{t=0}^{\infty}\left\{e^{-[slog(a)-\rho]t} + e^{-[slog(a)+\rho]t}\right\} \\ &= \frac{[slog(a)]}{s^r\left\{[slog(a)]^2 - \rho^2\right\}}. \end{aligned}$$

The proof is complete. □

(viii) If $f(t) = u(t-\beta)$ where $u(t-\beta)$ is a unit function defined by $u(t-\beta) = \begin{cases} 1, & if t \geq \beta, \\ 0, & if t < \beta, \end{cases}$ then

$$S_a\{u(t-\beta)\}_{(s)} = \frac{a^{-s\beta}}{s^r[slog(a)]}. \tag{4.8}$$

Proof. By definition of SUM transform in (2.1), we have

$$\begin{aligned} S_a\{u(t-\beta)\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\beta} 0\cdot a^{-st}dt + \frac{1}{s^r}\int_{t=\beta}^{\infty} 1\cdot a^{-st}dt \\ &= \frac{1}{s^r}\int_{t=\beta}^{\infty} a^{-st}dt = \frac{a^{-s\beta}}{s^r[slog(a)]}. \end{aligned}$$

The proof is complete. □

(ix) If a function $f(t)$ is m-times continuously differentiable on $[0,\infty)$ and of exponential order $\partial(>0)$, then $S_a\{f'(t)\}_{(s)}, S_a\{f''(t)\}_{(s)}, \cdots, S_a\{f^{(m)}(t)\}_{(s)}$ exist for $Re(s) > \frac{\partial}{log(a)}$ and

$$S_a\{f^{(m)}(t)\}_{(s)} = [slog(a)]^m S_a\{f(t)\}_{(s)} - \frac{1}{s^r}\sum_{w=0}^{(m-1)}[slog(a)]^{m-w-1}f^{(m-1)-w}(0). \tag{4.9}$$

Proof. According to the definition of the SUM transform in (2.1) for $m = 1$, we have

$$S_a\{f'(t)\}_{(s)} = \frac{1}{s^r}\int_{t=0}^{\infty} f'(t)a^{-st}dt. \tag{4.10}$$

Using integration by part and by setting $u = a^{-st}$ and $dv = f'(t)dt$, one can deduce that

$$S_a\{f'(t)\}_{(s)} = -\frac{f(0)}{s^r} + [slog(a)]\left[\frac{1}{s^r}\int_{t=0}^{\infty} f(t)a^{-st}dt\right]$$
$$= [slog(a)]S_a\{f(t)\}_{(s)} - \frac{f(0)}{s^r}. \quad (4.11)$$

Since (4.9) is true for $m = k$, then by (4.11) and the assumption of induction we can deduce

$$S_a\left\{\left(f^{(k)}(t)\right)'\right\}_{(s)} = [slog(a)]S_a\{f^{(k)}\}_{(s)} - \frac{f^{(k)}(0)}{s^r}$$
$$= [slog(a)]\left\{[slog(a)]^k S_a\{f(t)\}_{(s)} - \frac{1}{s^r}\sum_{w=0}^{(k-1)}[slog(a)]^{k-w-1}f^{(k-1)-w}(0)\right\}$$
$$= [slog(a)]^{k+1}S_a\{f(t)\}_{(s)} - \frac{1}{s^r}\sum_{w=0}^{(k)}[slog(a)]^{k-w}f^{(k-1)-w}(0),$$

this implies that (4.9) holds for $m = k + 1$. By the induction hypothesis, The proof is complete. □

Remark 4.1. Equation (4.9) can be represented in expanded form as follows

$$S_a\{f^{(m)}(t)\}_{(s)} = [slog(a)]^m S_a\{f(t)\}_{(s)} - [slog(a)]^{m-1}\frac{f(0)}{s^r} - \cdots$$
$$- [slog(a)]\frac{f^{m-2}(0)}{s^r} - \frac{f^{m-1}(0)}{s^r}.$$

5 Applications of the SUM integral transform

In this section, the SUM transform is used to solve some problems involving ordinary and partial differential equations.

5.1 Solving ordinary differential equations

This subsection covers examples that contain ordinary differential equations

Example 5.1. Consider the initial value problem [21]

$$\frac{dy}{dt} + y = 0 \quad with\ y(0) = 1. \quad (5.1)$$

Applying the SUM transform in (2.1) and linearity property in (3.1), gives

$$-\frac{y(0)}{s^r} + [slog(a)]y_a(s) + y_a(s) = 0.$$

Simplifying leads to

$$y_a(s) = \frac{1}{s^r\{[log(a)] + 1\}}.$$

Taking the inverse SUM transform, gives

$$y(t) = e^{-t}.$$

Example 5.2. For the following equation [21]:

$$\frac{dy}{dt} + 2y = t \quad with\ y(0) = 1. \quad (5.2)$$

Applying the SUM transform in (2.1) and linearity property in (3.1), yields

$$-\frac{y(0)}{s^r}+[slog(a)]y_a(s)+2y_a(s)=\frac{1}{s^r[slog(a)]}.$$

Simplifying

$$y_a(s)=\frac{1}{s^r[slog(a)]^2\{[slog(a)]+2\}}+\frac{1}{s^r\{[slog(a)]+2\}}.$$

Using partial fraction, leads to

$$y_a(s)=-\frac{1}{4}\frac{1}{s^r[slog(a)]}+\frac{1}{2}\frac{1}{s^r[slog(a)]^2}+\frac{5}{2}\frac{1}{s^r\{[slog(a)]+2\}}$$

Taking the inverse SUM transform, gives

$$y_a(t)=-\frac{1}{4}+\frac{1}{2}t+\frac{5}{4}e^{-2t}.$$

Example 5.3. Consider the following initial value problem [21]:

$$\frac{d^2y}{dt^2}+9y=\cos(2t)\quad with\ y(0)=1\ and\ y\left(\frac{\pi}{2}\right)=-1. \tag{5.3}$$

Applying the SUM transform and linearity property in equation 3.1 and setting $y'(0)=\alpha$ (where α is constant), yields

$$-\frac{y'(0)}{s^r}-[slog(a)]\frac{y(0)}{s^r}+[slog(a)]^2y_a(s)+9y_a(s)=\frac{[slog(a)]}{s^r\left\{[slog(a)]^2+4\right\}}.$$

Simplifying

$$y_a(s)=\frac{[slog(a)]}{s^r\left\{[slog(a)]^2+4\right\}\left\{[slog(a)]^2+9\right\}}+\frac{\alpha}{s^r\left\{[slog(a)]^2+9\right\}}+\frac{[slog(a)]}{s^r\left\{[slog(a)]^2+9\right\}}.$$

Using partial fraction, leads to

$$y_a(s)=\frac{1}{5}\frac{[slog(a)]}{s^r\left\{[slog(a)]^2+4\right\}}+\frac{\alpha}{3}\frac{3}{s^r\left\{[slog(a)]^2+9\right\}}+\frac{4}{5}\frac{[slog(a)]}{s^r\left\{[slog(a)]^2+9\right\}}.$$

Applying the inverse of the SUM transform, yields

$$y(t)=\frac{1}{5}\cos(2t)+\frac{\alpha}{3}\sin(3t)+\frac{4}{5}\cos(2t).$$

Considering $y\left(\frac{\pi}{2}\right)=-1$, then $\alpha=-\frac{12}{5}$ and so

$$y(t)=\frac{1}{5}\cos(2t)+\frac{4}{5}\sin(3t)+\frac{4}{5}\cos(2t).$$

Example 5.4. To obtain the solution of the 2^{nd} order linear differential equation [22]

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=4e^{3t}\quad with\ y(0)=-3\ and\ y'(0)=5. \tag{5.4}$$

Applying the SUM transform in (2.1) and linearity property in (3.1), leads to

$$\begin{aligned}&-\frac{y'(0)}{s^r}-[slog(a)]\frac{y(0)}{s^r}+[slog(a)]^2y_a(s)\\&-3\left\{-\frac{y(0)}{s^r}+[slog(a)]y_a(s)\right\}+2y_a(s)=\frac{4}{s^r\left\{[slog(a)]-3\right\}}.\end{aligned}$$

Simplifying

$$y_a(s) = \frac{4}{s^r \{[slog(a)] - 3\} \{[slog(a)] - 2\} \{[slog(a)] - 1\}}$$
$$+ \frac{14}{s^r \{[slog(a)] - 2\} \{[slog(a)] - 1\}} + \frac{5[slog(a)}{s^r \{[slog(a)] - 2\} \{[slog(a)] - 1\}}.$$

Taking the partial fraction, gives

$$y_a(s) = \frac{2}{s^r \{[slog(a)] - 3\}} + \frac{4}{s^r \{[slog(a)] - 2\}} - \frac{9}{s^r \{[slog(a)] - 1\}}.$$

Using the inverse of the SUM transform gives

$$y(t) = 2e^{3t} + 4e^{2t} - 9e^{t}.$$

Example 5.5. To solve the equation [31]

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 125 \quad with\ y(0) = y'(0) = 0. \tag{5.5}$$

Applying the SUM transform in (2.1) and linearity property in (3.1), gives

$$-\frac{y'(0)}{s^r} - [slog(a)]\frac{y(0)}{s^r} + [slog(a)]^2 y_a(s)$$
$$+ 8\left\{-\frac{y(0)}{s^r} + [slog(a)]y_a(s)\right\} + 25y_a(s) = \frac{150}{s^r \{[slog(a)]\}}.$$

Simplifying

$$y_a(s) = \frac{150}{s^r[slog(a)]\left\{[slog(a)]^2 + 8[slog(a)] + 125\right\}}.$$

Taking the partial fraction, gives

$$y_a(s) = \frac{6}{s^r[slog(a)]} - \frac{6\{[slog(a)] + 4\}}{s^r\left\{([slog(a)] + 4)^2 + 9\right\}} - \frac{24}{s^r\left\{([slog(a)] + 4)^2 + 9\right\}}.$$

Taking the inverse of the SUM transform leads to

$$y(t) = 6 - 6e^{-4t}\cos(3t) - 8e^{-4t}\sin(3t).$$

Example 5.6. For the equation [12, Example 12, p. 138]

$$2\frac{d^2y}{dt^2} + 8y = \alpha H(t - \beta) \quad \text{with}\ \ y(0) = 10, `y'(0) = 0. \tag{5.6}$$

Applying the SUM transform in (2.1) and linearity property in (3.1), yields

$$-\frac{y'(0)}{s^r} - [slog(a)]\frac{y(0)}{s^r} + [slog(a)]^2 y_a(s) + 8y_a(s) = \frac{\alpha a^{-s\beta}}{s^r[slog(a)]}.$$

Simplifying

$$y_a(s) = \frac{\alpha a^{-s\beta}}{2s^r \{[slog(a)] \{[slog(a)]^2 + 4\}\}} + \frac{10[slog(a)]}{2s^r \{[slog(a)]^2 + 4\}}.$$

Taking partial fraction, gives

$$y_a(s) = \frac{\alpha a^{-s\beta}}{8}\left\{\frac{1}{s^r[slog(a)]} - \frac{[slog(a)]}{s^r \{[slog(a)]^2 + 4\}}\right\} + \frac{10[slog(a)]}{\{[slog(a)]^2 + 4\}}.$$

Applying the SUM transform to both sides, gives

$$y(t) = \begin{cases} \frac{8}{\alpha}\{1 - \cos[2(t - \beta)]\} + 10\cos(2t), & if\ \ t \geq \beta, \\ 10\cos(2t), & if\ \ t < \beta. \end{cases}$$

Example 5.7. To solve the 3^{rd} order ODE [12, Example 11, pp. 137-138]

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t. \tag{5.7}$$

Applying the SUM transform in (2.1) and linearity property in (3.1), gives

$$\begin{aligned}&-3\left\{\frac{y'(0)}{s^r} - [slog(a)]^2\frac{y(0)}{s^r} + [slog(a)]^2 y_a(s)\right\} + 3\left\{-\frac{y(0)}{s^r} + [slog(a)]y_a(s)\right\}\\ &-\frac{y''(0)}{s^r} - [slog(a)]\frac{y'(0)}{s^r} - [slog(a)]^2\frac{y(0)}{s^r} + [slog(a)]^3 y_a(s) - y_a(s) = \frac{2}{s^r\{[sloa(a)]-1\}^3}\end{aligned}$$

Setting $y(0) = \alpha$, $y'(0) = \beta$, $y''(0) = \gamma$ where α, β, γ are arbitrary constant and simplify

$$y_a(s) = \frac{2}{s^r\{[sloa(a)]-1\}^6} + \frac{\alpha[slog(a)]^2 + (\beta - 3\alpha)[slog(a)] + (3\alpha - 3\beta + \gamma)}{s^r\{[sloa(a)]-1\}^3}$$

Considering partial fraction, gives

$$y_a(s) = \frac{2}{s^r\{[sloa(a)]-1\}^6} + \frac{A}{s^r\{[sloa(a)]-1\}^3} + \frac{B}{s^r\{[sloa(a)]-1\}^2} + \frac{C}{s^r\{[sloa(a)]-1\}},$$

where A, B and C are arbitrary constant, applying the inverse of the SUM transform, gives

$$\begin{aligned}y(t) &= \frac{t^5e^t}{60} + \frac{A}{2}t^5e^t + Bte^t + Ce^t\\ &= \frac{t^5e^t}{60} + A_1t^5e^t + Bte^t + Ce^t \quad \left(A_1 = \frac{A}{2}\right).\end{aligned}$$

Example 5.8. To solve the 4^{th} order differential equation [12, Example 2, p. 131]

$$\frac{d^4y}{dt^4} - y = 1 \text{ with } y(0) = y'(0) = y''(0) = y'''(0) = 0. \tag{5.8}$$

Applying the SUM transform in (2.1) and linearity property in (3.1), yields

$$\begin{aligned}&-\frac{y'''(0)}{s^r} - [slog(a)]\frac{y''(0)}{s^r} - [slog(a)]^2\frac{y'(0)}{s^r} - [slog(a)]^3\frac{y(0)}{s^r}\\ &+ [slog(a)]^4 y_a(s) - y_a(s) = \frac{1}{s^r[slog(a)]}.\end{aligned}$$

Simplifying

$$y_a(s) = \frac{1}{s^r[slog(a)]\{[slog(a)]^2 - 1\}\{[slog(a)]^2 + 1\}}.$$

Applying the partial fraction, gives

$$y_a(s) = -\frac{1}{[slog(a)]} + \frac{1}{2}\frac{[slog(a)]}{s^r\{[slog(a)]-1\}} + \frac{1}{2}\frac{[slog(a)]}{s^r\{[slog(a)]-1\}}.$$

Considering the inverse of the SUM transform, gives

$$y_a(s) = -1 + \frac{1}{2}[\cosh(t) + \cos(t)].$$

5.2 Solving partial differential equations

Solution of partial differential equations such as heat, Laplace, wave, telegraph and Klein-Gordan equations are considered in this subsection

Theorem 5.9. *Suppose the function $v(x,t)$ is defined for $x \in [a,b]$, $t > 0$ and $S_a\{v(x,t)\}_{(s)} = v_a(x,s)$, then*

$$S_a\left\{\frac{\partial v(x,t)}{\partial t}\right\}_{(s)} = -\frac{v(x,0)}{s^r} + [slog(a)]v_a(x,s), \tag{5.9}$$

$$S_a\left\{\frac{\partial^2 v(x,t)}{\partial t^2}\right\}_{(s)} = -\frac{1}{s^r}\frac{\partial v(x,0)}{\partial t} - [slog(a)]\frac{v(x,0)}{s^r} + [slog(a)]^2 v_a(x,s), \tag{5.10}$$

$$S_a\left\{\frac{\partial v(x,t)}{\partial x}\right\}_{(s)} = \frac{dv_a(x,t)}{dx}, \tag{5.11}$$

$$S_a\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}_{(s)} = \frac{d^2 v_a(x,t)}{d^2 x}. \tag{5.12}$$

Proof. Equation (5.9) can be obtained by using (2.1) gives

$$\begin{aligned}
S_a\left\{\frac{\partial v(x,t)}{\partial t}\right\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\infty}\frac{\partial v(x,t)}{\partial t}a^{-st}dt \\
&= \frac{1}{s^r}\left\{\lim_{z\to\infty}\left(\left[v(x,t)a^{-st}\right]_0^z + [slog(a)]\int_{t=0}^{z} v(x,t)a^{-st}dt\right)\right\} \\
&= -\frac{v(x,0)}{s^r} + [slog(a)]v_a(x,s).
\end{aligned}$$

Equation (5.10) is received by setting $\frac{dv}{dt} = w$, leads to

$$\begin{aligned}
S_a\left\{\frac{\partial^2 v(x,t)}{\partial t^2}\right\}_{(s)} = S_a\left\{\frac{\partial w(x,t)}{\partial t}\right\}_{(s)} &= -\frac{w(x,0)}{s^r} + [slog(a)]S_a\left\{w(x,t)\right\}_{(s)} \\
&= -\frac{1}{s^r}\frac{\partial v(x,0)}{\partial t} + [slog(a)]\left\{-\frac{v(x,0)}{s^r} + [slog(a)]v_a(x,s)\right\} \\
&= -\frac{1}{s^r}\frac{\partial v(x,0)}{\partial t} - [slog(a)]\frac{v(x,0)}{s^r} + [slog(a)]^2 v_a(x,s).
\end{aligned}$$

Equation (5.11) can be obtained by considering (2.1) yields

$$\begin{aligned}
S_a\left\{\frac{\partial v(x,t)}{\partial x}\right\}_{(s)} &= \frac{1}{s^r}\int_{t=0}^{\infty}\frac{\partial v(x,t)}{\partial x}a^{-st}dt \\
&= \frac{d}{dx}\left\{\frac{1}{s^r}\int_{t=0}^{\infty} v(x,t)a^{-st}dt\right\} = \frac{dv_a(x,s)}{dx}.
\end{aligned}$$

Equation (5.12) is obtained by substituting $\frac{dv}{dt} = w$, gives

$$S_a\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}_{(s)} = S_a\left\{\frac{\partial w(x,t)}{\partial x}\right\}_{(s)} = \frac{d}{dx}S_a\left\{w(x,t)\right\}_{(s)} = \frac{d^2 v_a(x,s)}{d^2 x}.$$

The proof is complete. □

Example 5.10. Solve the partial differential equation [31]

$$\frac{\partial v}{\partial x} = 2\frac{\partial v}{\partial t} + v, \quad t, x > 0. \tag{5.13}$$

With the condition $v(x,0) = 6e^{-3x}$.
Applying the SUM transform in (2.1) and linearity property in (3.1) to gives

$$\frac{dv_a(x,s)}{dx} - \left\{2[slog(a)] + 1\right\} v_a(x,s) = -\frac{12}{s^r}e^{-3x}.$$

This is linear ordinary differential equation which has the following solution

$$v_a(x,s) = \frac{6}{s^r\{[slog(a)]+2\}}e^{-3x} + Ae^{\{2[slog(a)]+1\}x},$$

where A is constant of integration, when $x \to \infty$, $v(x,t)$ is bounded and when $x \to \infty$, $v_a(x,s)$ should also be bounded and so $A = 0$ and taking inverse SUM transform gives

$$v(x,t) = 6e^{-(2t+3x)}.$$

Example 5.11. Solve the Laplace equation [31]

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} = 0, \quad t, x > 0. \tag{5.14}$$

With the condition $v(x,0) = 0$ and $\frac{\partial v(x,0)}{\partial t} = \cos(x)$.
Applying the SUM transform in (2.1) and linearity property in (3.1) to (5.14) gives

$$\frac{1}{[slog(a)]^2}\frac{d^2 v_a(x,s)}{dx^2} + v_a(x,s) = \frac{1}{s^r[slog(a)]^2}\cos(x).$$

Simplifying gives

$$v_a(x,s) = \frac{\frac{1}{s^r[slog(a)]^2}\cos(x)}{\frac{1}{[slog(a)]^2}D^2+1} = \frac{\frac{1}{s^r[slog(a)]^2}\cos(x)}{\frac{1}{[slog(a)]^2}(-1^2)+1} = \frac{1}{s^r\{[slog(a)]^2-1\}}\cos(x).$$

Taking the SUM transform inverse of both sides gives

$$v(x,t) = \sinh(t)\cos(x).$$

Example 5.12. Consider the following wave equation [31]

$$\frac{\partial^2 v}{\partial x^2} = 4\frac{\partial^2 v}{\partial t^2}, \quad t, x > 0. \tag{5.15}$$

With the condition $v(x,0) = \sin(\pi x)$ and $\frac{\partial v(x,0)}{\partial t} = 0$.
Applying the SUM transform in (2.1) and linearity property in (3.1) to (5.15) gives

$$\frac{1}{4[slog(a)]^2}\frac{d^2 v_a(x,s)}{dx^2} - v_a(x,s) = -\frac{1}{s^r[slog(a)]}\sin(\pi x).$$

Simplifying gives

$$v_a(x,s) = \frac{-\frac{1}{s^r[slog(a)]}\sin(\pi x)}{\frac{1}{4[slog(a)]^2}D^2-1} = \frac{-\frac{1}{s^r[slog(a)]}\sin(\pi x)}{\frac{1}{4[slog(a)]^2}(-\pi^2)-1} = \frac{[slog(a)]}{s^r\left\{[slog(a)]^2+\left(\frac{\pi}{2}\right)^2\right\}}\sin(\pi x).$$

Taking the SUM transform inverse of both sides gives

$$v(x,t) = \cos\left(\frac{\pi}{2}t\right)\sin(\pi x).$$

Example 5.13. Solve the differential equation [31]

$$\frac{\partial^2 v}{\partial x^2} = 4\frac{\partial v}{\partial t}, \quad t, x > 0. \tag{5.16}$$

With the condition $v(x,0) = \sin\left(\frac{\pi}{2}x\right)$.
Applying the SUM transform in (2.1) and linearity property in (3.1) to (5.16) gives

$$\frac{1}{4[slog(a)]}\frac{d^2 v_a(x,s)}{dx^2} - v_a(x,s) = -\frac{1}{s^r[slog(a)]}\sin\left(\frac{\pi}{2}x\right).$$

Simplifying gives

$$v_a(x,s) = \frac{-\frac{1}{s^r[slog(a)]}\sin(\frac{\pi}{2}x)}{\frac{1}{4[slog(a)]}D^2 - 1} = \frac{-\frac{1}{s^r[slog(a)]}\sin\left(\frac{\pi}{2}x\right)}{\frac{1}{4[slog(a)]}(-\frac{\pi^2}{4}) - 1} = \frac{1}{s^r\left\{[slog(a)] + \left(\frac{\pi^2}{16}\right)\right\}}\sin\left(\frac{\pi}{2}x\right).$$

Taking the SUM transform inverse of both sides gives

$$v(x,t) = e^{-\frac{\pi^2}{16}t}\sin\left(\frac{\pi}{2}x\right).$$

Example 5.14. Solve the telegraph equation [31]

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} + v. \tag{5.17}$$

With the condition $v(x,0) = e^x$ and $\frac{\partial v(x,0)}{\partial t} = -2e^x$.
Applying the SUM transform in (2.1) and linearity property in (3.1) to (5.17) gives

$$\frac{1}{\{[slog(a)]+1\}^2}\frac{d^2 v_a(x,s)}{dx^2} - v_a(x,s) = -\frac{[slog(a)]}{s^r\{[slog(a)]+1\}^2}e^x.$$

Simplifying gives

$$v_a(x,s) = \frac{-\frac{[slog(a)]}{s^r\{[slog(a)]+1\}^2}e^x}{\frac{1}{\{[slog(a)]+1\}^2}D^2 - 1} = \frac{-\frac{[slog(a)]}{s^r\{[slog(a)]+1\}^2}e^x}{\frac{1}{\{[slog(a)]+1\}^2}(1^2) - 1} = \frac{1}{s^r\{[slog(a)]+2\}}e^x.$$

Taking the SUM transform inverse of both sides gives

$$v(x,t) = e^{x-2t}.$$

Example 5.15. Consider the following Klein-Gordan equation [31]

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + 2v, \quad -\infty < x < \infty,\ t > 0. \tag{5.18}$$

With the condition $v(x,0) = e^x$ and $\frac{\partial v(x,0)}{\partial t} = 0$.
Applying the SUM transform in (2.1) and linearity property in (3.1) to (5.18) gives

$$\frac{1}{\{[slog(a)]^2-2\}}\frac{d^2 v_a(x,s)}{dx^2} + \frac{1}{\{[slog(a)]^2-2\}}\frac{dv_a(x,s)}{dx} - v_a(x,s) = -\frac{[slog(a)]}{s^r\{[slog(a)]^2-2\}}e^x.$$

Simplifying gives

$$v_a(x,s) = \frac{-\frac{[slog(a)]}{s^r\{[slog(a)]^2-2\}}e^x}{\frac{1}{\{[slog(a)]^2-2\}}D^2 + \frac{1}{\{[slog(a)]^2-2\}}D - 1} = \frac{1}{s^r\{[slog(a)]^2 - 2^2\}}e^x.$$

Taking the SUM transform inverse of both sides gives

$$v(x,t) = e^x\cosh(2t).$$

6 Conclusion

The SUM integral transform is introduced and some of its applications in solving ordinary and partial differential equations that appears in science and technology. Duality properties of the SUM integral transform with other integral transforms such as SEE, Modified Laplace, Laplace, Kamal, etc., can be obtained, see examples
SUM-SEE duality property

$$S_a\{f(t)\}_{(s)} = S\{f(t)\}_{(slog(a))},$$

SUM-Modified Laplace (due to Saif et al., [33]) duality property

$$S_a\{f(t)\}_{(s)} = \frac{1}{s^r} L_a\{f(t)\}_{(s)},$$

SUM-Laplace duality property

$$S_a\{f(t)\}_{(s)} = \frac{1}{s^r} L\{f(t)\}_{(slog(a))},$$

SUM-Kamal duality property

$$S_a\{f(t)\}_{(s)} = \frac{1}{[slog(a)]^r} K\{f(t)\}_{(s)}.$$

It is expected that the SUM integral transformation will have scientific and technological applications in areas such as pharmacokinetics, nuclear physics, population growth, electrical circuits, modeling of colonic and crypt cancer, beam deflection, mixing problems, Newton law of cooling, spring modeling and other fields of mechanics, see for example [7].

References

[1] M.M.M. Abderahim, The new integral transform Sawi transform, *Adv. Theor. Appl. Math.* **14**(1), 81–87 (2019).

[2] K.S. Aboodh, Application of new transform 'Aboodh transform' to partial differential equations, *Glob. J. Pure Appl. Math.* **10**(2), 249–254 (2014).

[3] K.S. Aboodh, Application of new transform 'Aboodh transform', *Glob. J. Pure Appl. Math.* **9**(1), 35–43 (2013).

[4] U.M. Abubakar, A comparative analysis of modified extended fractional derivative and integral operators via modified extended beta function with applications to generating functions, *Cankaya Univ. J. Sci. Tech.* **19**(1), 40–50 (2022).

[5] A. Atangana and A. Kilicman, A novel integral operator transform and its applications to some FODE and FPDE with some kind of singularities, *Math. Prob. Eng.,* (2013).

[6] U. Duran, Modified Sumudu transform and its properties, *Sakarya Univ. J. Sci.* **28**(2), 389–396 (2021).

[7] A.I. El-Mesady, Y.S. Hamed, A.M. Alshafif, Jafari transformation for solving a system of ordinary differential equations with medical application, *Fractal Fract.* **5**(130), 1–12 (2021).

[8] T.M. Elzaki, The new integral transform Elzaki transform, *Glob. J. Pure Appl. Math.* **7**(1), 57–64 (2011).

[9] A. Gaur and G.A. Agarwal, On β-Laplace integral transforms and their properties, *Interdiscip. J. Adv. Sci. Tech.* **29**, 1487–1491 (2020).

[10] A. Gaur and G.A. Agarwal, Applications β-Laplace integral transforms, *Talent Develop. Excel.* **12**(3S), 8653–8659 (2020).

[11] A. Gaur and G.A. Agarwal, Fractional β-Laplace integral transforms, *AIP Conf. Proc.* **1706**(2020)012043, 1–9 (2020).

[12] S.P. Goyal and A.K. Goyal, *Integral transform*, Jaipur Publishing House, India (2014).

[13] R. Gupta and D. Verma, Application of novel integral transform: Gupta transform to mechanical engineering and electrical oscillators, *ASIO J. Chem. Phy. Math. Appl. Sci.* **4**(1), 4–7 (2020).

[14] R. Gupta, A novel integral transform: Rohit transform and its application to boundary value problem, *ASIO J. Chem. Phy. Math. Appl. Sci.* **4**(1), 8–13 (2020).

[15] H.S. Kadhem and S.Q. Hassan, Solving some fractional partial differential equations by invariant subspace and double Sumudu transform methods, *Ibn Al Haitham J. Pure Appl. Sci.* **33**(3), 127–139 (2020).

[16] A. Kamal, H. Sedeeg, The new integral transform 'Kamal transform', *Adv. Theor. and Appl. Math.* **11**(4), 451–458 (2016).

[17] B.N. Kharrat and G.A. Toma, A new integral transform: Kharrat-Toma transform and its properties, *World Appl. Sci. J.* **38**(5), 436–443 (2020).

[18] H. Kim, On the form and properties of an integral transform with strength in integral transforms, *Far East J. Math. Sci.* **102**(11), 2831–2822 (2017).

[19] E.A. Kuffi, E.S. Abbas and S.O. Mebayet, Al-Tememe transform in solving develop missile automatic problem, *Inter. J. Eng. Tech.* **7**(4), 6565–6567 (2019).

[20] E.A. Kuffi and E.S. Abbas, Applying Al-Zughair transform into some engineering fields, 3^{rd} *Inter. Sci. Conf. of Alkafeel Univ.*, 1–7 (2021).

[21] E.A. Kuffi, F. Karaaslan and G.S. Sadkhan, The complex EFG integral transform and its applications, *Inter. J. Heal. Sci.* **6**(53), 537—547 (2022).

[22] E.A. Kuffi and S.F. Maktoof, ”Emad-Falih transform” a new integral transform, *J. Interd. Math.* **24**(5), 2381–2390 (2021).

[23] A. Kumar, S. Bansal and S. Aggarwal, A new noval integral transform ”Anuj transform” with applications, *Des. Eng.* **9**, 12741–12751 (2022).

[24] X. Liang, G. Liu, and S. Su, Solving some fractional partial differential equations by invariant subspace and double Sumudu transform methods, *Ibn Al Haitham J. Pure Appl. Sci.* **33**(3), 127–139 (2020)

[25] A. Mahgoub, The new integral transform ’Mahgoub transform’, *Adv. Theor. Appl. Math.* **11**(4), 391–398 (2016).

[26] S.F. Maktoof, E. Kuffi, E.S. Abbas, ’Emad-Sara transform’ a new integral transform, *J. Interdiscip. Math.* **24**(4), 1–11 (2021).

[27] M. Mohand and A. Mahgoub, The new integral transform ”Mohand transfrom”, *Adv. Theor. Appl. Math.* **12**(2), 113–120 (2017).

[28] E.A. Monsour, E.A. Kuffi and S.A. Mehdi, Solving population and decay problems using complex SEE transform, 7^{th} *Inter. Conf. Contem. Infor. Tech. Math., Mosul, Iraq,*, 243–246 (2021).

[29] E.A. Monsour, E.A. Kuffi and S.A. Mehdi, Applying complex SEE transform in cryptography, *MJPS* **8**(2), 99–104 (2021).

[30] E.A. Monsour, S.A. Mehdi and E.A. Kuffi, The new integral transform and its application, *Inter. J. Nonlinear Anal. Appl.* **12**(2), 1477–1483 (2021).

[31] E.A. Monsour, E.A. Kuffi and S.A. Mehdi, The new integral transform ”SEE transform” and it properties, *Period. Eng. Nat. Sci.* **9**(2), 1016–1029 (2021).

[32] R. Saadeh, A. Qazza and A. Burqan, A new integral transform: ARA transform and its properties and applications, *Symmetry* **12**(925), 1–15 (2021).

[33] M. Saif, A. Khan, K.S. Nisar and S. Araci, Modified Laplace transform and its properties, *J. Math. Comput. Sci.* **21**, 127–135 (2020).

[34] S. Samah, M.K. Kothawade, M.D. Shinde and A. Maheshwari, Application of Laplace transform in mechanical engineering, *Inter. Res. J. Moder. Eng. Sci.* **4**(3), 1257–1264 (2022).

[35] D. Verma, Putting forward a novel integral transform Dinish Verma transform and its Applications, *Inter. J. Sci. Res. Math. Sta. Sci.* **7**(2), 139–145 (2020).

[36] G.K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problem, *Inter. J. Math. Educ. Sci. Tech.* **24**(1), 35–43 (1993).

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