Palestine Journal of Mathematics

Vol. 12(Special Issue I)(2023) , 59–72

NON–EXISTENCE OF POLYNOMIAL FIRST INTEGRALS Of A FAMILY OF THREE-DIMENSIONAL DIFFERENTIAL SYSTEMS

Chiman Qadir, Waleed Aziz, Ibrahim Hamad

Communicated by Rostam Kareem Saeed

MSC 2010 Classifications: Primary 34A05.

Keywords and phrases: Analytic first integral, polynomial first integral, Lotka–Volterra systems.

Abstract The work in this paper is a continuation of the recent work in [10] that the local integrability of a three dimensional Lotka–Volterra system is studied. More precisely, necessary and sufficient conditions are given for the existence of two independent analytic first integrals of the considered system. Here, in this paper, for particular parametric subsystems of three-dimensional Lotka-Volterra systems in [10], the non-existence of polynomial first integrals are investigated. Moreover, we mainly used the contradiction technique to prove that the given subsystems admits no polynomial first integrals.

1 Introduction

The characterisation of first integrals of polynomial differential systems is one of the interesting and difficult problems in the qualitative theory of differential equations. For two dimensional differential equations, the existence of one first integral will classify all its trajectories in phase plane. In general, to characterize the phase portrait of n dimensional differential systems, $n-1$ functionally independent first integrals are required.

This study has been motivated to the recent work in [10] which investigated the necessary conditions under which the three dimensional Lotka–Volterra system

$$\begin{aligned}\dot{x} &= x(1+ax+by+cz) = P(x,y,z),\\ \dot{y} &= y(-3+dx+ey+fz) = Q(x,y,z),\\ \dot{z} &= z(1+gx+hy+kz) = R(x,y,z),\end{aligned} \tag{1.1}$$

admits two independent analytic first integrals. Note that the existence of an analytic first integral of a given polynomial differential system does not imply the existence of a polynomial first integral, it may have or may have not. Hence having ideas on the existence analytic first integrals; it would be also a good idea to have knowledge on the existence of polynomial first integrals. So, here, we distinguish a family of differential systems in which they admit no polynomial first integrals.

The investigation of polynomial first integrals for two dimensional polynomial differential systems was considered by several authors in [11] and [4]. Other works on two dimensional quasi homogeneous polynomial differential systems and a family of three dimensional systems can be found in [3] and [6] respectively. They gave necessary conditions for which the given differential systems admits a polynomial, rational or analytic first integral. Recently, the authors in [1], gave necessary and sufficient conditions in order that a subfamily of three dimensional Lotka–Volterra systems has no polynomial first integrals. Other related works can be found in [5, 6, 8, 9] and references therein.

Let U be an open subset of $\mathbb{C}^3$, we recall that the non-constant function $H : U \to \mathbb{C}$ is a first integral if H is constant on all its solutions of system (1.1) contained on U. That means, the function $H(x, y, z)$ satisfies

$$P\frac{\partial H(x,y,z)}{\partial x} + Q\frac{\partial H(x,y,z)}{\partial y} + R\frac{\partial H(x,y,z)}{\partial z} = 0, \tag{1.2}$$

for all $(x, y, z) \in U$. We say that H is a polynomial first integral when H is a polynomial. That is, $H \in \mathbb{C}[x, y, z]$ where $\mathbb{C}[x, y, z]$ denotes the ring of all polynomials in the variables x, y and z with coefficients in $\mathbb{C}$.

Given a polynomial $F \in \mathbb{C}[x, y, z]$, we say that $F = 0$ is a Darboux polynomial (an invariant algebraic surface) of system (1.1), if it satisfies

$$P\,\partial_x F + Q\,\partial_y F + R\,\partial_z F = KF, \tag{1.3}$$

for some polynomial $K = K(x, y, z) \in \mathbb{C}[x, y, z]$, called the cofactor of $F = 0$ and for system (1.1) has at most degree one. Note that ∂_x, ∂_y and ∂_z denotes the partial derivatives with respect to x, y and z respectively. If the cofactor is zero then the Darboux polynomial reduces to polynomial first integral [2, 7].

The paper is organized as follows: in Section 2 we give an approach for proving the non-existence of polynomial first integrals for a family of subcases in system (1.1). Section 3 is devoted to conclude the ideas and results of this paper.

2 The non-existence of the polynomial first integral.

In this section, we mainly study the non-existence of polynomial first integrals for some subfamilies of Lotka–Volterra system (1.1) in which those subfamilies admit two independent analytic first integrals. All considered subfamilies are obtained in integrability conditions in Theorem 5.1, in [10]. Note that, Theorem 2.1 to Theorem 2.11 correspond to integrability conditions (1, 2*, 5, 6, 7, 11, 12, 34, 39, 40) of Theorem 5.1 in [10] respectively.

Theorem 2.1. *The subsystem*

$$\begin{aligned} \dot{x} &= x(1 + ax + by), \\ \dot{y} &= y(-3 + ey), \\ \dot{z} &= z(1 + hy + kz), \end{aligned} \tag{2.1}$$

has no polynomial first integrals if $a = 0$.

Proof. Assume that $H = H(x, y, z)$ is a polynomial first integral of system (2.1). Then, this implies

$$x(1+ax+by)\frac{\partial H(x,y,z)}{\partial x} + y(-3+ey)\frac{\partial H(x,y,z)}{\partial y} + z(1+hy+kz)\frac{\partial H(x,y,z)}{\partial z} = 0. \tag{2.2}$$

We can also assume $H(x, y, z) = \sum_{i=0}^{n} H_i(y, z)x^i$, where $H_i(y, z)$ is a polynomial in the variables y and z for each i and $n \in \mathbb{N} \cup \{0\}$. The terms of degree $n + 1$ of variable x in equation (2.2), satisfy

$$naH_n(y, z) = 0. \tag{2.3}$$

We consider the following two cases.

(i) If $a = 0$, we compute the terms of x^n in (2.2) which satisfy the partial differential equation

$$y(-3+ey)\frac{\partial H_n(x,y)}{\partial y} + z(1+hy+kz)\frac{\partial H_n(x,y)}{\partial z} + n(1+by)H_n(y,z) = 0,$$

and its solution by a software system Maple is

$$H_n(y,z) = y^{\frac{n}{3}}(-3+ey)^{\frac{-n(e+3b)}{3e}} F_n\Big(\frac{(-1)^{1-\frac{h}{e}} 3^{\frac{3h+e}{3e}} kz\,{}_2\mathrm{F}_1(-\frac{1}{3}, \frac{2e-3h}{3e}; \frac{2}{3}; \frac{ey}{3}) + (-3+ey)^{\frac{3h+e}{3e}}}{y^{\frac{1}{3}} z}\Big), \tag{2.4}$$

where F_n is a function in the variables y and z and ${}_2\mathrm{F}_1(-\frac{1}{3}, \frac{2e-3h}{3e}; \frac{2}{3}; \frac{ey}{3})$ is a hypergeometric function. Since H_n is a polynomial, then from the solution (2.4) it must be $F_n = 0$ and hence $H_n = 0$. This is a contradiction.

(ii) If $n = 0$, then the variable x does not involve in the polynomial first integral and $H = H_0(y, z)$. The relation (2.2) implies,

$$y(-3 + ey)\frac{\partial H_0(y, z)}{\partial y} + z(1 + hy + kz)\frac{\partial H_0(y, z)}{\partial z} = 0.$$

The function

$$H_0(y, z) = F_0\Big(\frac{(-1)^{1-\frac{h}{e}} 3^{\frac{3h+e}{3e}} kz {}_2\mathrm{F}_1(-\frac{1}{3}, \frac{2e-3h}{3e}; \frac{2}{3}; \frac{ey}{3}) + (-3 + ey)^{\frac{3h+e}{3e}}}{y^{\frac{1}{3}} z}\Big),$$

satisfies the last partial differential equation where F_0 is a function in the variables y and z. Since H_0 is a polynomial, then F_0 must be a constant. So, the function $H = H_0$ must also be a constant. This is a contradiction.

Note that, if both $e = -3b$ and $a = 0$, then (2.1) has a polynomial first integral $H_0 = F_0(x^3 y)$. □

Theorem 2.2. *The system*

$$\begin{aligned} \dot{x} &= x(1 + ax + by - kz), \\ \dot{y} &= y(-3 + ey), \\ \dot{z} &= z(1 + hy + kz), \end{aligned} \tag{2.5}$$

admits no polynomial first integrals.

Proof. Let $H = H(x, y, z)$ be a polynomial first integral of system (2.5). Then H satisfies the partial differential equation

$$x(1+ax+by-kz)\frac{\partial H(x, y, z)}{\partial x} + y(-3+ey)\frac{\partial H(x, y, z)}{\partial y} + z(1+hy+kz)\frac{\partial H(x, y, z)}{\partial z} = 0. \tag{2.6}$$

Without lost of generality we can write $H(x, y, z) = \sum_{i=0}^{n} H_i(x, y)z^i$, where for each i, $H_i(x, y)$ is a polynomial in the variables x and y and the degree $n \in \mathbb{N} \cup \{0\}$. In equation (2.6), the terms in z^{n+1} satisfy

$$k\Big(x\frac{\partial H_n(x, y)}{\partial y} - nH_n(x, y)\Big) = 0. \tag{2.7}$$

We distinguish the following two cases.

(i) If $k = 0$. The terms of variable z of degree n in equation (2.6) we have

$$x(1 + ax + by)\frac{\partial H_n(x, y)}{\partial x} + y(-3 + ey)\frac{\partial H_n(x, y)}{\partial y} + n(1 + hy)H_n(x, y) = 0.$$

The solution of the equation above is

$$H_n(x, y) = y^{\frac{n}{3}}(-3+ey)^{\frac{-n(e+3h)}{3e}} F_n\Big(\frac{(-1)^{1-\frac{b}{e}} 3^{\frac{3b+e}{3e}} ax {}_2\mathrm{F}_1(-\frac{1}{3}, \frac{2e-3b}{3e}; \frac{2}{3}; \frac{ey}{3}) + (-3 + ey)^{\frac{3b+e}{3e}}}{y^{\frac{1}{3}} x}\Big),$$

where F_n is a function in the variables x and y. We now proceed as Case (i) in Theorem 2.1.

(ii) If $k \neq 0$, then

$$H_n(x, y) = F_n(y)x^n, \tag{2.8}$$

where F_n is an arbitrary function in y. We first consider the case if $n = 0$, and from equation (2.6), we have

$$x(1 + ax + by)\frac{\partial H_0(x, y)}{\partial x} + y(-3 + ey)\frac{\partial H_0(x, y)}{\partial y} = 0.$$

The solution of the equation above is

$$H_0(x,y) = F_0\Big(\frac{(-1)^{1-\frac{b}{e}}3^{\frac{3b+e}{3e}}ax_2\mathrm{F}_1(-\frac{1}{3},\frac{2e-3b}{3e};\frac{2}{3};\frac{ey}{3}) + (-3+ey)^{\frac{3b+e}{3e}}}{y^{\frac{1}{3}}x}\Big), \tag{2.9}$$

where F_0 is a function of the variables x and y. Since F_0 is not a polynomial then it must be constant. Therefore, the function H_0 must also be constant and this is a contradiction.

We now investigate the case where the first integral H is of degree $n > 0$. From equation (2.8), the function $F_n(y)$ must be a constant C_n and the solution becomes

$$H_n(x,y) = C_n x^n.$$

Next, computing the terms in z^n in equation (2.6) which is

$$k\Big(-x\frac{\partial H_{n-1}(x,y)}{\partial x} + (n-1)H_{n-1}(x,y)\Big) + nC_n\Big(ax + (2+(b+h)y)\Big)x^n = 0. \tag{2.10}$$

The function

$$H_{n-1}(x,y) = \big(\frac{n}{k}C_n(\frac{a}{2}x^2 + (b+h)xy + 2x) + F_{n-1}(y)\big)x^{n-1}, \tag{2.11}$$

satisfies the equation (2.10). Since H_{n-1} is a polynomial of degree $n-1$, then must the term $\frac{n}{k}C_n(\frac{a}{2}x^2 + (b+h)xy + 2x)$ be zero and this only holds if $C_n = 0$ which is a contradiction.

□

Theorem 2.3. *Consider the three parametric family*

$$\begin{aligned} \dot{x} &= x(1-gx-hy), \\ \dot{y} &= y(-3-gx-hy), \\ \dot{z} &= z(1+gx+hy+kz). \end{aligned} \tag{2.12}$$

Then the system has no polynomial first integrals.

Proof. Assume $H(x,y,z) = \sum_{i=0}^{n} H_i(x,z)y^i$ is a polynomial first integral of system (2.12), where each $H_i(x,z)$ is a polynomial in the variables x and z of degree i. Then, $H(x,y,z)$ satisfies

$$x(1-gx-hy)\frac{\partial H(x,y,z)}{\partial x} + y(-3-gx-hy)\frac{\partial H(x,y,z)}{\partial y} + z(1+gx+hy+kz)\frac{\partial H(x,y,z)}{\partial z} = 0. \tag{2.13}$$

The coefficient of y^{n+1} in equation (2.13) is

$$h\Big(-x\frac{\partial H_n(x,z)}{\partial x} + z\frac{\partial H_n(x,z)}{\partial z} - nkH_n(x,z)\Big) = 0. \tag{2.14}$$

We consider two cases.

(i) When $h = 0$, the coefficient of y^n in equation (2.13) is

$$x(1-gx)\frac{\partial H_n(x,z)}{\partial x} + z(1+gx+kz)\frac{\partial H_n(x,z)}{\partial z} - n(3+gx)H_n(x,z) = 0.$$

It has a solution

$$H_n(x,z) = x^{3n}(-1+gx)^{-4n}F_n\big(\frac{2gx+kz}{2gz(1-gx)^2}\big),$$

where F_n is an arbitrary function in the variables x and z. Thus, $H_n(x,z)$ is not a polynomial of degree n. This is a contradiction.

(ii) When $h \neq 0$, equation (2.14) has a solution

$$H_n(x,z) = F_n(xz)x^{-n},$$

where F_n is an arbitrary function in the variable xz.
There is a possibility that the variable y is missing in the polynomial first integral H_n where $n = 0$. Then, $H = H_0(x,z)$ and it satisfies

$$x(1-gx)\frac{\partial H_0(x,z)}{\partial x} + z(1+gx+kz)\frac{\partial H_0(x,y)}{\partial z} = 0.$$

Solving equation above, we obtain

$$H_0(x,z) = F_0(\frac{2gx+kz}{2gz(1-gx)^2}),$$

where F_0 is a function in the variable $\frac{2gx+kz}{2gz(1-gx)^2}$ and it is not a polynomial. This is a contradiction.
For $n > 0$, the coefficient of y^n in equation (2.13) is

$$h(-x\frac{\partial H_{n-1}(x,z)}{\partial x} + z\frac{\partial H_{n-1}(x,z)}{\partial z} - (n-1)H_{n-1}(x,z)) + x^{1-n}z(2+kz)DF_n(xz) \\ - 4nF_n(xz)x^{-n} = 0,$$

and this differential equation has a solution

$$H_{n-1}(x,z) = \frac{x^{-n}}{h}\Big(-2xz(1+\frac{kz}{2})DF_n(xz) + 4nF_n(xz)\Big) + F_{n-1}(xz)x^{1-n},$$

where F_{n-1} is a function in the variable xz. Since $x^{-n}F_n(xz)$ is also in H_n, then must $F_n(xz) = 0$ and consequently $H_n = 0$. This is a contradiction.

□

Theorem 2.4. *The system*

$$\begin{aligned} \dot{x} &= x(1+dx+ey), \\ \dot{y} &= y(-3+dx+ey), \\ \dot{z} &= z(1-2dx-2ey+kz), \end{aligned} \tag{2.15}$$

admits no polynomial first integrals.

Proof. Let $H(x,y,z)$ be a polynomial first integral of system (2.15). We write $H(x,y,z)$ as a polynomial in the variable x as $H(x,y,z) = \sum_{i=0}^{n} H_i(y,z)x^i$, where each $H_i(y,z) \in \mathbb{C}[y,z]$. The coefficient of x^{n+1} in

$$x(1+dx+ey)\frac{\partial H(x,y,z)}{\partial x} + y(-3+dx+ey)\frac{\partial H(x,y,z)}{\partial y} + z(1-2dx-2dy+kz) \\ \frac{\partial H(x,y,z)}{\partial z} = 0, \tag{2.16}$$

is

$$d(y\frac{\partial H_n(y,z)}{\partial y} - 2z\frac{\partial H_n(y,z)}{\partial z} + nkH_n(y,z)) = 0. \tag{2.17}$$

We first consider the case when $d = 0$. Computing the coefficient of x^n in (2.16)

$$y(-3+ey)\frac{\partial H_n(y,z)}{\partial y} + z(1-2ey+kz)\frac{\partial H_n(y,z)}{\partial z} + n(1+ey)H_n(y,z) = 0. \tag{2.18}$$

The function

$$H_n(y,z) = y^{\frac{n}{3}}(ey-3)^{\frac{-4n}{3}} F_n\Big(\frac{kz((ey)^2-5ey+5)+5}{5(ey-3)^{\frac{5}{3}} zy^{\frac{n}{3}}}\Big),$$

satisfies (2.18) where F_n is a function in the variables y and z. It is clear that for any value of n, $H_n(y,z)$ is not a polynomial. This is a contradiction.
Now if $d \neq 0$ then the solution of equation (2.17) is

$$H_n(y,z) = F_n(zy^2)y^{-n},$$

where F_n is an arbitrary function in the variable zy^2.
If $n = 0$, we obtain $H(x,y,z) = H_0(y,z)$ and it satisfies

$$y(-3+ey)\frac{\partial H_0(y,z)}{\partial y} + z(1-2ey+kz)\frac{\partial H_0(y,z)}{\partial z} = 0.$$

Solving this differential equation we obtain

$$H_0(y,z) = F_0\Big(\frac{5+kz((ey)^2-4ey+5)}{5zy^{\frac{1}{3}}(ey-3)^{\frac{5}{3}}}\Big),$$

where F_0 is a function of variable $\frac{5+kz((ey)^2-4ey+5)}{5zy^{\frac{1}{3}}(ey-3)^{\frac{5}{3}}}$ which is not a polynomial. This is also a contradiction.
Now if $n > 0$ the coefficient of x^n in equation (2.16) is

$$d(y\frac{\partial H_{n-1}(y,z)}{\partial y} - 2z\frac{\partial H_{n-1}(y,z)}{\partial z} + (n-1)H_{n-1}(y,z)) + y^{2-n}z(-5+kz)DF_n(zy^2) \\ + 4nF_n(zy^2)y^{-n} = 0.$$

It has a solution

$$H_{n-1}(y,z) = \frac{y^{-n}}{3d}\Big(zy^2(-5+kz)DF_n(zy^2) + 12nF_n(zy^2)\Big) + F_{n-1}(zy^2)y^{1-n},$$

where F_{n-1} is a function in the variable zy^2.

Since $y^{-n}F_n(zy^2)$ is also in H_n, then must $F_n(zy^2) = 0$ and eventually $H_n = 0$. This is a contradiction.

□

Theorem 2.5. *The system*

$$\begin{aligned} \dot{x} &= x(1+ax+hy+cz), \\ \dot{y} &= y(-3+ey), \\ \dot{z} &= z(1+gx+hy+kz), \end{aligned} \tag{2.19}$$

has no polynomial first integrals.

Proof. We assume that the polynomial first integral of system (2.19) is $H(x,y,z) = \sum_{i=0}^{n} H_n(x,y)z^i$, where for each i, $H_i(x,y)$ is a polynomial in the variables x and y. Then it satisfies

$$x(1+ax+hy+cz)\frac{\partial H(x,y,z)}{\partial x} + y(-3+ey)\frac{\partial H(x,y,z)}{\partial y} + z(1+gx+hy+kz) \\ \frac{\partial H(x,y,z)}{\partial z} = 0. \tag{2.20}$$

The coefficient of z^{n+1} in equation (2.20) is

$$cx\frac{\partial H_n(x,y)}{\partial x} + nkH_n(x,y) = 0.$$

Solving this partial differential equation, we obtain

$$H_n(x,y) = F_n(y)x^{\frac{-nk}{c}}, \tag{2.21}$$

where F_n is a function in y.
The following cases are considered.

(i) If $k = 0$ and $n \neq 0$ then $H_n(x,y) = F_n(y)$ is a function in the variable y alone. The coefficient of z^n in equation (2.20) is

$$cx\frac{\partial H_{n-1}(x,y)}{\partial x} + y(-3+ey)\frac{dF_n(y)}{dy} + n(1+gx+hy)F_n(y) = 0,$$

which has a solution

$$H_{n-1}(x,y) = \frac{1}{c}(y\ln(x)(-3+ey)\frac{dF_n(y)}{dy} - n(\ln(x)(1+hy)+gx)F_n(y)) + F_{n-1}(y),$$

where F_{n-1} is an arbitrary function in the variable y. Since $H_{n-1}(x,y)$ is a polynomial of degree $n-1$, then $ngxF_n(y) = 0$ and

$$y(-3+ey)\frac{dF_n(y)}{dy} - n(1+hy)F_n(y) = 0. \tag{2.22}$$

From $ngxF_n(y) = 0$ must $F_n(y) = 0$ if $g \neq 0$ and this implies (2.22) vanishes as well and consequently $H_n = 0$. This is a contradiction.
If $g = 0$, we solve equation (2.22) and its solution is

$$F_n(y) = C_n y^{\frac{n}{3}}(-3+ey)^{-\frac{n}{3}(\frac{e+3h}{e})},$$

where C_n ia an arbitrary constant. Since $n \neq 0$, then $F_n(y)$ is not a polynomial and eventually $H_n(x,y)$ is not a polynomial. This is also a contradiction.

(ii) If $n = 0$, then the variable z will not appear in the polynomial first integral and $H = H_0(x,y)$. Then it satisfies the partial differential equation

$$x(1+ax+hy)\frac{\partial H_0(x,y)}{\partial x} + y(-3+ey)\frac{\partial H_0(x,y)}{\partial y} = 0,$$

and its solution is

$$H_0(x,y) = F_0(\frac{1}{xy^{\frac{1}{3}}}((-3+ey)^{\frac{e+3h}{3e}} - ax(-1)^{\frac{2e-3h}{3e}}3^{\frac{e+3h}{3e}}{}_2\mathrm{F}_1(-\frac{1}{3},\frac{2e-3h}{3e};\frac{2}{3};\frac{ey}{3}))),$$

where F_0 is a function in the variables x and y and the function ${}_2\mathrm{F}_1(-\frac{1}{3},\frac{2e-3h}{3e};\frac{2}{3};\frac{ey}{3})$ is a hypergeometric function. Therefore, it is not a polynomial. This is a contradiction.

(iii) If $k = -c$. Then (2.21) becomes

$$H_n(x,y) = F_n(y)x^n,$$

where F_n is a polynomial in the variable y.
The proof is the same as Case(ii) in Theorem (2.2).

□

Theorem 2.6. *The system*

$$\begin{aligned} \dot{x} &= x(1+ax+by-2kz), \\ \dot{y} &= y(-3-3ax-3by), \\ \dot{z} &= z(1+4ax-8by+kz), \end{aligned} \tag{2.23}$$

has no polynomial first integrals.

Proof. Let $H(x,y,z)$ be a polynomial first integral of system (2.23). Without lost of generality, we can write $H(x,y,z) = \sum_{i=0}^{n} H_i(x,y)z^i$ where for each i, $H_i(x,y)$ is a polynomial in the variables x and y. Then $H(x,y,z)$ satisfies

$$x(1+ax+by-2kz)\frac{\partial H(x,y,z)}{\partial x} + y(-3-3ax-3by)\frac{\partial H(x,y,z)}{\partial y} + z(1+4ax-8by+kz)\frac{\partial H(x,y,z)}{\partial z} = 0. \quad (2.24)$$

The terms of degree $n+1$ in equation (2.24) are

$$k\Big(-2x\frac{\partial H_n(x,y)}{\partial x} + nH_n(x,y)\Big) = 0. \quad (2.25)$$

First we consider the case when $k=0$.
The terms of degree n in equation (2.24) are

$$x(1+ax+by)\frac{\partial H_n(x,y)}{\partial x} - 3y(1+ax+by)\frac{\partial H_n(x,y)}{\partial y} + n(1+4ax-8by)H_n(x,y) = 0. \quad (2.26)$$

The solution of equation (2.26) is

$$H_n(x,y) = F_n(yx^3)(1+ax+by)^{-3n}x^{-n}, \quad (2.27)$$

where F_n is a polynomial in the variable yx^3. We see that the function $H_n(x,y)$ is not a polynomial. When $n=0$ we can obtain a polynomial first integral of the form $H_0(x,y) = F_0(yx^3)$.
Now, if $k \neq 0$, then the solution of equation (2.25) is

$$H_n(x,y) = F_n(y)x^{\frac{n}{2}},$$

where F_n is a function in y.
If $n \neq 0$, we get a fractional power and this is a contradiction.
If $n=0$, then equation (2.24) becomes

$$(1+ax+by)\Big(x\frac{\partial H_0(x,y)}{\partial x} - 3y\frac{\partial H_0(x,y)}{\partial y}\Big) = 0,$$

the equation above have the polynomial first integral $H_0(x,y) = F_0(yx^3)$.

□

Theorem 2.7. *If $e \neq -3h$ then the system*

$$\begin{aligned} \dot{x} &= x(1+ax), \\ \dot{y} &= y(-3-2ax+ey), \\ \dot{z} &= z(1+gx+hy), \end{aligned} \quad (2.28)$$

has no polynomial first integrals.

Proof. We propose that a polynomial first integral $H(x,y,z)$ exists of the system (2.28). Therefore, it can be expressed as $H(x,y,z) = \sum_{i=0}^{n} H_i(y,z)x^i$, where each $H_i(y,z)$ are polynomials in (y,z), $H_n \neq 0$. Then $H(x,y,z)$ satisfies

$$x(1+ax)\frac{\partial H(x,y,z)}{\partial x} + y(-3-2ax+ey)\frac{\partial H(x,y,z)}{\partial y} + z(1+gx+hy)\frac{\partial H(x,y,z)}{\partial z} = 0. \quad (2.29)$$

The terms of degree $n+1$ in equation (2.29) become

$$-2ay\frac{\partial H_n(y,z)}{\partial y} + gz\frac{\partial H_n(y,z)}{\partial z} + anH_n(y,z) = 0,$$

and whose solution is

$$H_n(y,z) = F_n(zy^{\frac{g}{2a}})y^{\frac{n}{2}}, \quad (2.30)$$

where F_n is a function in the variable $zy^{\frac{g}{2a}}$.
We consider the following two cases.

(i) If $n = 0$, then the variable x will disappear in the polynomial first integral and $H = H_0(x, y)$. Then it satisfies the differential equation

$$y(ey-3)\frac{\partial H_0(y,z)}{\partial y} + z(1+hy)\frac{\partial H_0(y,z)}{\partial z} = 0. \tag{2.31}$$

Solving it, we obtain

$$H_0(y,z) = F_0(z^3 y(ey-3)^{-\frac{3h+e}{e}}). \tag{2.32}$$

Note that H_0 will be a polynomial only when $e = -3h$, but since $e \neq -3h$, so we get a contradiction.

(ii) If $n > 0$ and $g = ma$, where $m \in \mathbb{Z} \setminus \mathbb{Z}^-$ then we have two subcases.

1) If $m = 0$, then $H_n(y,z) = F_n(z)y^{\frac{n}{2}}$. Since $n \neq 0$, we get a fractional power and this is a contradiction.

2) If m is positive integer then $H_n(y,z) = F_n(zy^{\frac{m}{2}})y^{\frac{n}{2}}$.
The coefficient of x^n in equation (2.29) is

$$a(-2y\frac{\partial H_{n-1}(y,z)}{\partial y} + mz\frac{\partial H_{n-1}(y,z)}{\partial z} + (n-1)H_{n-1}(y,z)) + \frac{z}{2}(2-3m+(2h+em)y)$$
$$DF_n(zy^{\frac{m}{2}})y^{\frac{m+n}{2}} - \frac{n}{2}(1-ey)F_n(zy^{\frac{m}{2}})y^{\frac{n}{2}} = 0.$$

Solving the equation above yields

$$H_{n-1}(y,z) = \frac{1}{6a}(z((em+2h)y-3(3m-2))DF_n(zy^{\frac{m}{2}})y^{\frac{n+m}{2}} + n(ey-3)F_n(zy^{\frac{m}{2}})y^{\frac{n}{2}}).$$

Since H_{n-1} is a polynomial of degree $n-1$ then must be

$$z((em+2h)y-3(3m-2))DF_n(zy^{\frac{m}{2}})y^{\frac{n+m}{2}} + n(ey-3)F_n(zy^{\frac{m}{2}})y^{\frac{n}{2}} = 0.$$

The transformation $u = zy^{\frac{m}{2}}$ gives

$$(((em+2h)y-9m+6)uDF_n(u) + n(ey-3)F_n(u))y^{\frac{n}{2}} = 0,$$

it has a solution

$$F_n(u) = C_n u^{\frac{-n(ey-3)}{(em+2h)y-3(3m-2)}},$$

and pull back to the solution, we obtain

$$F_n(zy^{\frac{m}{2}}) = C_n(zy^{\frac{m}{2}})^{\frac{-n(ey-3)}{(em+2h)y-3(3m-2)}},$$

then

$$H_n(y,z) = C_n(zy^{\frac{m}{2}})^{\frac{-n(ey-3)}{(em+2h)y-3(3m-2)}}y^{\frac{n}{2}}.$$

It is clearly that $H_n(y,z)$ is not a polynomial.

□

Theorem 2.8. *If $d \neq -3a$, then the differential system*

$$\begin{aligned} \dot{x} &= x(1+ax+cz), \\ \dot{y} &= y(-3+dx+fz), \\ \dot{z} &= z(1+gx+kz), \end{aligned} \tag{2.33}$$

has no polynomial first integrals.

Proof. Let the system (2.33) has a polynomial first integral $H(x,y,z)$. It can be written as a polynomial in the variable z. That is, $H(x,y,z)=\sum_{i=0}^{n} H_i(x,y)z^i$ where $H_i(x,y)$ are polynomials in the remaining variables. So $H(x,y,z)$ satisfies

$$x(1+ax+cz)\frac{\partial H(x,y,z)}{\partial x}+y(-3+dx+fz)\frac{\partial H(x,y,z)}{\partial y}+z(1+gx+kz)\frac{\partial H(x,y,z)}{\partial z}=0. \quad (2.34)$$

The coefficient of z^{n+1} in equation (2.34) is

$$cx\frac{\partial H_n(x,y)}{\partial x}+fy\frac{\partial H_n(x,y)}{\partial y}+nkH_n(x,y)=0,$$

whose solution is

$$H_n(x,y)=F_n(yx^{-\frac{f}{c}})x^{-\frac{kn}{c}}, \quad (2.35)$$

where F_n is a function in the variable $yx^{-\frac{f}{c}}$. The following cases are considered.

(i) When $k=0$ we have several subcases.

(1) If $f=-c$ then

$$H_n(x,y)=F_n(xy),$$

where F_n is an arbitrary function in the variable xy.
The coefficient of z^n in equation (2.34) is

$$c(x\frac{\partial H_{n-1}(x,y)}{\partial x}-y\frac{\partial H_{n-1}(x,y)}{\partial y})+xy(-2+x(a+d))DF_n(xy)+n(1+gx)F_n(xy)=0.$$

It has a solution

$$H_{n-1}(x,y)=F_{n-1}(xy)-\frac{1}{c}(((a+d)yx^2-2xy\ln x)DF_n(xy)+n(\ln x+gx)F_n(xy)),$$

where F_{n-1} is an arbitrary function in the variable xy. Since the power of $H_{n-1}(x,y)$ is higher than $n-1$ then must be

$$-\frac{1}{c}(((a+d)yx^2-2xy\ln x)DF_n(xy)+n(\ln x+gx)F_n(xy))=0. \quad (2.36)$$

If we let $u=xy$ then the differential equation above reduces to

$$\frac{1}{cy}(u(2y\ln\frac{u}{y}-(a+d)u)DF_n(u)-n(y\ln\frac{u}{y}+gu)F_n(u))=0. \quad (2.37)$$

Solving equation (2.37), we get

$$F_n(u)=C_n\exp^{\int(\frac{n(y\ln\frac{u}{y}+gu)}{u(2y\ln\frac{u}{y}-(a+d)u)})du}.$$

Where C_n is a constant and F_n is not a polynomial, then we get a contradiction.

(2) If $f=0$ then equation (2.35) becomes

$$H_n(x,y)=F_n(y),$$

where F_n is a function in y. The coefficient of z^n in equation (2.34) is

$$cx\frac{\partial H_{n-1}(x,y)}{\partial x}+y(-3+ey)DF_n(y)+n(1+gx)F_n(y)=0.$$

It has a solution

$$H_{n-1}(x,y) = F_{n-1}(y) - \frac{1}{c}(y(dx - 3\ln x)DF_n(y) + n(\ln x + gx)F_n(y)),$$

where F_{n-1} is an arbitrary function in the variable y. Since the powers of $H_{n-1}(x,y)$ is higher then $n-1$ then must be

$$-\frac{1}{c}(y(dx - 3\ln x)DF_n(y) + n(\ln x + gx)F_n(y)) = 0.$$

The solution of the differential equation above is

$$F_n(y) = C_n y^{\frac{n(gx+\ln(x))}{-dx+3\ln(x)}},$$

we get that $F_n(y)$ is not a polynomial. Consequently the function $H_n(x,y)$ is not a polynomial which contradict with our assumption.

(ii) If $n = 0$ and $k \neq 0$ then the equation (2.34) does not depend in variable z. Then it becomes

$$x(1+ax)\frac{\partial H_0(x,y)}{\partial x} + y(-3+dx)\frac{\partial H_0(x,y)}{\partial y} = 0,$$

and whose solution is

$$H_0(x,y) = F_0(\frac{yx^3(1+ax)^{-\frac{d}{a}}}{(1+ax)^3}), \tag{2.38}$$

where F_0 is a function in the variables x and y. Since $d \neq -3a$, then $H_0(x,y)$ is not a polynomial. This is a contradiction.

(iii) If $k = -c$, then

$$H_n(x,y) = F_n(yx^{\frac{-f}{c}})x^n, \tag{2.39}$$

where F_n is an arbitrary function in the variable $yx^{\frac{-f}{c}}$. Since $H_n(x,y)$ is a polynomial of degree n then $F_n(yx^{-\frac{f}{c}})$ must be a constant and let it is equal to C_n.
The coefficient of z^n in equation (2.34) is

$$c(x\frac{\partial H_{n-1}(x,y)}{\partial x} - y\frac{\partial H_{n-1}(x,y)}{\partial y} - (n-1)H_{n-1}(x,y)) + n(x(g+a)+2)C_n x^n = 0.$$

This equation has a solution

$$H_{n-1}(x,y) = F_{n-1}(yx^{\frac{-f}{c}})x^{(n-1)} - \frac{nC_n}{2c}(4+(a+g)x)x^n. \tag{2.40}$$

Where F_{n-1} is an arbitrary function in the variable $yx^{\frac{-f}{c}}$. Since the power of $H_{n-1}(x,y)$ is higher then $n-1$ then the term $\frac{nC_n}{2c}(4+(a+g)x)$ must be zero and it is hold only when $C_n = 0$. It is contradict with assumption $H_n(x,y)$ is a polynomial of degree n.

Remark 2.9. The system (2.33) where $n = 0$ and $k \neq 0$ with additional condition $d = -3a$, admits a polynomial first integral of form yx^3.

□

Theorem 2.10. *The subfamily*

$$\begin{aligned} \dot{x} &= x(1+ax-8hy+4kz), \\ \dot{y} &= y(-3+hy+kz), \\ \dot{z} &= z(1-2ax+hy+kz), \end{aligned} \tag{2.41}$$

admits no polynomial first integrals if $a \neq 0$.

Proof. Suppose that a polynomial $H = H(x,y,z)$ is a first integral of (2.41), and we write $H(x,y,z) = \sum_{i=0}^{n} H_i(y,z)x^i$, where each $H_i(y,z)$ is a polynomial. Then $H(x,y,z)$ satisfies

$$x(1+ax-8hy+4kz)\frac{\partial H(x,y,z)}{\partial x} + y(-3+hy+kz)\frac{\partial H(x,y,z)}{\partial y} + z(1-2ax+hy+kz)\frac{\partial H(x,y,z)}{\partial z} = 0. \quad (2.42)$$

The coefficient of x^{n+1} in equation (2.42) is

$$a(-2z\frac{\partial H_n(y,z)}{\partial z} + nH_n(y,z)) = 0. \quad (2.43)$$

Solving equation (2.43), we obtain

$$H_n(y,z) = F_n(y)z^{\frac{n}{2}},$$

where $a \neq 0$ and F_n is an arbitrary function in the variable y.
If $n > 0$, we get a fractional power and this is a contradiction.
If $n = 0$ then equation (2.42) reduces to

$$y(-3+hy+kz)\frac{\partial H_0(y,z)}{\partial y} + z(1+hy+kz)\frac{\partial H_0(y,z)}{\partial z} = 0, \quad (2.44)$$

and it has a solution

$$H_0(y,z) = F_0(\frac{-4096yz^3k^3}{875(-3+hy-3kz)^4}), \quad (2.45)$$

which is not a polynomial first integral. □

Theorem 2.11. *The system*

$$\begin{aligned}\dot{x} &= x(1-gx-5hy+3kz),\\ \dot{y} &= y(-3+gx-3hy-3kz),\\ \dot{z} &= z(1+gx+hy+kz).\end{aligned} \quad (2.46)$$

has no polynomial first integrals.

Proof. Consider $H(x,y,z)$ is a polynomial first integral of system (2.46), and suppose $H(x,y,z) = \sum_{i=0}^{n} H_i(y,z)x^i$, where each $H_i(y,z)$ is a polynomial in the variables y and z. Then it satisfies

$$x(1-gx-5hy+3kz)\frac{\partial H(x,y,z)}{\partial x} + y(-3+gx-3hy-3kz)\frac{\partial H(x,y,z)}{\partial y} + z(1+gx+hy+kz)\frac{\partial H(x,y,z)}{\partial z} = 0. \quad (2.47)$$

The terms of degree $n+1$ in the variable x in equation (2.47) obtain

$$g(y\frac{\partial H_n(y,z)}{\partial y} + z\frac{\partial H_n(y,z)}{\partial z} - nH_n(y,z)) = 0. \quad (2.48)$$

If $g = 0$ then the terms of degree n in the variable x in equation (2.47) become

$$-3y(1+hy+kz)\frac{\partial H_n(y,z)}{\partial y} + z(1+hy+kz)\frac{\partial H_n(y,z)}{\partial z} + n(1-5hy+3kz)H_n(y,z) = 0. \quad (2.49)$$

It has a solution

$$H_n(y,z) = F_n(z^3y)z^n(1+hy+kz)^{2n}. \quad (2.50)$$

Note that if $n = 0$, we can get the polynomial first integral of the form $H_0(y,z) = F_0(z^3y)$ and if $n \neq 0$ we get a contradiction.

More precisely, it is clear that from (2.46), the polynomial first integral is $H_0(y,z) = F_0(z^3y)$.
Now for $g \neq 0$ then the solution of the partial differential equation (2.48) is

$$H_n(y,z) = F_n(\frac{z}{y})y^n, \tag{2.51}$$

where F_n is an arbitrary function in the variable $\frac{z}{y}$.
We have two possibilities.
If $n = 0$ and from equation (2.47) we can get a polynomial first integral of the form

$$H_0(y,z) = F_0(z^3y).$$

If $n \neq 0$, we calculate the coefficient of x^n in equation (2.47) is

$$g(y\frac{\partial H_{n-1}(y,z)}{\partial y} + z\frac{\partial H_{n-1}(y,z)}{\partial z} - (n-1)H_{n-1}(y,z)) + 4z(1+hy+kz)y^{n-1}DF_n(\frac{z}{y}) \\ - 2n(1+4hy)y^nF_n(\frac{z}{y})) = 0.$$

It has a solution

$$H_{n-1}(y,z) = F_{n-1}(\frac{z}{y})y^{n-1} + \frac{y^{n-1}}{g}(-2z(2+hy+kz)DF_n(\frac{z}{y}) + 2n(1+2hy)F_n(\frac{z}{y})), \tag{2.52}$$

where F_{n-1} is an arbitrary function in the variable $\frac{z}{y}$.
Since $H_{n-1}(x,y)$ is a polynomial of degree $n-1$ then must

$$-2z(2+hy+kz)DF_n(\frac{z}{y}) + 2n(1+2hy)F_n(\frac{z}{y}) = 0, \tag{2.53}$$

and it has a solution

$$F_n(\frac{z}{y}) = C_n(2+hy+kz)^{\frac{-n(1+2hy)}{2+hy}}(zy^{-1})^{\frac{n(1+2hy)}{2+hy}},$$

where C_n is a constant. Since $n \neq 0$ then $F_n(\frac{z}{y})$ is not a polynomial. Therefore, $H_n(y,z)$. This is a contradiction. □

3 Conclusion

In this study, we proved that some subfamilies of three-dimensional Lotka-Volterra system (1.1) have no polynomial first integrals even though they admit two independent first integrals. Then we conclude that the existence of analytic first integrals does not guarantee the existence of polynomial first integrals. Some subsystems which have analytic first integrals and without any extra conditions have shown that they also admit no polynomial first integrals. However, in some cases, one has assumed extra condition to prove the non-existence of polynomial first integrals.

References

[1] W. Aziz, A. Amen, and C. Pantazi. Integrability and linearizability of a family of three-dimensional quadratic systems. Dynamical Systems 36, no. 2 (2021), 317-331.

[2] W. Aziz, C .Christopher, J. Llibre and C. Pantazi. Three-Dimensional Lotka–Volterra Systems with 3 : −1 : 2-Resonance. Mediterranean Journal of Mathematics 18, no. 4 (2021), 1-24.

[3] J. Giné, M. Grau, and J. Llibre. Polynomial and rational first integrals for planar quasi-homogeneous polynomial differential systems. Discrete and continuous dynamical systems Volume 33, Number 10 (2013), 4531-4547.

[4] J. Giné, M. Grau, and J. Llibre. Polynomial and rational first integrals for planar homogeneous polynomial differential systems. Publicacions matemàtiques (2014), 255-278. Proceedings of New Trends in Dynamical Systems. Salou, 2012.

[5] J. Llibre, and C. Valls. On the polynomial integrability of the Hoyer systems. Journal of Mathematical Analysis and Applications 399, no. 1 (2013), 411-417.

[6] J. Llibre, and C. Valls. Polynomial, rational and analytic first integrals for a family of 3-dimensional Lotka-Volterra systems. Zeitschrift für angewandte Mathematik und Physik 62, no. 5 (2011), 761-777.

[7] J. Llibre, and C. Valls. "Proper rational and analytic first integrals for asymmetric 3-dimensional Lotka-Volterra systems." Journal of Nonlinear Mathematical Physics 24, no. 3 (2017): 393-404.

[8] J. Llibre, and X. Zhang. Polynomial first integrals of quadratic systems. The Rocky Mountain journal of mathematics (2001) 1317-1371.

[9] J. Moulin-Ollagnier. Polynomial first integrals of the Lotka–Volterra system. Bulletin des Sciences Mathématiques 121 (1997) 463-476.

[10] C. Qadir, W. Aziz and I. Hamad. Rank Two Integral Aspects of Three Dimensional Lotka–Volterra Equations with Nonstandard Analysis. Submitted. https://arxiv.org.

[11] A. Tsygvintsev. On the existence of polynomial first integrals of quadratic homogeneous systems of ordinary differential equations. Journal of Physics A: Mathematical and General 34 (2001), 2185-2193.

Author information

Chiman Qadir, Waleed Aziz, Ibrahim Hamad, Department of Mathematics, College of Science, Salahaddin University–Erbil, Kurdistan Region–Iraq, IRAQ.
E-mail: chiman.qadir@su.edu.krd, waleed.aziz@plymouth.ac.uk, ibrahim.hamad@su.edu.krd