Palestine Journal of Mathematics

Vol. 12(Special Issue I)(2023) , 5–9

The lcm minus gcd and relations to the zeta function

Jamal Farokhi

Communicated by Rostam Kareem Saeed

MSC 2010 Classifications: 11A05 ,11S40 ,11A99, 11N05.

Keywords and phrases: Greatest common divisor, Zeta function, Prime number, Polylogarithm function.

Abstract In this paper, we introduce a two-variable arithmetic function with some essential properties $d(n,m) = lcm(n,m) - gcd(n,m)$ and then try to find some relations by infinity series connected to zeta function, Euler ϕ function, divisor function and some another foundation of number theory concepts and formulas.

1 Introduction

For a pair of nature numbers for example n and m, the greatest common divisor and least common multiple are denote respectively by gcd(n,m) and $lcm(n,m)$[1]. We can look at them as functions from $\mathbf{N}^2$ to $\mathbf{N}$.

2 Preliminary

In this section, we remind some basic number theory definitions that are necessary to continue our aims. In number theory, Euler's totient function counts the positive integers up to a given integer n that are relatively prime to n. It is written using the Greek letter phi as $\phi(n)$.

Definition 2.1. For each $k \in \mathbf{N}$, Jordan's totient function $\phi_k(n)$ is multiplicative and may be evaluated as

$$\phi_k(n) := \sum_{(i,n)=1} i^k$$

Definition 2.2. The sum of positive divisors function $\sigma_x(n)$, for a real or complex number x, is defined as the sum of the x-th powers of the positive divisors of n. It can be expressed in sigma notation as

$$\sigma_x(n) = \sum_{d|n} d^x.$$

where $d \mid n$ is shorthand for " d divides n. In special case $x = 0, \sigma_0(n)$ is the cardinal of n divisor set.

Definition 2.3. The Polylogarithm function is defined by a power series in $a(n) : \mathbf{N} \to \mathbf{C}$, which is also a Dirichlet series in s:

$$Li_s(a(n)) := \sum_{k=1}^{\infty} \frac{a(k)^k}{k^s} = a(1) + \frac{a(2)^2}{4} + \frac{a(3)^3}{9} + \dots$$

In case $z = 1, Li_s(1) = \zeta(s)$ is Riemann zeta function.

Definition 2.4. Suppose that $n, m \in \mathbf{N}$ are two arbitrary number, then we define a function $d : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that

$$d(n,m) := lcm(m,n) - \text{gcd}(n,m)$$

as lcm and gcd are well defined, then d is well defined and also d is a projection bout, not an injection.

Proposition 2.5. *For all* $n, m, r, s, h \in \mathbf{N}$ *:*

(i) $d(n,m) = d(m,n)$.

(ii) $d(hn, hm) = hd(n,m)$.

(iii) $n \mid m \Longrightarrow d(n,m) = m - n$.

(iv) $d(n,m) = \frac{\left(mn - gcd(m,n)^2\right)}{\gcd(n,m)}$.

(v) $gcd(n,m) = 1 \Longleftrightarrow d(n,m) = nm - 1$.

(vi) *If* $r \leq s \Longrightarrow d\left(n^s, n^r\right) = n^s - n^r$.

(vii) $d(n,m) = 0 \Longleftrightarrow n = m$.

Definition 2.6. Assume that $n, k \in \mathbf{N}$ are fixed numbers. then we define these three functions as follows:

(i) $\lambda_k(n) := \sum_{i|n} d(i,n)^k$.

(ii) $\gamma_k(n) := \sum_{\gcd(i,n)=1} d(i,n)^k$

(iii) $\omega_k(n) := \sum_{i=1}^{n} d(i,n)^k$

3 Main result

In this section, we prove some theorems by the above definitions.

Theorem 3.1. *Assume that* $n, k \in \mathbf{N}$ *are fixed numbers. Then:*

(i) $\lambda_k(n) = \sum_{j=0}^{k} n^j \begin{pmatrix} k \\ j \end{pmatrix} (-1)^{k-j} \sigma_{k-j}(n)$.

(ii) $\gamma_k(n) = \sum_{j=0}^{k} n^j \begin{pmatrix} k \\ j \end{pmatrix} (-1)^{k-j} \phi_j(n)$.

Proof. (i)

$$\begin{aligned}
\lambda_k(n) &= \sum_{i|n} d(i,n)^k \\
&= \sum_{i|n} (n-i)^k \\
&= \sum_{i|n} \left(\sum_{j=0}^{k} n^j (-i)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} \right) \\
&= \sum_{j=0}^{k} n^j (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} \Bigg) \left(\sum_{i|n} i^{k-j} \right) \\
&= \sum_{j=0}^{k} n^j \begin{pmatrix} k \\ j \end{pmatrix} (-1)^{k-j} \sigma_{k-j}(n)
\end{aligned}$$

(ii)

$$\begin{aligned}
\gamma_k(n) &= \sum_{\gcd(i,n)=1} d(i,n)^k \\
&= \sum_{i|n} (in-1)^k \\
&= \sum_{i|n} \left(\sum_{j=0}^{k} n^j (-i)^{k-j} \left(\begin{array}{c} k \\ j \end{array} \right) \right) \\
&= \sum_{j=0}^{k} n^j (-1)^{k-j} \left(\begin{array}{c} k \\ j \end{array} \right) \Bigg) \left(\sum_{\gcd(i,n)=1} i^j \right) \\
&= \sum_{j=0}^{k} n^j \left(\begin{array}{c} k \\ j \end{array} \right) (-1)^{k-j} \phi_j(n)
\end{aligned}$$

□

Theorem 3.2. *Suppose that p is a prime number and $k \in \mathbf{N}$, then:*
$\omega\left(p^k\right) = p^{k-1}\left(k - kp + \frac{p^{2k+2}-p^2}{2p+2}\right)$

Remark 3.3. Suppose that p is a prime number and $k \in \mathbf{N}$, then: In above theorem, for $k = 1$ we can simplify summations as $\lambda_1(n) = n\tau(n) - \sigma(n)$ and $\gamma_1(n) = \phi(n)\left(\frac{1}{2}n^2 - 1\right)$ such that $\tau(n) = J_0(n), \sigma(n) = J_1(n), \phi(n) \equiv \phi_1(n)$ and $\omega_1(n) = \omega(n)$. Also we denote $\lambda_1(n) = \lambda(n)$ and $\gamma_1(n) = \gamma(n)$. If $n = p$, be a prime number, then $\lambda_k(p) = (p-1)^k$ and $\gamma(n) = (p-1)\left(\frac{1}{2}n^2 - 1\right)$.

Theorem 3.4. *If $k \in \mathbf{N}$ and p be an arbitrary prime number, Then:*

(i) $\lambda\left(p^k\right) = \frac{p^k(kp-k-1)+1}{p-1}$.

(ii) $\gamma\left(p^k\right) = p^{k-1}\left(1 - p - \frac{1}{2}p^{2k} + \frac{1}{2}p^{2k} + 1\right)$.

Theorem 3.5. *If $n, m \in \mathbf{N}$, then:*

(i) $\lambda(nm) = \lambda(m)\lambda(n) + \tau(n)\tau(m)(m+n)$.

(ii) $\gamma(nm) = \gamma(m)\gamma(n) + \frac{1}{2}\phi(n)\phi(m)\left(\frac{1}{2}n^2m^2 - n^2 - m^2\right)$.

Proof. For the first equality, since τ and σ are multiplicative so:

$$\lambda(nm) = nm\tau(n)\tau(m) - \sigma(n)\sigma(m).(1)$$

Also by definition and (1):

$$\lambda(n)\lambda(m) = nm\tau(n)\tau(m) - n\tau(m)\tau(n) - m\tau(m)\tau(n) - \tau(m)\tau(n) \tag{3.1}$$
$$= \lambda(nm) - \tau(nm)(m+n) \tag{3.2}$$

So $\lambda(nm) = \lambda(m)\lambda(n) + \tau(n)\tau(m)(m+n)$.
For the second as ϕ is multiplicative:

$$\gamma(nm) = \frac{1}{2}n^2m^2\phi(n)\phi(m) - \phi(n)\phi(m).(2)$$

Also by definition and (2):

$$\gamma(n)\gamma(m) = \left(\frac{1}{2}n^2\phi(n) - \phi(n)\right)\left(\frac{1}{2}m^2\phi(m) - \phi(m)\right) \tag{3.3}$$
$$= \frac{1}{4}n^2m^2\phi(n)\phi(m) - \frac{1}{2}n^2\phi(n)\phi(m) \tag{3.4}$$
$$- \frac{1}{2}m^2\phi(n)\phi(m) - \phi(n)\phi(m) \tag{3.5}$$

So
$\gamma(nm) = \gamma(m)\gamma(n) + \frac{1}{2}\phi(n)\phi(m)\left(\frac{1}{2}n^2m^2 - n^2 - m^2\right)$ □

Theorem 3.6. *Lets that* p *be a prime number, so:*

(i) $\sum_{k=1}^{\infty} \frac{\lambda(p^k)}{k^s} = \frac{(p^2-2)Li_s(p)+\zeta(p)}{p-1}$.

(ii) $\sum_{k=1}^{\infty} \frac{\gamma(p^k)}{k^s} = \left(\frac{p-1}{p}\right)\left(\frac{1}{2}Li_s\left(p^3\right) - Li_s(p)\right)$.

(iii) $\sum_{k=1}^{\infty} \frac{\omega(p^k)}{k^s} = \left(\frac{1-p}{p}\right)Li_{s-1}(p) + \left(\frac{p}{2+2p}\right)\left(Li_s\left(p^3\right) - Li_s(p)\right)$.

Proof. (i)

$$\sum_{k=1}^{\infty} \frac{\lambda\left(p^k\right)}{k^s} = \left(\frac{1}{p-1}\right)\sum_{k=1}^{\infty}\left(\frac{kp^{k+1} - kp^k - p^k + 1}{k^s}\right) \tag{3.6}$$

$$= \left(\frac{p}{p-1}\right)\sum_{k=1}^{\infty}\frac{p^k}{k^{s-1}} - \left(\frac{1}{p-1}\right)\sum_{k=1}^{\infty}\frac{p^k}{k^{s-1}} \tag{3.7}$$

$$-\left(\frac{1}{p-1}\right)\sum_{k=1}^{\infty}\frac{p^k}{k^s} + \left(\frac{1}{p-1}\right)\sum_{k=1}^{\infty}\frac{1}{k^s} \tag{3.8}$$

$$= \left(\frac{p}{p-1}\right)Li_{s-1}(p) - \left(\frac{1}{p-1}\right)Li_{s-1}(p) \tag{3.9}$$

$$-\left(\frac{1}{p-1}\right)Li_{s-1}(p) + \left(\frac{1}{p-1}\right)\zeta(s) \tag{3.10}$$

$$= Li_{s-1}(p) + \left(\frac{1}{p-1}\right)\left(\zeta(s) - Li_s(p)\right) \tag{3.11}$$

(ii)

$$\sum_{k=1}^{\infty} \frac{\gamma\left(p^k\right)}{k^s} = \sum_{k=1}^{\infty}\left(\frac{p^{k-1} - p^k - \frac{1}{2}p^{3k-1} + \frac{1}{2}p^{3k}}{k^s}\right) \tag{3.12}$$

$$= \left(\frac{1}{p}\right)\sum_{k=1}^{\infty}\frac{p^k}{k^s} - \sum_{k=1}^{\infty}\frac{p^k}{k^s} - \left(\frac{1}{2p}\right)\sum_{k=1}^{\infty}\frac{\left(p^3\right)^k}{k^s} \tag{3.13}$$

$$+\left(\frac{1}{2}\right)\sum_{k=1}^{\infty}\frac{\left(p^3\right)^k}{k^s} \tag{3.14}$$

$$= \frac{1}{p}Li_s(p) - Li_s(p) - \frac{1}{2p}Li_s\left(p^3\right) - \frac{1}{2}Li_s\left(p^3\right) \tag{3.15}$$

$$= \left(\frac{p-1}{p}\right)\left(\frac{1}{2}Li_s\left(p^3\right) - Li_s(p)\right) \tag{3.16}$$

(iii)

$$\sum_{k=1}^{\infty}\frac{\omega\left(p^k\right)}{k^s}=\sum_{k=1}^{\infty}\left(\frac{kp^{k-1}-kp^k-\frac{p^{k+1}}{2p+2}+\frac{p^{3k+1}}{2p+2}}{k^s}\right) \tag{3.17}$$

$$=\left(\frac{1}{p}\right)\sum_{k=1}^{\infty}\frac{p^k}{k^{s-1}}-\sum_{k=1}^{\infty}\frac{p^k}{k^{s-1}}-\left(\frac{p}{2p+2}\right)\sum_{k=1}^{\infty}\frac{p^k}{k^s} \tag{3.18}$$

$$+\left(\frac{p}{2p+2}\right)\sum_{k=1}^{\infty}\frac{\left(p^3\right)^k}{k^s} \tag{3.19}$$

$$=\frac{1}{p}Li_{s-1}(p)-Li_{s-1}(p)-\frac{p}{2p+2}Li_s(p)+\frac{p}{2p+2}Li_s\left(p^3\right) \tag{3.20}$$

$$=\left(\frac{1-p}{p}\right)Li_{s-1}(p)+\left(\frac{p}{2p+2p}\right)\left(Li_s\left(p^3\right)-Li_s(p)\right) \tag{3.21}$$

□

Theorem 3.7. *Lets that $k \in \mathbf{N}$ be an arbitrary number and $s \in \mathbf{C}$ such that the real part of s is more significant than one, so:*

$$\sum_{n=1}^{\infty}\frac{\lambda_k(n)}{n^s}=\zeta(s-k)\sum_{j=0}^{k}\left(\begin{array}{c} j \\ k \end{array}\right)(-1)^{k-j}\zeta(s-j)$$

References

[1] Hardy, G. H.; Wright, E. M. (2008) [1938], An Introduction to the Theory of Numbers, Revised by D. R. Heath-Brown and J. H. Silverman. Foreword by Andrew Wiles. (6th ed.), Oxford

[2] Apostol, Tom M. (1976), Introduction to analytic number theory, Undergraduate Texts in Mathematics, New YorkHeidelberg: Springer-Verlag 3.

[3] Bach, Eric; Shallit, Jeffrey, Algorithmic Number Theory, volume 1, 1996, MIT Press.

[4] Ivi, Aleksandar (1985), The Riemann zeta-function. The theory of the Riemann zeta-function with applications, A Wiley-Interscience Publication, New York, etc.: John Wiley Sons, p p. 385 $\rightarrow$ 440

Author information

Jamal Farokhi, Department of Mathematics, Shahid Beheshti university, Evin, Tehran., IRI.
E-mail: `jamalfarokhi70@gmail.com`