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Estimating the Rate of Occurrence of Exponential Process Using Intelligence and Classical Methods with Application

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Abstract Non-Homogeneous Poisson Process models are widely used for modeling the stochastic phenomena in which the main focus is on the parameter estimation. In this paper, to estimate the parameter of the rate of occurrence described by the Exponential function for Poisson Process, the Firefly Algorithm is proposed. It is shown that the implementation of Firefly technique enables more accurate results and ensures the maximum value for the function in less iterations in comparison with each of Particle Swarm Optimization and Maximum Likelihood Estimation method. Furthermore, by using simulation techniques, it is declared that the Firefly approach is more appropriate for estimating the Exponential Process parameters; it provides more precise results than the other methods, Maximum Likelihood Estimation and Particle Swarm Optimization. The study involved a real application that deals with successive operating periods between two stops for two consecutive days for the mill of raw materials from the General Company for Northern Cement / Badush Cement Factories during the period from 01/04/2018 to 31/01/2019.

1 Introduction

Non homogeneous Poisson Process (NHPP), which is considered as the best-known generalization of the Poisson Process models, are mainly used for analyzing and modelling the failure data in recoverable systems. These models assume that the point process $N(t), t \geq 0$, with independent increments, is distributed as Poisson distribution and the occurrence of events in the Poisson process is random and monotonous during a certain period of time and with a fixed incidence rate per unit time denoted by the symbol λ, while the rate at which events occur in a non homogeneous Poisson process is time-varying t with the ratio called the time rate of occurrence or intensity function; it is denoted by the symbol $\lambda(t)$. The Exponential Process is a special model for NHPP that has appeared as a special case of the Weibull distribution; it plays a key role in modelling and analyzing failure data accumulated over time [12]. In this paper, Exponential distribution, which is commonly used to measure the expected time for an event to occur, is introduced and different methods for estimating its parameters are presented [15].

1.1 Exponential Process

Assume that the Poisson process $\{X\,(t)\,,\ t \geq 0\}$ represents a NHPP; the number of events that occur over a period of time $(0, t]$ following the Poisson distribution with a probability density function:

$$p\left[X\,(t) = x\right] = \frac{\left[\lambda(t)\right]^{x} e^{-m(t)}}{x!} \qquad , \quad x = 1, 2, 3, \ldots \tag{1.1}$$

$m(t)$ represents the cumulative function of the time rate of occurrence. It is defined by the following formula [8]:

$$m(t) = \int_{0}^{t} \lambda(u)du \qquad , \quad 0 < t < \infty \tag{1.2}$$

where $\lambda(u)$ represents the time rate of occurrence or intensity with the Exponential function for a non homogeneous Poisson process where:

$$\lambda(t) = \alpha e^{-\beta t}, 0 \leq \ t \ \leq \ \infty, \alpha, \beta > 0 \tag{1.3}$$

α, β are the two parameters for the intensity function and the problem is how to estimate these parameters. Parameter estimation process has extensively been studied and numerous techniques have been applied [9]. In this study, each of Firefly Algorithm (FFA), Particle Swarm Optimization (PSO) and Maximum Likelihood Estimation (MLE) are proposed to estimate parameters in Equation 1.3.

1.2 Firefly Algorithm (FFA)

The FFA was first proposed by Xin-She Yang at Cambridge University [17], the technique of this algorithm is based on the flashing pattern and the behavior of the fireflies at night. For optimization problems to gain optimal solutions, the objective function uses the differences in the light intensity and the formulation of attractiveness. Both help the fireflies to move towards brighter and more attractive locations. To construct the FFA, these flashing characteristics can be idealized based on the following three rules [5]:

- all fireflies are unisex so that one firefly is attracted to another one regardless of their sex;
- the attractiveness of each firefly is proportional to its light intensity which decreases as the distance from the other firefly increases. Thus, for any two types of blinking fireflies, the less bright one will move towards the brighter one and both will go down as the distance between them increases. If one firefly is not brighter than the other, it moves randomly;
- The brightness of the firefly will be affected or determined by the value of the target function to be improved; the brightness can simply be proportional to the objective function. Other forms of brightness can also be defined in a manner similar to the fitness function in genetic or bacterial foraging algorithms (BFA).

Therefore, by the second rule, a given firefly will be attracted by the brighter ones and the attractiveness of a firefly can be determined by its brightness or light intensity which in turn is associated with the value of the objective function. In the simplest case for maximum optimization problems, the light intensity $I(x)$ of a firefly representing the solution at a particular location x is proportional to the value of fitness function, $I(x) \ \propto \ f(x)$. The attractiveness of a fireflies, which is denoted by β , is proportional to their light intensities $I(r)$ thus, it should vary with the distance r_{ij} between firefly i and firefly j. As light intensity decreases with the distance from its source, and light is also absorbed in the media, so the attractiveness would vary based on the degree of absorption. In the simplest form, the light intensity $I(r)$ varies with the distance r monotonically and exponentially [11]:

$$I(r) \ = I_0 e^{-\gamma r} \tag{1.4}$$

Where I_0 represents the original light intensity and γ indicates the light absorption coefficient.

From elementary physics, it is clear that the intensity of light is inversely proportional to the square of the distance say r from the source. Thus, the variation of attractiveness β with the distance r can be defined by:

$$\beta r = \beta_0 e^{-\gamma r^2} \tag{1.5}$$

It is worth pointing out that the exponent γr can be replaced by other functions such as γr^m when $m > 0$, hence, the FFA can be summarized as the pseudo code.
Then, the distance between any two fireflies i and j at x_i and x_j can be the cartesian distance $r_{ij} \ = ||x_i \ - x_j \ ||_2$ or the ℓ_2-norm. The movement of a firefly i is attracted to another more

attractive (brighter) firefly j and the position update of the firefly located at X_i will be as follows:

$$X_{i+1} = X_i^t + \beta_0 e^{-\gamma r_{ij}^2} \left(X_j^t - X_i^t\right) + \alpha_t \epsilon_i^t \tag{1.6}$$

In Equation 1.6, the second term is due to the attractiveness towards X_j while the last term is a randomness term with α_t being a randomization parameter with $0 \leq \alpha_t \leq 1$ and ϵ_i^t is a vector of random numbers drawn from a Gaussian or Uniform or other distribution at time t.
In summary, FFA is controlled by three parameters: the randomization parameter α, the attractiveness β, and the absorption coefficient [6].

Furthermore, the following would be satisfied:

1 If $\beta_0 = 0$, it becomes a random walk where: $X_{i+1} = X_i^t + \alpha_t \epsilon_i^t$.

2 If $\gamma = 0$, the FFA reduces to standard PSO.

However, for implementation, various cases can be considered; for example, when β_0=1, $\alpha \in [0, 1]$, and $\gamma = 1$.

1.3 Particle Swarm Optimization Algorithm (PSO)

Particle Swarm Optimization (PSO) technique, which was first proposed by Eberhart and Kennedy in 1995 [4], is a recent approach for solving various optimization cases; it is a population-based stochastic optimization algorithm, with particle position and velocity iteratively updated. PSO is inspired by intelligence and social behavior of a collective behavior of some animals such as flocks of birds or schools of fish. Specifically, PSO algorithm maintains a population of particles, each of which represents a potential solution for an optimization problem in which each solution consists of a set of parameters in multidimensional space. The position of the particle denotes a feasible, if not the best, solution to the problem. The optimum progress is required to move the particle position in order to improve the value of objective function. Therefore, the main concept of PSO algorithm is that a population (swarm) of particles keeps moving around in a search-space to find a solution; each particle say i of the swarm is expressed by its position denoted by x_i and velocity denoted by v_i according to a simple formula.

The PSO algorithm can be described by the following steps:

- Initialize randomly the position, by assigning a random position, and velocity for each particle;
- Compute the fitness value of each particle;
- Determine the position of the best fitness value, the local best (pbest), of each particle and updated if it is better than the previous one;
- Calculate the position of best fitness value of all particle, the global best (gbest), and updated if it is better than the previous one;
- Repeat all steps until termination criteria are satisfied.

The following formulas are used to update the position and velocity for each particle:

$$V_i(t+1) = \omega V_i(t) + C_1 r_1 (P_i - X_i(t)) + C_2 r_2 (G - X_i(t)) \tag{1.7}$$

$$X_i(t+1) = X_i(t) + V_i(t+1) \tag{1.8}$$

Where $V_i(t)$ represents the velocity vector of particle s in t time; $X_i(t)$ represents the position vector of particles s in t time; P_i is the personal best position of particle s;G is the best position of the particle found at present; ω represents inertia weight; C_1, C_2 are two acceleration constants, called cognitive and social parameters, respectively; $r_1$and r_2 are two random functions in the range [1].

1.4 Maximum Likelihood Estimation (MLE) Method

The MLE is one of the most widely used methods for estimating the parameters of stochastic models in such a way that the estimators are most consistent with the observed data in addition to its good properties, including stability and Minimum Variance Unbiased Estimator [14].

1.5 Measures of the Accuracy

Measuring the accuracy for the estimation models show the closeness of the estimated result to its actual value. Accuracy is measured either by evaluating the difference between the estimated and the actual value or by the ratio. Measuring accuracy for estimated parameters are carried out using various measurement methods; one of these measures is the Maximum Percentage Error (MPE); it is considered as one of the measurements used to analyze the accuracy of the estimation by comparing its value with the estimated one. In this study MPE is used to compare each of MLE method, the FFA and the PSO algorithm applied to the Exponential Process Parameter Estimation; this measure is considered as fitness function for FFA algorithm and PSO algorithm. if S_i and $\widehat{S}_i$ are defined as follows [12],[4]:

$$S_i = \sum_{j=1}^{i} X_j \quad , \qquad \text{and} \quad \widehat{S}_i = \sum_{j=1}^{i} \widehat{X}_j \tag{1.9}$$

Then MPE is evaluated by:

$$MPE = \sum_{1 \le i \le n}^{\max} 2mu \left[\left| S_i - \widehat{S}_i \right| / S_i \right] \tag{1.10}$$

2 Parameters Estimation for Exponential Process

2.1 Using MLE Method

Assume that $\{X(t), \ t \ge 0\}$ is a NHPP with the time rate of occurrence defined by formula in Equation 1.3, then the joint probability function of the occurrence times $t_1, t_2, \ldots, \ t_n)$ in which $(0 < t_1 \le t_2 \le \ldots \le \ t_n \le t_0)$ is defined by the following [2];[9]:

$$f_n(t_1, t_2, \ldots, \ t_n) = \prod_{i=1}^{n} \lambda(t_i) \, e^{-m(t_0)}$$

Therefore, the cumulative function of the time rate of occurrence, which is a parameter of the Exponential process and defined as:

$$\begin{aligned} m(t) &= \int_0^t \lambda(u) du \\ &= \int_0^t \alpha e^{-\beta u} du \\ &= -\frac{\alpha}{\beta} \left(\ e^{-\beta u} \right|_0^t \\ &= -\frac{\alpha}{\beta} (1 - \ e^{-\beta t_0}) \quad 0 \le t \le t_0 \end{aligned} \tag{2.1}$$

Hence, the Likelihood function Lf_n for the Exponential process for the period $(0, t]$ with the rate time $\lambda(t)$ is

$$Lf_n = \prod_{i=1}^{n} \alpha e^{-\beta t_i} e^{-\frac{\alpha}{\beta}(1 - e^{-\beta t_0}})\tag{2.2}$$

The formula 2.2 becomes as follows:

$$Lf_n = (\alpha)^n \ e^{-\beta \sum_{i=1}^{n} t_i} e^{-\frac{\alpha}{\beta}(1 - \ e^{-\beta t_0})} \tag{2.3}$$

To simplify the calculations, the natural logarithm of the maximum function is taken instead of the maximum function itself represented by the Equation 2.3. Hence, the following formula is obtained:

$$\ln L = n \ln(\alpha) - \beta \sum_{i=1}^{n} t_i - \frac{\alpha}{\beta} \ e^{(1 - \beta t_0)} \tag{2.4}$$

Then, the two parameters α and β of the function $\lambda(t)$ are estimated using the following equations:

$$\frac{\partial \ln L}{\partial \alpha}|_{\alpha=\widehat{\alpha}} = 0 \;\rightarrow\; \frac{n}{\widehat{\alpha}} - \frac{1}{\widehat{\beta}}(1 - e^{-\beta t_0}) = 0 \tag{2.5}$$

Therefore, the maximum likelihood estimator for the parameter α for Exponential process is:

$$\widehat{\alpha} = \frac{n\widehat{\beta}}{1 - e^{-\widehat{\beta} t_0}} \tag{2.6}$$

The distribution of parameter b can be inferred through the conditional distribution of the variable $S = \sum_{i=1}^{n} t_i$ conditional on the number of incidents n the reason is that the observations in the potential function of the non homogeneous Poisson process only come from n, $\sum_{i=1}^{n} t_i$. In order to find the maximum likelihood estimator for parameter b, we need to find the probability distribution for it, which represents the conditional distribution of the variable $S = \sum_{i=1}^{n} t_i$ conditional on the number of events n. To get the probability distribution of the variable S conditioned by the number of incidents n, this is done by dividing the potential function of the non homogeneous Poisson process in the formula 1.1 as follows [4]; [16]:

$$\begin{aligned} f\left[S|N\left(t\right)=n\right] &= \frac{L}{\frac{e^{-m(t)}}{n!}\left(m(t)\right)^{n}} \\ &= \frac{\prod_{i=1}^{n} \lambda\left(t_i\right) e^{-m(t_0)}}{\frac{e^{-m(t)}}{n!}\left(m(t)\right)^{n}} \\ &= \frac{n! \prod_{i=1}^{n} \lambda(t_i)}{\left(m\left(t_0\right)\right)^{n}} \end{aligned} \tag{2.7}$$

If the probability distribution of the variable $S = \sum_{i=1}^{n} t_i$ is conditional on the number of events n it becomes a conditional probability density function:

$$f[S|N(t) = n] = \frac{n! \prod_{i=1}^{n} \lambda(t_i)}{\left(m\left(t_0\right)\right)^{n}} \qquad ;\; n = 0, 1, 2, \ldots \tag{2.8}$$

In the non homogeneous Poisson process with a time rate of occurrence described by an exponential function according to the formula 1.3, the conditional maximum function of parameter β when α is fixed conditioned by the number of events n is as follows:

$$\begin{aligned} L[S|N(t) = n] &= \frac{n! \prod_{i=1}^{n} \alpha\, e^{-\beta t_i}}{\left(\frac{\alpha}{\beta}(1 - \; e^{-\beta t_0})\right)^{n}} \\ &= \frac{n! \beta^{n}\; e^{-\beta \sum_{i=1}^{n} t_i}}{\left(1 - e^{-\beta t_0}\right)^{n}} \;,\; \beta \neq 0 \end{aligned} \tag{2.9}$$

And when $\beta = 0$, the time rate of occurrence will be in the following form:

$$\lambda(t) = \alpha \tag{2.10}$$

Therefore, the conditional maximum function for parameter β will be as follows:

$$\begin{aligned} L[S|N(t) = n] &= \frac{n! \prod_{i=1}^{n} \alpha}{\left(\alpha t\right)^{n}} \\ &= \frac{n!}{\left(t_0\right)^{n}} \;,\; \beta = 0 \end{aligned} \tag{2.11}$$

When parameter α is constant and limited to n occurrences, then the probability function of $N(t)$ for the parameter β in the Exponential process is represented as:

$$L[S|N(t) = n] = \begin{cases} \frac{n! \beta^{n}\; e^{-\beta \sum_{i=1}^{n} s_i}}{\left(1 - e^{-\beta s_0}\right)^{n}} & \beta \neq 0 \\ \frac{n!}{t_0^{n}} & \beta = 0 \end{cases} \tag{2.12}$$

when $\beta = 0$ the probability density function over time is $(0, t_0]$ represent Uniform Distribution, either in case $\beta \neq 0$ then the probability density function for the same time period and taken for a random sample size (n) is from a distributed community Truncated Exponential Distribution, usually this process Order Statistics. The probability density function of a random variable (t) is in the event that there is only one view of the time rate of occurrence of accidents $\lambda(t)$ described by the Exponential function is:
The log-likelihood function for formula 2.12 is expressed as follows:

$$\ln L = \begin{cases} \ln(n!) + n\, ln\beta - \beta \sum_{i=1}^{n} t_i + n\, ln(1 - e^{-\beta t_0}) & \beta \neq 0 \\ \ln(n!) - n \ln t_0 & \beta = 0 \end{cases} \tag{2.13}$$

The derivative of the logarithm of the possible function with respect to the parameter is found β as follows:

$$\frac{\partial lnL}{\partial \beta} = \begin{cases} \frac{n}{\beta} - \sum_{i=1}^{n} t_i - \frac{nt_0}{(e^{\beta t_0} - 1)} & \beta \neq 0 \\ -\frac{1}{2} n t_0 + \sum_{i=1}^{n} t_i & \beta = 0 \end{cases} \tag{2.14}$$

As much as possible for the parameter β it can be found by solving the following equation:

$$\left. \frac{\partial lnL}{\partial \beta} \right|_{\beta = \widehat{\beta}} = 0 \tag{2.15}$$

It was shown from the Equation 2.14 there is no analytical solution to it when $\beta \neq 0$, Therefore, numerical methods were used to obtain a number of estimated values for the parameter β, and prepare Newton's Method one of the methods used to solve the equations of the maximum likelihood method, through which the estimator was obtained, as this method is one of the numerical methods important which is used to find the capabilities of the maximum likelihood in the absence of a solution by the following Ordinary algebraic methods and as in the formula:
For parameter β , Taylor's theorem is used to find it in the following form:

$$0 = \left. \frac{\partial lnL}{\partial \beta} \right|_{\beta = \widehat{\beta}} \cong \frac{\partial lnL}{\partial \beta} + \left(\widehat{\beta} - \beta \right) \frac{\partial^2 \ln L}{\partial \beta^2} \tag{2.16}$$

Then

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{\partial^2 \ln L}{\partial \beta_i\, \partial \beta_j} \ ;\ i \neq j\ , \quad i, j = 1, 2, 3, \ldots$$

$$\left(\widehat{\beta} - \beta \right) \frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{\partial lnL}{\partial \beta} \tag{2.17}$$

Then the maximum likelihood estimator for the parameter β is:

$$\widehat{\beta} = \beta - \frac{\frac{\partial lnL}{\partial \beta}}{\frac{\partial^2 \ln L}{\partial \beta^2}} \tag{2.18}$$

And by converting the formula 2.18 to a sequential formula, then we get the following:

$$\widehat{\beta}_i = \beta_{i-1} - \frac{\left. \frac{\partial lnL}{\partial \beta} \right|_{\beta_{i-1} = \widehat{\beta}_{i-1}}}{\left. \frac{\partial^2 \ln L}{\partial \beta^2} \right|_{\beta_{i-1} = \widehat{\beta}_{i-1}}} \quad ; i = 1, 2, 3, \ldots \tag{2.19}$$

In order to complete the formula 2.19 which is a general formula to find the maximum likelihood estimator for parameter β in the direct method, the formula 2.14 is derived to get the second derivative as follows:

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n}{\beta^2} - \frac{n t^2{}_0\, e^{\beta t_0}}{\left(e^{\beta t_0} - 1\right)^2} \tag{2.20}$$

And by substituting into the formula 2.12 then the maximum likelihood parameter β is obtained as follows:

$$\widehat{\beta} = \beta - \frac{\frac{n}{\beta} - \sum_{i=1}^{n} t_i - \frac{n t_0}{\left(e^{\beta t_0} - 1\right)}}{-\frac{n}{\beta^2} - \frac{n s^2{}_0\, e^{\beta t_0}}{\left(e^{\beta t_0} - 1\right)^2}} \tag{2.21}$$

And by converting the last formula to a successive form, as in the formula 2.21 the sequential estimator is obtained:

$$\widehat{\beta}_i = \beta_{i-1} - \left(\frac{n}{\widehat{\beta}_{i-1}} - \sum_{i=1}^{n} t_i - \frac{nt_0}{(e^{\beta_{i-1}t_0} - 1)} \right) \left(-\frac{n}{\beta_{i-1}{}^2} - \frac{nt^2{}_0\, e^{\beta_{i-1}t_0}}{(e^{\beta_{i-1}t_0} - 1)^2} \right)^{-1} \tag{2.22}$$

Therefore, the estimator of the maximum likelihood for parameter β using the method of maximum likelihood is:

$$\widehat{\beta}_{\text{MLE}} = \widehat{\beta}_i \tag{2.23}$$

Substitution formula 2.23 in formula 2.6, we get:

$$\widehat{\alpha}_{\text{MLE}} = \frac{n\widehat{\beta}_{\text{MLE}}}{1 - e^{-\widehat{\beta}_{\text{MLE}}t_0}} \tag{2.24}$$

Remark 1. In order to find different possible values for the estimator for the parameter β, the time rate of occurrence of the Exponential Process, a computer program written in MATLAB/R2019b is used.

2.2 Estimating parameter of Exponential Process using FFA

In this section, Exponential process parameters α,β which represents the parameter for the time rate of occurrence, is estimated using FFA algorithm. For the estimation process, an algorithm, Algorithm (1), is proposed; it is described as follows:

Algorithm (1): Algorithm for estimating Exponential Process Parameter:

Step1: Set the number of particles $N = 50$ and the number of iterations with $i_{\text{max}} = 100$.

Step2: Identify the input parameters:

alpha = 0.5, beta0 = 0.2, gamma = 1, n= Number of observations;

Step3: Set the position of each particle, where each position represents an estimation for the Exponential Process parameters α and β;

Step4: Determine the fitness function, which is represented by the MPE, in which $MPE = \sum_{1\leq i\leq n}^{\text{max}} 2mu \left[\left| S_i - \widehat{S}_j \right| / S_i \right]$;

Step5: Implement the FFA algorithm using the following values:

$delta =\ 1 - ((10^{-4})/0.9)\hat{}(1/i)$

$$alpha1 = (1 - delta) * \text{alpha}$$

$T = alpha1 * (rand(1, N) - 0.5)$

$$r = norm(xx(i,:) - xx(j,:))$$

$$beta =\ ((beta0) * (exp(-1 * \text{gamma} * r\hat{}2)))$$

$$x1(i,:) =\ xx(i,:) + (beta * (xx(j,:) - xx(i,:))) + T(1,:)$$

Step6: Calculate each of α and β value, depending on the resultant value of the objective function MPE.

Step7: Repeat steps 4 - 5 until an i_{max} is reached.

Remark:

- For implementing Algorithm 1, a written program in MATLAB/R2019b is used.
- The following flowchart represents a schematic diagram of the FFA algorithm to estimate Exponential process parameter:

2.3 Estimation of Exponential Process using PSO

In this section, Exponential process parameters are estimated using PSO algorithm. For the estimation process, an algorithm, Algorithm (2), is proposed; it is described as follows:

Algorithm (2): Algorithm for estimating Exponential Process Parameters
For this algorithm, the following steps are set:

Step1: Identify each of: the number of particles (N=50); the number of iterations with $i_{\max} = 100$; the acceleration coefficients C_1, C_2 in which $C_1 = C_2 = 1$, $(r_1 = r_2 = 0.1)$; further the minimum and the maximum values for the inertial weight are: $\gamma_{\max} = 0.9$ and $\gamma_{\min} = 0.4$;

Step2: Determine randomly the position of each particle, where each position represents an estimation for the Exponential process parameter and the initial position of each particle is generated from a uniform distribution within the range $[0, 1]$;

Step3: Generate the initial velocity for each particle from a uniform distribution.

Step4: Set the fitness function as the MPE, in which MPE $= \sum_{1\leq i\leq n}^{\max} 2mu \left[\left|S_i - \widehat{S}_i\right| / S_i\right]$;

Step5: Adjust, depending on the resultant value of the function MPE, the Parameter estimator $\widehat{\beta}$ for the process under study by updating the speed (V_i) according to the following equation:

$$V_j^{(i)} = \gamma V_j^{(i-1)} + C_1 r_1 \left[P_{\text{best},j} - X_j^{(i-1)}\right] + C_2 r_2 \left[G_{best,j} - X_j^{(i-1)}\right] \quad j = 1, 2 \ldots, N \tag{2.25}$$

As well as updating sites X_i depending on the equation:

$$X_j^{(i)} = X_j^{(i-1)} + V_j^{(i)}; j = 1, 2, \ldots, N \tag{2.26}$$

Step6: Repeat steps 4 - 5 until an $i_{\max}$ is reached.

3 Simulation

Simulation is a computer-based methodology that enables experimentation on a valid digital representation; it can be used to imitate the process of an existing or a proposed system and then analyze its behaviour by considering different possible values for the main parameters of the problem. An algorithm, which is written as a computer program, describes the discipline of a designing model for a process, which is under study, and then executing it on a digital computer, then analyzing the obtained output. furthermore, simulation is considered as one of the best methods used for generating random variables from specific distribution functions. The importance of the simulation lies in using simulated random numbers to estimate parameters in which different random numbers, which are independent from each other, are used for different experiments. For statistical analysis, the most widely used simulation method is the Monte Carlo simulation method; it is used to estimate parameters or statistical measures and then tests the properties of the estimates. In this section, a comprehensive simulation study was performed to compare between the three estimation methods to reach at the best estimate for the Exponential parameters. To design the simulation experiments, the following four steps are described [7]:

3.1 Assigning the Virtual Values

In this stage, which is considered as the main one for the other steps, in generating random numbers the required virtual values are specified. These values are:

1. **Sample Size (n)**

   In generating random numbers, to obtain an efficient and accurate results, the sample size plays an important role. Therefore, in this study, various sample sizes are used; for example: small-sized sample with (N=20); middle-sized sample with (N=50) and a large-sized sample with (N=100).

2. **Parameter Values for Exponential Process**

   The probability distribution function for Exponential processes used to generate random variables using different uniform random numbers assuming values for $\widehat{\alpha} = 0.5$, 0.8 ; $\widehat{\beta}$ =0.8 [1];[6].

3. **Sample Repetition Size (i)** In order to obtain high homogenous results, the experiments were repeated for (i=50) once for each experiment.

3.2 Generating random variables

Random variables are generated from the Exponential distribution function based on each value of the virtual parametric values and the assumed sample size N as follows:

(i) Generate random numbers that follow continuous uniform distribution over $[0, 1]$.

(ii) Change the generated uniform data into data that follow Exponential probability distribution by using the cumulative density function and according to the inverse transform method; this method is one of the simplest simulation techniques and the most important methods to get random variables from continuous and discrete distributions. Then, the algorithm for generating random variables from Exponential distribution using inverse transform method is described by:

(iii) From the cumulative probability function for the Exponential distribution, which is defined as:

$$m(t) = \int_0^t \lambda(u)\, du \tag{3.1}$$

Where

$$\lambda(t) = \alpha e^{-\beta t} 0 \leq \ t \ \leq \ \infty, \ \ \alpha, \beta > 0 \tag{3.2}$$

Then

$$m(t) = \int_0^t \alpha e^{-\beta u} du = -\frac{\alpha}{\beta}\left(e^{-\beta t} - 1\right) = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \tag{3.3}$$

Since $U = m(t)$ where $U \ \sim \ U(0, 1)$ $u = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$

then $e^{-\beta t} = 1 - \frac{\beta}{\alpha} u$

By taking the natural logarithm of both sides, the following is obtained:

$-\beta t = \ln\left(1 - \frac{\beta}{\alpha} u\right)$

The random generator for the Exponential process by the following equation:

$$t = -\frac{1}{\beta} \ln\left(1 - \frac{\beta}{\alpha} u\right) \tag{3.4}$$

Hence, from implementing Inverse Transform Method, various random variables, $t(i)$ are obtained based on the Exponential function using the MATHLAB program:

Hence, $t(i) = -\frac{1}{\beta} * \log\left(1 - \frac{\beta}{\alpha} * u(i)\right)$ for $i = 1, 2, \ldots N$

Where $U(i) \ \sim \ U(0, 1)$

3.3 Comparing Estimators

When parameters are estimated using different methods then comparison stage is performed based on the Root MeanSquares Error (RMSE). To implement RMSE, the following form is used [10].

$$RMSE = \sqrt{\frac{\sum_{i=1}^{Q} \left(\widehat{\theta}_i - \theta\right)^2}{Q}} \tag{3.5}$$

$\widehat{\theta_i}$: Represents value of the parameter estimated in iteration i. θ : Represents the real parameter value.

Q : Represents number of iterations.

3.4 Numerical Computations

The following results, which is shown by Table 1, represent the results of implementing simulation to generate different random variables from the stochastic Exponential process using different sample sizes $(n = 20, 50, 100)$, for two values for Exponential process parameters α, β $(\alpha = 0.5\ , 0.8\ and\ \beta = 0.8)$ for the three methods, MLE and the proposed intelligent methods namely FFA and PSO.

Table1. RMSE for the ML, FFA and PSO obtained from Exponential process

α	β	n	Methods	RMSE($\hat{\alpha}$)	RMSE($\hat{\beta}$)
0.5	0.8	20	MLE	13.9204	0.1118
			FFA	0.1105*	0.1117*
			PSO	0.1106	0.9465
		50	MLE	8.8040	0.0707
			FFA	0.0699*	0.0711*
			PSO	0.0670	0.5986
			MLE	6.2254	0.0500
		100	PSO	0.0495	0.4233
			FFA	0.0494*	0.0502*
0.8	0.8	20	MLE	25.5407	0.1789
			FFA	0.1781*	0.1665*
			PSO	0.1782	1.0373
		50	MLE	16.1534	0.1131
			FFA	0.1127*	0.1053*
			PSO	0.1128	0.6560
		100	MLE	11.4222	0.0800
			FFA	0.0797*	0.0745*
			PSO	0.0798	0.4639

Numerical results in Table1 show that, for estimating the Exponential process parameters for different values of n, FFA method gives more efficient results than the other two techniques, hence, it is more efficient than the MLE and PSO methods.

4 Application

In this study, in order to evaluate the applicability of the three approaches, real data from Badush Cement Factory was used. The new Badush Cement Factory in Nineveh Governorate is one of the most important factories for the General Cement Company in north of Iraq; it is the main source for cement production for the governorates of Iraq in general and Nineveh Governorate in particular. The data represent the successive operating periods in days between two successive stops for the cement production during the period from 1/4/2018 to 31/1/2019. To ensure the adequacy fit of data, a test of goodness of fit is needed; it is explained below

4.1 Test of the Homogeneity of Exponential Process

The Exponential process is considered as a non homogeneous process when the time rate of accidents varies with time change$t$ and the behaviour of the parameter β is affected by time change t $(\beta \neq 0)$, while the Exponential process is homogeneous in the case $\beta = 0$. To conduct a test process to specify the homogeneity of the process, the following hypothesis is tested [4]:

$$H_0 : \beta = 0$$
$$H_1 : \beta \neq 0$$

The statistical laboratory used to test the above hypothesis is:

$$Z = \frac{-\sum_{i=1}^{n} t_i - \frac{1}{2} n t_0}{\sqrt{\frac{n {t_0}^2}{12}}} \tag{4.1}$$

Where:
$\sum_{i=1}^{n} t_i$ is the sum of the accident times for a time period $(0, t_0]$.
n : Represents the number of accidents that occur in a period of time.$(0, t_0]$.

4.2 The consistency test of the data under study

When the homogeneity of the data under study is tested based on the value for the statistical test then the consistency test can be verified. Since the obtained value for Z-test is calculated and found to be: $Z = 74.4596$, then as this calculated value is too large compared to its tabular value (1.96) at the level of significance (0.05), therefore, the null hypothesis is rejected and the alternative accepted, meaning that the process under study is heterogeneous.

4.3 Parameter Estimation using Data under Study

In order to assess the performance for each of the intelligence methods, the FFA and PSO, with the classical method MLE, in estimating the Exponential Process Parameters, the parameters of the process under study, the real data, which represent number of days for operating periods between two successive stops and the times of occurrence, is used. The data is raw materials for a laboratory of the new Badush cement factory in Mosul in Iraq. To implement the required algorithms, a specific Matlab program is used. The following table, Table 2, displays the results for parameter estimation using different methods based on real data under study.

Table2. Estimation of Exponential process parameters based on real data under study

Methods	Parameters estimator $\widehat{\alpha}$	Parameters estimator $\widehat{\beta}$
MLE	0.4077	2.0000
FFA	1.7677	0.0001
PSO	1.9153	0.0444

Table (2) shows the estimations of Exponential process parameters for the operating periods between two consecutive stops in days for the mill for raw materials using the proposed estimation methods FFA, PSO and MLE; several runs were conducted in the estimation process and different values for the parameters were used. Then, the best estimated values for the $\widehat{\alpha}, \widehat{\beta}$ were obtained based on the following values:

- Consecutive operating periods between two successive stops of the raw materials mill in days during the extended period of time from 1/4/2018 to 31/1/2019, which represent 53 runs / day are: t = [3 8 2 4 1 1 2 3 1 1 1 1 3 2 3 1 1 1 2 3 5 6 5 2 1 1 4 1 4 3 1 3 1 1 7 2 5 1 2 1 1 3 3 1 6 1 2 3 3 1 3 2 1]

4.4 Results and Discussion

To draw accurate conclusions about the performance of various approaches, estimating inaccuracy or systematic error for these methods plays an important role in comparing these methods to select the most efficient one based on real data under study. In this study, to estimate the parameters of the Exponential Process, the MPE measure described by formula 2.12 for the three methods are determined as the criterion for measuring error for each approach. A MATLAB computer program calculates the MPE between the real and the estimated values of the average plant shutdown time for the three methods using the raw materials mill for the new Badush Cement Factory during the period under study; it is shown by the following table:

Table.3 MPE values for the three methods used to estimate Exponential parameters

Methods	MPE
MLE	0.9969
FFA	0.7674*
PSO	0.7932

Table 3 explains that the obtained MPE value for FFA approach is less than that for each of MLE and PSO. This indicates that the smart method FFA is more efficient than others for the estimation of Exponential Process Parameters.
Furthermore, the following figure represents the estimation for the EP parameters using conventional and intelligent methods used in the study, compared with the real cumulative values that represent the successive operating periods between two successive stops of the raw materials mill for the new Badush Cement Factory:

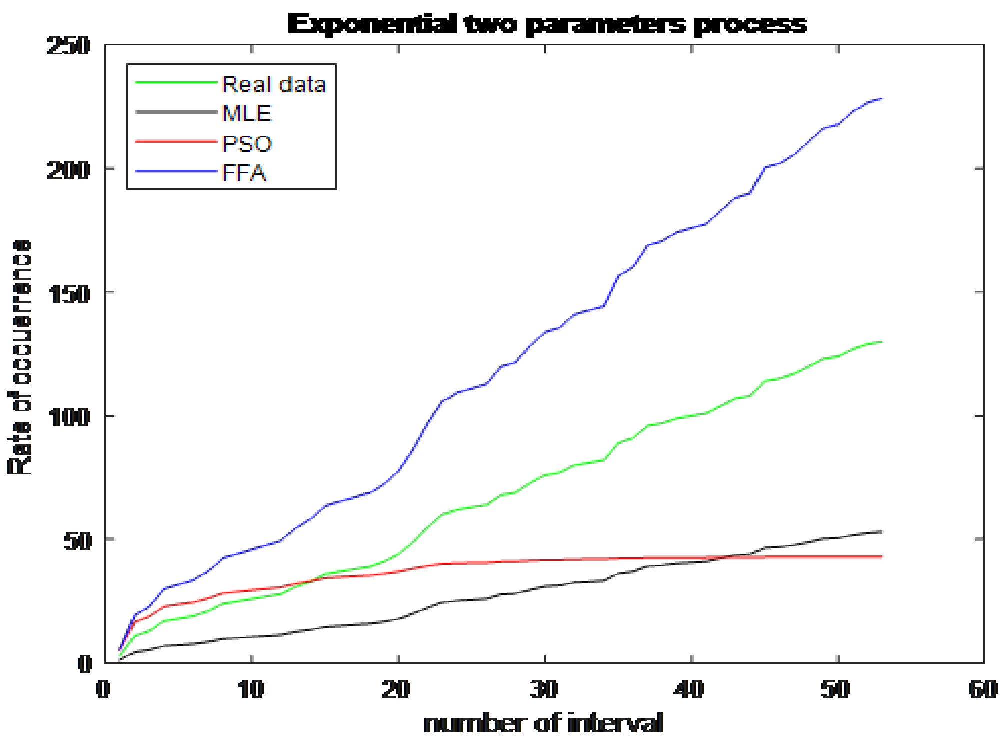

Figure 1. Estimated functions for the cumulative number of successive operating periods between two successive mill stops using different methods.

Figure (1)display the estimated functions of the cumulative number of successive operating periods between two successive mill stops raw materials for the new Badush cement plant using estimation methods compared to real data. This shows, graphically, that the FFA method gives a better representation compared to other methods; it explains that the FFA method is more efficient than the other two methods namely the MLE and PSO.

Conclusion

In this study, the main aim is to propose an efficient parameter estimation technique described by FFA and PSO algorithms for NHPP model with rate of occurrence defined by Exponential function. It is concluded that FFA, PSO would provide with more accurate estimate for the parameter in addition to its fast convergence for maximum value for the function compared with the classical MLE method based on the results for the calculated values for the MPE measure. Moreover, by using simulation it is shown that, for different values and for many iterations, FFA algorithm can guarantee more efficient results and fastest in getting a robust convergence than MLE, PSO methods. Finally, when both approaches are applied to the real data, the graphical distribution for the cumulative number of days for operating periods between two successive stops and the times of occurrence on the logarithmic scale of the data under study shows that the FFA, PSO allows a linear relationship and hence the possibility of modelling such data with Exponential function. It is recommended to apply data from other fields to see the availability of these algorithms.

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